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A THEOREM ON NILPOTENCY IN NEAR-RINGS

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Throughout this paper a near-ring N will satisfy the distributive law $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$ for all α, β and γ in N. We shall also assume that $0\alpha=0$ for all α in N.

We prove the following theorem.

Theorem. *Let N be a near-ring, M a right N-subgroup of N and R a right ideal of N. If M and R are nilpotent, then so is $M+R$.*

Proof. We first show that there exists a finite sequence

$$N=T_0\geqslant T_1\geqslant T_2\geqslant\cdots\geqslant T_k=\{0\}$$

of ideals of N, such that, $MT_i\subseteq T_{i+1}$ for $i=0,\ldots,k-1$.

Set $T_0=N$. For $i=1,2,\ldots$, define $T_i=T(MT_{i-1})$, where $T(MT_{i-1})$ denotes the ideal of N generated by the subset MT_{i-1} of N. It follows from the definition of T_i that $MT_i\subseteq T_{i-1}$. Also $MT_i\subseteq T_i$, and thus $T_{i+1}=T(MT_i)\leqslant T_i$. It remains to prove that $T_k=\{0\}$ for some positive integer k. Take k to be the smallest integer such that $M^k=\{0\}$. If $k=1$, then $MT_0=\{0\}=T(MT_0)=T_1$ and the result follows. Assume $k\geqslant 2$ and let r be in $\{0,\ldots,k-1\}$. We shall show by induction on r that $M^{k-r}T_r=\{0\}$. Since $M^kN=\{0\}$, the statement is true for $r=0$. If $M^{k-r+1}T_{r-1}=\{0\}$, then $M^{k-r}MT_{r-1}=\{0\}$ and $MT_{r-1}\subseteq(0:M^{k-r})$. But $(0:M^{k-r})$ is an ideal of N. Thus $T_r=T(MT^{r-1})\leqslant(0:M^{k-r})$ and $M^{k-r}T_r=\{0\}$. In particular we have $MT_{k-1}=\{0\}$ and $T(MT_{k-1})=T_k=\{0\}$.

Now suppose that $N=A_0\geqslant A_1\geqslant\cdots\geqslant A_s=\{0\}$ is a finite sequence of ideals of N such that $MA_i\subseteq A_{i+1}$ for all i in $\{0,\ldots,s-1\}$. Assume further that s is minimal. We shall show by induction on s that $M+R$ is nilpotent. If $s=1$, then $MN\subseteq A_1=\{0\}$. Hence

$$(M+R)(M+R)\subseteq(M+R)N\subseteq R.$$

If $R^k=\{0\}$ for some positive integer k, then

$$(M+R)^{2k}=[(M+R)^2]^k\subseteq R^k=\{0\}$$

and $M+R$ is nilpotent. Suppose the theorem holds for $s-1$. Now $(M+A_{s-1})/A_{s-1}$ and $(R+A_{s-1})/A_{s-1}$ are nilpotent in N/A_{s-1}. Also

$$(M+A_{s-1})/A_{s-1}\cdot A_i/A_{s-1}\subseteq(MA_i+A_{s-1})/A_{s-1}\subseteq(A_{i+1}+A_{s-1})/A_{s-1}$$

for $i=\{0,\ldots,s-1\}$. Thus $(M+A_{s-1})/A_{s-1}$ has a finite sequence, as above, of length $s-1$. Since

$$(M+R+A_{s-1})/A_{s-1}=(M+A_{s-1})/A_{s-1}+(R+A_{s-1})/A_{s-1},$$

we may assume that $(M+R+A_{s-1})/A_{s-1}$ is nilpotent in N/A_{s-1}. Thus there exists a positive integer p such that $(M+R)^p \subseteq A_{s-1}$. Now

$$(M+R)^{p+1} \subseteq (M+R)A_{s-1} \subseteq MA_{s-1}+R.$$

But $MA_{s-1} \subseteq A_s = \{0\}$ and thus $(M+R)^{p+1} \subseteq R$. Since $R^k = \{0\}$, it follows that $(M+R)^{(p+1)k} = \{0\}$. Hence $M+R$ is nilpotent and the proof is complete.

Corollary. *Let N be a near-ring. A finite sum of nilpotent right ideals of N is nilpotent.*

For ideals the above corrollary is easily proved (see (**1**) or (**2**, Corollary 3.2)).

REFERENCES

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