A NEW QUANTITY IN RIEMANN-FINSLER GEOMETRY*

XIAOHUAN MO

Key Laboratory of Pure and Applied Mathematics
School of Mathematical Sciences
Peking University, Beijing 100871, China
e-mail: moxh@pku.edu.cn

ZHONGMIN SHEN

Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 N. Blackford Street
Indianapolis, IN 46202-3216, USA
e-mail: zshen@math.iupui.edu

and HUAIFU LIU

College of Applied Science
Beijing University of Technology, Beijing 100022, China
e-mail: liufu369@emails.bjut.edu.cn

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Abstract. In this note, we study a new Finslerian quantity \( \hat{C} \) defined by the Riemannian curvature. We prove that the new Finslerian quantity is a non-Riemannian quantity for a Finsler manifold with dimension \( n = 3 \). Then we study Finsler metrics of scalar curvature. We find that the \( \hat{C} \)-curvature is closely related to the flag curvature and the \( H \)-curvature. We show that \( \hat{C} \)-curvature gives, a measure of the failure of a Finsler metric to be of weakly isotropic flag curvature. We also give a simple proof of the Najafi-Shen-Tayebi' theorem.

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1. Introduction. In Finsler geometry, there are several important geometric quantities: the flag curvature, the (mean) Cartan torsion and the (mean) Berwald curvature, etc. (cf. [6, 10]). In [1], H. Arbar-Zadeh considered a Finslerian quantity \( H \), which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. Arbar-Zadeh proved that for a Finsler metric of scalar flag curvature with dimension \( n \geq 3 \), the flag curvature is constant on the manifold if and only if \( H = 0 \).

Recently, a great progress has been made in studying Finsler metrics of weakly isotropic flag curvature. These Finsler metrics are of scalar curvature whose flag curvature is in a special form \( K = \theta/F + \sigma \) where \( \theta \) is a 1-form and \( \sigma \) is a scalar function on \( M \). Finsler metrics of weakly isotropic flag curvature not only include Finsler metrics of constant flag curvature, but also include Finsler metrics of (almost) isotropic \( S \)-curvature and of scalar flag curvature [4, 6, 13]. Cheng and Shen have

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classified Finsler metrics of Randers type with weakly isotropic flag curvature via the navigation problem in Riemannian manifolds \([5, 12]\). Chen-Zhao constructed explicitly Finsler metrics are of scalar curvature such that they are not of weakly isotropic flag curvature \([3]\). A natural task for us is to give a geometric quantity on a Finsler manifold, which characterises the Finsler metrics of weakly isotropic flag curvature.

In this paper, we find the desired quantity (see Section 2 below) and call it the \(\hat{C}\)-curvature. We show that the \(\hat{C}\)-curvature gives a measure of the failure of an \(n\)-dimensional Finsler metric of scalar curvature to be of weakly isotropic flag curvature if \(n \geq 3\). Precisely we prove the following:

**Theorem 1.1.** Let \((M, F)\) be an \((n \geq 3)\)-dimensional Finsler manifold of scalar curvature with flag curvature \(K(x, y)\). Then \(K\) is weakly isotropic if and only if the \(\hat{C}\)-curvature vanishes.

Recall that a Finsler metric \(F\) is said to be of *scalar curvature* if the flag curvature \(K = K(x, y)\) is a scalar function on the slit tangent bundle \(TM\{0\}\). For these Finsler metrics, we establish an equation between the flag curvature \(K\), the \(\hat{C}\)-curvature \(\hat{C}\) and the \(H\)-curvature \(H\) (see Proposition 3.2 below), therefore the \(\hat{C}\)-curvature is subtly related to the flag curvature and the \(H\)-curvature.

Recently, Najafi-Shen-Tayebi extended Arbar-Zadeh’s characterisation for Finsler metrics of constant flag curvature and proved the following \([8, 11, 13]\): 

**Theorem 1.2.** Let \(F\) be a Finsler metric of scalar flag curvature on an \((n \geq 3)\)-dimensional manifold \(M\). For a 1-form \(\theta\), the flag curvature is weakly isotropic given by (2.10) if and only if the \(H\)-curvature satisfies the following:

\[
H_{ij} = \frac{n + 1}{6} \theta F_{y^j y^i},
\]

(1.1)

where \(\theta = \theta_i(x)y^i\) is a 1-form on \(M\).

See Section 2 for the definition of the \(H\)-curvature. Say a Finsler metric \(F\) has *almost vanishing* \(H\)-curvature if its \(H\)-curvature is given by (1.1) \([11, 13]\). By using Theorem 1.1 and Proposition 3.2, we obtain a new and simple proof of Theorem 1.2 (see Section 5).

Riemannian metrics are a special case of Finsler metrics, namely Finsler metrics with the quadratic restriction. Call a geometric quantity on a Finsler manifold *non-Riemannian* if it vanishes for a Riemannian metric. For instance, the (mean) Cartan torsion, the \(S\)-curvature and the \(H\)-curvature are all non-Riemannian \([6, 7]\). In Section 6, we show the following:

**Theorem 1.3.** For a Finsler manifold with dimension \(n = 3\), the \(\hat{C}\)-curvature is a non-Riemannian quantity.

2. **Preliminaries.** Let \((M, F)\) be a Finsler manifold of dimension \(n \geq 3\). In a standard local coordinate system \((x^i, y^j)\) in \(TM\), \(F = F(x, y)\) is a function of \((x^i, y^j)\). Let

\[
g_\theta(x, y) := \frac{1}{2} [F^2]_{y^j y^i}(x, y)
\]
and \((g^\parallel) := (g_\parallel)^{-1}\). Let \(R^i_j\) denote the Riemannian curvature of \(F\) [7]. The \(\hat{C}\)-curvature is defined by 
\[\hat{C} = \hat{C}_i dx^i \otimes dx^i\] where
\[
\hat{C}_i = R_{ij} + \frac{1}{n-2} \left( y_i \text{Ric}_j + y_j \text{Ric}_i - g_{\parallel} \text{Ric} - \text{Ric}_{\parallel} F^2 \right) + \frac{SF^2}{(n-1)(n-2)} h_{\parallel},
\] (2.1)
where \(R_{ij} := g_{ik} R^k_j\) is the flag curvature tensor of \(F\), \(h_{\parallel} := FF_{y'y'}\) is the angular metric of \(F\) [10] and
\[
\text{Ric} := R^i_j, \quad (2.2)
\]
\[
\text{Ric}_j := \frac{1}{2} \text{Ric}_{y'^j}, \quad (2.3)
\]
\[
\text{Ric}_{\parallel} = \frac{1}{2} \text{Ric}_{y'y'}, \quad (2.4)
\]
\[
y_i := g_{\parallel} y^i, \quad (2.5)
\]
\[
S := g^k \text{Ric}_{\parallel}. \quad (2.6)
\]

\(F\) is of scalar flag curvature with flag curvature \(K\) is equivalent to the following equation (see [6, page 110]):
\[
R^i_j = K F^2 h^i_j, \quad (2.7)
\]
where
\[
h^i_j = \delta^i_j - F^{-2} g_{jk} y^k y^j = g^{dk} h_{kj}. \quad (2.8)
\]
It is easy to see that (2.7) holds if and only if
\[
R_{\parallel} = K F^2 h_{\parallel}. \quad (2.9)
\]

In Sections 4 and 5, we will consider Finsler metrics of weakly isotropic flag curvature defined as follows:
\[
K = \frac{\theta}{F} + \sigma, \quad (2.10)
\]
where \(\sigma = \sigma(x)\) is a scalar function and \(\theta = \lambda_i(x) y^i\) is a 1-form.

The \(H\)-curvature \(H_{y} = H_{y} dx^i \otimes dx^i\) is defined by
\[
H_{y} = E_{y[k]y^k}
\]
where “\(\mid\)” denotes the covariant horizontal derivatives and \(E_{y}\) denote the mean Berwald curvature of \(F\) [8, 13]. The \(H\)-curvature vanishes for a \(R\)-quadratic Finsler metric [7, 9].

3. Finsler metrics of scalar flag curvature. Assume that \(F\) is of scalar curvature, that is, the flag curvature \(K = K(x, y)\) is a scalar function on \(TM \setminus \{0\}\). Using (2.2) and (2.7), we obtain the Ricci scalar \(\text{Ric}\) is given by
\[
\text{Ric} = (n - 1) K F^2, \quad (3.1)
\]
where we have used $F^{-2}g_{ij}y^iy^j = 1$. We use the following notations:

$$K_i = K_{iy^i}, \quad K_y = K_{y^iy^j}.$$ 

By (2.3) and (3.1) we get

$$Ric_j = \frac{n-1}{2}[KF^2]_{y^j} = \frac{n-1}{2}(K_jF^2 + 2KY_j),$$

(3.2)

where we have used the fact

$$y_j = FF_{y^j} = \left( \frac{F^2}{2} \right)_{y^j}.$$ \quad (3.3)

Together with (2.4) we obtain

$$Ric_y = \frac{n-1}{2}(K_yF^2 + 2K_iy_j + 2K_jy_i + 2Kg_{ij}),$$

(3.4)

where we have used $(y_j)_{y^j} = \left( \frac{F^2}{2} \right)_{y^j} = g_{ij}$. Note that the flag curvature $K$ is homogeneous of degree zero with respect to $y$. It follows that

$$K_{jy^j} = 0.$$ \quad (3.5)

Using (2.6), (3.4) and (3.5), we obtain the scalar curvature $S$ is determined by

$$S = \frac{n-1}{2}(g^yK_yF^2 + 4K_jy_j + 2nK) = n(n-1)K + \frac{(n-1)^2}{2}F^2\Psi,$$

where $\Psi := \frac{1}{n-1}g^yK_y$. Together with (2.1), (2.9), (3.2) and (3.4), we get

$$\hat{\nabla}^2 g_{ij} = \frac{n-1}{2(n-2)}(4K_yy_j - K_jy_jF^2 - K_jy_iF^2 - 4KF^2g_{ij} - K_yF^4)$$

$$+ \frac{n-1}{2(n-2)}\Psi F^4h_{ij} + \frac{nK}{n-2}F^2h_{ij}.$$ \quad (3.6)

Note that $y_jy^j - F^2g_{ij} = -F^2h_{ij}$. It follows that

$$\hat{\nabla}^2 g_{ij} = \frac{n-1}{2(n-2)}(4Kh_{ij} + K_iy_j + K_jy_i + K_yF^2)$$

$$+ \frac{n-1}{2(n-2)}\Psi F^4h_{ij} + \frac{nK}{n-2}F^2h_{ij}$$

$$= \frac{n-1}{2(n-2)}\left( \Psi F^2h_{ij} - K_iy_j - K_jy_i - K_yF^2 \right)F^2.$$ \quad (3.6)

Hence, we have the following:

**Lemma 3.1.** Let $(M, F)$ be an $n(\geq 3)$-dimensional Finsler manifold of scalar curvature with flag curvature $K(x, y)$. Then $F$ has vanishing $\hat{\nabla}$-curvature if and only if in any standard local coordinate system

$$\Psi F^2h_{ij} = K_iy_j + K_jy_i + K_yF^2,$$

(3.7)

where $\Psi := \frac{1}{n-1}g^yK_y$. In particular, $F$ has vanishing $\hat{\nabla}$-curvature if $F$ has isotropic (or constant) flag curvature.
A Finsler metric $F$ is said to be of isotropic curvature if the flag curvature $K(P, y) = K(x)$ is a scalar function on $M$. In particular, $F$ is said to have constant (flag) curvature if the flag curvature $K(P, y) = \text{constant}$. Now we are going to establish an important equation between the flag curvature, the $\hat{C}$-curvature and the $\mathcal{H}$-curvature.

**Proposition 3.2.** Let $F$ be a Finsler metric of scalar flag curvature on an $n$-manifold $M$. Then the flag curvature, $\hat{C}$-curvature and the $\mathcal{H}$-curvature satisfy

$$\hat{C}_{ij} = \frac{n - 1}{2(n - 2)} \left( \Psi F^3 F_{y^iy^j} + \frac{6H_{ij}}{n + 1} \right) F^2,$$

where $\Psi := \frac{1}{n - 1} g^{ij} K_{ij}$.

**Proof.** A direct calculation yields (cf [7, (3.26)])

$$0 = 6H_{ij} + (n + 1)F[(FK)_{y^iy^j} - KF_{y^iy^j}] = 6H_{ij} + (n + 1)(K_{ij}y_j + K_{ji}y_i + K_{ij}F^2).$$

It follows that

$$-(K_{ij}y_j + K_{ji}y_i + K_{ij}F^2) = \frac{6H_{ij}}{n + 1}.$$ 

Plugging this into (3.6) yields (3.8). \qed

**4. Proof of theorem 1.1.** First suppose that $F$ has weakly isotropic flag curvature, i.e. (2.10) holds. Differentiating (2.10) with respect to $y^i$, we obtain

$$K_i = \frac{\lambda_i}{F} + \frac{\theta l_i}{F^2},$$

where $l_i := F_{y^i} = F^{-1}y_i$. Moreover

$$K_{ij} = \frac{-\lambda_i l_j}{F^2} - \frac{\lambda_j l_i + \theta F_{y^iy^j}}{F^2} + \frac{2\theta l_i l_j}{F^3} = \frac{\theta(3l_i l_j - g_{ij}) - (\lambda_i l_j + \lambda_j l_i)}{F^3},$$

where we have used the fact

$$g_{ij} = h_{ij} + l_i l_j.$$ 

It follows that

$$\Psi = -\frac{\theta}{F^3}.$$ 

From (4.1), (4.2) and (4.3) we get

$$K_{ij}y_j + K_{ji}y_i + K_{ij}F^2 = \frac{\theta(3l_i l_j - g_{ij}) - 2\theta l_i l_j}{F} = -\frac{\theta}{F} h_{ij}.$$ 

It follows from (4.4) and (4.5) that

$$\Psi F^2 h_{ij} = -\frac{\theta}{F} h_{ij} = K_{ij}y_j + K_{ji}y_i + K_{ij}F^2.$$ 

By Lemma 3.1, $F$ has vanishing $\hat{C}$-curvature.
Conversely, suppose that $\hat{C} = 0$. Then (3.7) holds. Differentiating (3.7) with respect to $y^k$, we obtain
\[
\Psi_{j^k} F^2 h_{j^k} + 2\Psi h_{j^k} y^k + \Psi F^2 (h_{j^k})_{j^k} = g_{jk} K_j + y^l K_{jk} + g_{jk} K_i + y^l K_{ik} + 2y^k K_{ji} + F^2 K_{jk},
\]
where
\[
K_{jk} := (K_{j^k})_{j^k} = K_{y^j y^k}
\]
is totally symmetric. Direct computations yield
\[
g^{\hat{j}} h_{\hat{j}} = n - 1,
\]
\[
K_{\hat{j} y^j} = -K_{\hat{j}},
\]
\[
(h_{\hat{j}})_{j^k} = 2C_{\hat{j} k} - F^{-2} y^j h_{ik} - F^{-2} y^j h_{jk},
\]
where $C_{\hat{j} k}$ is the Cartan torsion [6]. Contracting (4.10) with $g^{\hat{j}}$ yields
\[
g^{\hat{j}} (h_{\hat{j}})_{j^k} = 2I_{\hat{k}}
\]
where $I_{\hat{k}}$ is the mean Cartan torsion [6] and we have used the fact
\[
h_{\hat{j} y^j} = 0.
\]
Contracting (4.6) with $g^{\hat{j}}$ gives, by (4.8), (4.9) and (4.11),
\[
(n - 1) \Psi_{j^k} + 2\Psi I_{\hat{k}} = g^{\hat{j}} K_{\hat{j} k}.
\]
Since $\Psi$ is homogeneous of degree $-2$ with respect to $y$, we have $\Psi_{j^k} y^k = -2\Psi$. Together with (2.8) yields
\[
\Psi_{j^k} h^{kj} = \Psi_{\hat{j} \hat{k}} + 2F^{-2} y^j \Psi.
\]
Contracting (4.10) with $g^{ik}$ yields
\[
g^{ik} (h_{\hat{j}})_{j^k} = 2I_{i} - (n - 1)F^{-2} y^j,
\]
where we have made use of (4.8) and (4.12). Contracting (4.6) with $g^{ik}$ gives, by (3.5), (4.9) and (4.12),
\[
\Psi_{j^k} F^2 h^{kj} + \Psi F^2 g^{ik} (h_{\hat{j}})_{j^k} = nK_j + (n - 1)y^j \Psi - 2K_{\hat{j}} + F^2 g^{ik} K_{\hat{j} k}.
\]
Plugging (4.14) and (4.15) into (4.16) yields
\[
F^2 (\Psi_{\hat{j} \hat{k}} + 2F^{-2} y^j \Psi) + \Psi F^2 [2I_{i} - (n - 1)F^{-2} y^j] = (n - 2)K_j + (n - 1)y^j \Psi - F^2 g^{ik} K_{\hat{j} k}.
\]
Taking this together with (4.13) yields
\[
(n - 2)(\Psi_{\hat{j} \hat{k}} + K_j + 2y^j \Psi) = 0.
\]
It follows from (4.17) that
\[
(F^2 \Psi + K)_{j^i} = F^2 \Psi_{j^i} + 2y_j \Psi + K_{j^i} = 0.
\]
Thus
\[
\sigma := F^2 \Psi + K \tag{4.18}
\]
is a scalar function on \( M \). Plugging (4.18) into (3.7) yields
\[
[\sigma(x) - K] h_{ij} = K_{i}j^j + K_{j}i^i + K_{g}F^2. \tag{4.19}
\]
By using (3.3), (4.19) and the definition of the angular metric we get
\[
F_{j^i} K_{i^j} + F_{i^j} K_{j^i} + FK_{i^j}j^i + (K - \sigma)F_{j^i j^i} = 0.
\]
This implies that
\[
[(K - \sigma)F]_{j^i j^i} = [K_{i}F + (K - \sigma)F_{i}j^i]_{j^i} = K_{i}F_j^j + K_{j}F_{i}^i + K_{g}F^2 + (K - \sigma)F_{j^i j^i} = 0. \tag{4.20}
\]
Note that \((K - \sigma)F\) is homogeneous of degree one with respect to \( y \). Together with (4.20) we obtain \((K - \sigma)F\) is a 1-form
\[
(K - \sigma)F = \lambda_i(x) y^i = \theta.
\]
We get that \( K = \theta/F + \sigma \).

5. An alternative proof of the Najafi-Shen-Tayebi’ theorem. In this section, we are going to give a new proof of Theorem 1.2 (see Section 1) using Theorem 1.1 and the important identity (3.8).

5.1. Proof of Theorem 1.2. First suppose that \( H \) is almost vanishing given by (1.1). Plugging (1.1) into (3.9) yields
\[
\theta F_{j^i j^i} + K_i j^i + K_j i^i + K_{g}F^2 = 0. \tag{5.1}
\]
Contracting (5.1) with \( g_{ij} \) gives, by (3.5) and (4.8), \((n - 1) \left( \frac{\theta}{F} + F^2 \Psi \right) = 0. \) It follows that
\[
\theta = -F^3 \Psi. \tag{5.2}
\]
Plugging this into (1.1) yields
\[
H_{g} = -\frac{n+1}{6} \Psi F^3 F_{j^i j^i}. \tag{5.1}
\]
Substituting this into (3.8) gives \( \dot{C} = 0. \) By Theorem 1.1,
\[
K = \frac{\dot{\theta}}{F} + \sigma,
\]
where \( \sigma = \sigma(x) \) is a scalar function and \( \dot{\theta} = a_i(x) y^i \) is a 1-form on \( M \). By (4.4) we arrive at the following identity
\[
\Psi = -\frac{\dot{\theta}}{F^3}.
\]
Plugging this into (5.2) yields \( \hat{\theta} = \theta \). We conclude that the flag curvature is weakly isotropic given by (2.10).

Conversely, suppose that \( K = \frac{\theta}{F} + \sigma \) where \( \sigma = \sigma(x) \) is a scalar function and \( \theta = \lambda_i(x)y^i \) is a 1-form on \( M \). From (4.4) we deduce that

\[
\Psi = -\frac{\theta}{F^3}.
\]  

(5.3)

By Theorem 1.1, \( F \) has vanishing \( \hat{C} \)-curvature. Together with (3.8) we obtain

\[
H_{ij} = -\frac{n + 1}{6} \Psi F^3 F_{ij}.
\]  

(5.4)

Plugging (5.3) into (5.4) yields (1.1).

\[
\square
\]

6. Three-dimensional Finsler manifold. In Finsler geometry, there are several important non-Riemannian quantities: the mean Cartan torsion \( \mathbf{I} \), the Cartan torsion \( \mathbf{C} \) and the \( H \)-curvature \( \mathbf{H} \), etc [6, 7]. They all vanish for Riemannian metrics, hence they said to be non-Riemannian. In this section, we are going to show the following:

\textbf{Theorem 6.1.} For a Finsler manifold with dimension \( n = 3 \), the \( \hat{C} \)-curvature is a non-Riemannian quantity.

\textit{Proof.} Assume that \((M, F)\) is a Riemannian manifold. Then then the flag curvature tensor \( R_g \) is given by

\[
R_{ij} = R_{kijl}(x)y^k y^l.
\]  

(6.1)

where \( R_{kijl}(x) \) is the Riemannian curvature of \( F \). It follows that, from (2.2) and (2.4),

\[
Ric = R_{ij}(x)y^i y^j.
\]  

(6.2)

and

\[
Ric_{ij} = R_g(x),
\]  

(6.3)

where \( R_g(x) \) is the Ricci tensor of \( F \). By using (2.6) and (6.1) we have

\[
S = R,
\]  

(6.4)

where \( R \) is the scalar curvature of Riemannian metric \( F \). Plugging (6.1)–(6.4) into (2.1) yields

\[
\hat{C}_{ij} = R_{kijl}(x)y^k y^l + \frac{R}{(n-1)(n-2)} (g_{gk}g_{lj}y^k y^l - \Psi g_{gjl}g_{kik}y^l y^j)
\]

\[
- \frac{1}{n-2} (R_{ij}F^2 - R_{gij}g_{lj}y^l - R_{gij}g_{lk}y^l y^j + R_{gij}g_{lk}y^l y^j) = C_{kijl}y^k y^l,
\]

where \( C_{kijl} \) is the Weyl conformal curvature tensor [2]. Now our conclusion is an immediate consequence of \( C_{kijl} \equiv 0 \) for a 3-dimensional Riemannian manifold [2, Proposition 3.3.9].

\[
\square
\]
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