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ON THE BEHAVIOUR OF THE BACKWARD INTERPRETATION OF FEYNMAN–KAC FORMULAE UNDER VERIFIABLE CONDITIONS

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Abstract

We consider the time behaviour associated to the sequential Monte Carlo estimate of the backward interpretation of Feynman–Kac formulae. This is particularly of interest in the context of performing smoothing for hidden Markov models. We prove a central limit theorem under weaker assumptions than adopted in the literature. We then show that the associated asymptotic variance expression for additive functionals grows at most linearly in time under hypotheses that are weaker than those currently existing in the literature. The assumptions are verified for some hidden Markov models.

Keywords: Particle filter; central limit theorem; smoothing

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1. Introduction

Feynman–Kac formulae provide a very general description of several models, such as hidden Markov models (HMMs), used in statistics, physics, and many other fields; see [1]. For a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, $f\colon \mathcal{X} \to \mathbb{R}$ (bounded for now), the Feynman–Kac formula associated to the n-time marginal, $n \geq 1$ is: $\eta_n(f) := \gamma_n(f)/\gamma_n(1)$ with, for μ a probability measure on $\mathcal{X}$, $G_n\colon \mathcal{X} \to \mathbb{R}_+$ (bounded), $n \geq 0$, $M_n\colon \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to [0, 1]$, $n \geq 1$,

$$\gamma_n(f) := \int_{\mathcal{X}^{n+1}} f(x_n)\left[\prod_{p=0}^{n-1} G_p(x_p)\right]\mu(\mathrm{d}x_0)\prod_{p=1}^{n} M_p(x_{p-1}, \mathrm{d}x_p). \tag{1}$$

We take $\eta_0 = \mu$. In the context of HMMs, η_n represents the predictor, equivalently, the conditional distribution of the signal given the observations up to time $n-1$. In many practical applications, such as the smoothing problem in HMMs, one is interested in the formula for $F_n\colon \mathcal{X}^{n+1} \to \mathbb{R}$ (bounded for now),

$$\mathbb{Q}_n(F_n) = \frac{\int_{\mathcal{X}^{n+1}} F_n(x_0, \ldots, x_n)[\prod_{p=0}^{n-1} G_p(x_p)]\mu(\mathrm{d}x_0)\prod_{p=1}^{n} M_p(x_{p-1}, \mathrm{d}x_p)}{\int_{\mathcal{X}^{n+1}}[\prod_{p=0}^{n-1} G_p(x_p)]\mu(\mathrm{d}x_0)\prod_{p=1}^{n} M_p(x_{p-1}, \mathrm{d}x_p)}. \tag{2}$$

In practice $\eta_n(f)$ and $\mathbb{Q}_n(F_n)$ are unavailable analytically and we must resort to numerical approximation procedures in order to compute it. We remark that $\mathbb{Q}_n(F_n)$ is of interest, not only for smoothing for HMMs, but many other application areas; see, for instance [3]. In this article we focus on the numerical approximation of $\mathbb{Q}_n(F_n)$ and simultaneously $\eta_n(f)$. The latter task is often achieved quite well using sequential Monte Carlo (SMC) methods.

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SMC methods are designed to approximate a sequence of probability distributions of increasing dimension. The method uses $N \geq 1$ samples (or particles) that are generated in parallel, and are propagated via importance sampling and resampling methods. Several convergence results, as N grows, have been proved (see, e.g. [1], [7]) along with the stability in time of the error of the algorithm [5], [12] in the context of filtering for HMMs. These latter results are of particular importance due to the temporal sequential nature of the inference; we do not want the errors over time to accumulate.

As noted above, SMCs can be very useful for approximating $\eta_n(f)$. However, it is well-known due to the path degeneracy problem (see [8]) that the standard SMC approach of cost $\mathcal{O}(N)$ per time step for approximating $\mathbb{Q}_n(F_n)$ performs very badly. For example, consider the central limit theorem (CLT) for the standard SMC approximation of $\mathbb{Q}_n(F_n)$, call it $\mathbb{Q}_n^{N,S}(F_n)$ with $F_n(x_0, \ldots, x_n) = \sum_{p=0}^{n} f_p(x_p)$, $f_p \colon \mathcal{X} \to \mathbb{R}$ (additive functionals—this is of particular interest in application areas),

$$\sqrt{N}[\mathbb{Q}_n^{N,S}(F_n) - \mathbb{Q}_n(F_n)] \xrightarrow{\text{D}} \mathcal{N}(0, \sigma_n^{2,S}(F_n)),$$

where '$\xrightarrow{\text{D}}$' denotes convergence in distribution as $N \to +\infty$ and $\mathcal{N}(0, \sigma_n^{2,S}(F_n))$ is a one-dimensional Gaussian distribution with 0 mean and variance $\sigma_n^{2,S}(F_n)$. Poyiadjis *et al.* [11] showed that, under strong assumptions, $\sigma_n^{2,S}(F_n) \geq c(n)$, with $c(n)$, $\mathcal{O}(n^2)$.

One SMC approach designed to deal with these aforementioned issues is that of the forward filtering backward smoothing algorithm (FFBS) of [8] and [10] and in particular the SMC approximation of the backward interpretation of Feynman–Kac formulae, written as $\mathbb{Q}_n^N(F_n)$. This is a 'forward only' approximation of the FFBS algorithm, which is of cost $\mathcal{O}(N^2)$ per time step. Several convergence results for this algorithm (and FFBS), including a CLT are proved in [3], [7], and [9]; the assumptions used are fairly strong and do not always apply on noncompact state-spaces $\mathcal{X}$. The $\mathcal{O}(N^2)$ cost per time step is counterbalanced by the time-behaviour of (an appropriateley defined) an error in approximating $\mathbb{Q}_n(F_n)$ for F_n additive; it can be no worse than linear in time (see, e.g. [9]), versus the $\mathcal{O}(n^2)$ for standard SMC. For instance, Del Moral *et al.* [3] showed that for F_n additive, as $\sqrt{N}[\mathbb{Q}_n^N(F_n) - \mathbb{Q}_n(F_n)] \xrightarrow{\text{D}} \mathcal{N}(0, \sigma_n^2(F_n))$, under some strong hypotheses: $\sigma_n^2(F_n) \leq c(n+1)$ with $c < +\infty$ not depending upon n. In this paper we weaken the hypotheses previously used in the literature (such as [3], [7], and [9]). A related idea, the forward filtering backward simulation algorithm in [7], has cost $\mathcal{O}(N)$ but we do not consider it in this paper.

In the analysis of SMC algorithms, time-stability is often posed as follows. Writing $\eta_n^N(f)$ as the SMC approximation of $\eta_n(f)$, we have under minimal assumptions that $\sqrt{N}[\eta_n^N(f) - \eta_n(f)] \xrightarrow{\text{D}} \mathcal{N}(0, \vartheta_n^2(f))$. In the literature an often proved result, under additional assumptions, is that $\vartheta_n^2(f) \leq c$ where c does not depend upon n. The time-stability of an SMC has been addressed in many papers (e.g. [2]), but, only recently have the assumptions been weakened, for example, in [5] and [12]. The assumptions used in the early work of Del Moral *et al.* [2] relied on very strong mixing assumptions associated to the underlying Markov chain of the Feynman–Kac formula. Significant efforts were made to weaken this assumption (see [5] and [12]). We use similar assumptions to [12] in order to weaken the assumptions used in [3] and [4] to provide the framework for our approach. First, proving a CLT for the SMC approximation of the backward interpretation of Feynman–Kac formulae (Theorem 1), that is $\sqrt{N}[\mathbb{Q}_n^N(F_n) - \mathbb{Q}_n(F_n)] \xrightarrow{\text{D}} \mathcal{N}(0, \sigma_n^2(F_n))$. Secondly, providing a linear-in-time bound on the associated asymptotic variance expression when the function is additive (Theorem 2), that is for $F_n(x_0, \ldots, x_n) = \sum_{p=0}^{n} f_p(x_p)$, $\sigma_n^2(F_n) \leq c(n+1)$ where c does not depend upon n.

This paper is structured as follows. In Section 2 we state our notation, the algorithm, and estimates along with our assumptions. In Section 3 the CLT is proved. In Section 4 we prove $\sigma_n^2(F_n) \le c(n+1)$. In Section 5 we verify our assumptions. The appendices contain technical results for the proofs of the CLT and asymptotic variance.

2. Preliminaries

2.1. Notation

For a kernel $M\colon \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and σ-finite measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ $\mu M(\cdot) := \int_{\mathcal{X}} \mu(\mathrm{d}x) M(x, \cdot)$. For a function $\varphi\colon \mathcal{X} \to \mathbb{R}$ and kernel M (respectively signed measure μ), $M(\varphi)(x) := \int_{\mathcal{X}} \varphi(y) M(x, \mathrm{d}y)$ (respectively $\mu(\varphi) := \int \varphi(y)\mu(\mathrm{d}y)$). For a given function $V\colon \mathcal{X} \mapsto [1, \infty)$, we denote by $\mathcal{L}_V$ the class of functions $\varphi\colon \mathcal{X} \to \mathbb{R}$ for which $\|\varphi\|_V := \sup_{x\in\mathcal{X}} |\varphi(x)|/V(x) < +\infty$. When $V \equiv 1$ we write $\|\varphi\|_\infty := \sup_{x\in\mathcal{X}} |\varphi(x)|$. We also denote, for a probability measure μ, $\|\mu\|_V := \sup_{|\varphi|\le V} |\mu(\varphi)|$. The probability measures on $\mathcal{X}$ are denoted by $\mathcal{P}$. For $\mu \in \mathcal{P}$ such that $\mu(V) < +\infty$, we denote $\mu \in \mathcal{P}_V$. Throughout c is used to denote a constant whose meaning may change, depending upon the context; any (important) dependencies are written as $c(\cdot)$. The bounded, real-valued and measurable functions on a space $\mathcal{Z}$ are written as $\mathbb{B}_b(\mathcal{Z})$. The notation $x_{k:n} = (x_k, \dots, x_n)$ is used, with $k < n$.

Recall (1) which is defined in terms of potentials G_n and Markov kernels M_n. Throughout the paper it is assumed for a σ-finite measure λ on $\mathcal{X}$ (typically Lebesgue) and each $n \ge 1$ that $M_n(x_{n-1}, \mathrm{d}x_n) = H_n(x_{n-1}, x_n)\lambda(\mathrm{d}x_n)$ where $H_n\colon \mathcal{X}^2 \to \mathbb{R}_+$, with $\int_{\mathcal{X}} H_n(x_{n-1}, x_n)\lambda(\mathrm{d}x_n) = 1$ for all $x_{n-1} \in \mathcal{X}$. A semi-group for $n \ge 1$ is defined as

$$Q_n(x_{n-1}, \mathrm{d}x_n) := G_{n-1}(x_{n-1}) M_n(x_{n-1}, \mathrm{d}x_n)$$

with for $0 \le p \le n$, $f\colon \mathcal{X} \to \mathbb{R}$, $Q_{p,n}(f)(x) := \int f(x_n) \prod_{q=p+1}^{n} Q_q(x_{q-1}, \mathrm{d}x_q)$ with the convention $Q_{p,p} = Id$, the identity operator. We use this semi-group notation for operators that are introduced later on. Sometimes we abuse the notation and write $Q_n(x_{n-1}, x_n) = G_{n-1}(x_{n-1}) H_n(x_{n-1}, x_n)$. We write the d-dimensional Gaussian distribution with mean μ and covariance Σ as $\mathcal{N}_d(\mu, \Sigma)$ and if $d = 1$ we drop subscript d. We end by noting that throughout this paper for every $n \ge 0$, $\|G_n\|_\infty < +\infty$.

2.2. Algorithm and estimate

The SMC algorithm samples from the joint law

$$\mathbb{P}(\mathrm{d}(x_0^{1:N}, x_1^{1:N}, \dots, x_n^{1:N})) = \left(\prod_{i=1}^{N} \eta_0(\mathrm{d}x_0^i)\right) \prod_{p=1}^{n} \prod_{i=1}^{N} \Phi_p(\eta_{p-1}^N)(\mathrm{d}x_p^i),$$

where $x_q^{1:N} = (x_q^1, \dots, x_q^N) \in \mathcal{X}^N$ $(0 \le q \le n)$, $\eta_n^N = (1/N)\sum_{i=1}^N \delta_{x_n^i}$, and the operator $\Phi_n\colon \mathcal{P} \to \mathcal{P}$ is defined by $\Phi_n(\mu)(\mathrm{d}y) = \mu(G_{n-1}M_n)(\mathrm{d}y)/\mu(G_{n-1})$. The estimate of $\gamma_n(f)$ is $\gamma_n^N(f) = [\prod_{q=0}^{n-1} \eta_q^N(G_q)]\eta_n^N(f)$.

We let $F_n\colon \mathcal{X}^{n+1} \to \mathbb{R}$ in order to study the SMC approximation of (2). Now the backward interpretation (see, e.g. [3]) is

$$\mathbb{Q}_n(F_n) = \int_{\mathcal{X}^{n+1}} F_n(x_{0:\,n}) \eta_n(\mathrm{d}x_n) \mathcal{M}_n(x_n, \mathrm{d}x_{0:\,n-1}),$$

where

$$\mathcal{M}_n(x_n, \mathrm{d}x_{0:\,n-1}) = \prod_{q=1}^{n} M_{q,\eta_{q-1}}(x_q, \mathrm{d}x_{q-1}), \tag{3}$$

For

$$M_{q,\eta_{q-1}}(x_q, \mathrm{d}x_{q-1}) = \frac{G_{q-1}(x_{q-1})H_q(x_{q-1}, x_q)\eta_{q-1}(\mathrm{d}x_{q-1})}{\eta_{q-1}(G_{q-1}H_q(\cdot, x_q))},$$

we write $\mathcal{M}_n^N$ in (3) for when $\eta_0, \ldots, \eta_{n-1}$ are replaced by $\eta_0^N, \ldots, \eta_{n-1}^N$. The SMC approximation of $\mathbb{Q}_n(\cdot)$, written as $\mathbb{Q}_n^N(\cdot)$ is $\mathbb{Q}_n^N(\mathrm{d}x_{0:n}) = \eta_n^N(\mathrm{d}x_n)\prod_{q=1}^n M_{q,\eta_{q-1}^N}(x_q, \mathrm{d}x_{q-1})$. If $F_n(x_{0:n}) = \sum_{p=0}^n f_p(x_p)$, $f_p \colon \mathcal{X} \to \mathbb{R}$, then setting $F_0^N = f_0$ the $\mathcal{O}(N^2)$ approximation is $\mathbb{Q}_n^N(F_n) = \eta_n^N(F_n^N)$, where $F_n^N(x) = f_n(x) + \eta_{n-1}^N(Q_n(\cdot, x)F_{n-1}^N)/\eta_{n-1}^N(Q_n(\cdot, x))$.

2.3. Assumptions

We make the following assumptions.

(A1) There exists a $V \colon \mathcal{X} \to [1, \infty)$ unbounded and constants $\delta \in (0, 1)$ and $\underline{d} \geq 1$ with the following properties. For each $d \in (\underline{d}, +\infty)$ there exists a $b_d < +\infty$ such that for all $x \in \mathcal{X}$, $\sup_{n\geq 1} Q_n(\mathrm{e}^V)(x) \leq \mathrm{e}^{(1-\delta)V(x)+b_d \mathbf{1}_{\{C_d\}}(x)}$, where $C_d = \{x \in \mathcal{X} \colon V(x) \leq d\}$.

(A2) It holds that $\mu \in \mathcal{P}_v$, with $v = \mathrm{e}^V$.

(A3) For every $\alpha \in (0, \frac{1}{2})$, $\sup_{n\geq 1} G_{n-1}(x)H_n(x, y)/\eta_{n-1}(G_{n-1}H_n(\cdot, y)) \in \mathcal{L}_{\bar{v}^\alpha}$, with $\bar{v}(x, y)^\alpha = v(x)^\alpha v(y)^\alpha$.

(A4) With $\underline{d}$ as in (A1) for each $d \in [\underline{d}, \infty)$, $G_{n-1}(x)H_n(x, y) > 0$ for all $x, y \in \mathcal{X}, n \geq 1$ with $0 < \int_{C_d} \lambda(\mathrm{d}y) < +\infty$ and there exist $\tilde{\epsilon}_d^- > 0$ such that, $\inf_{n\geq 1} G_{n-1}(x)H_n(x, y) \geq \tilde{\epsilon}_d^-$ for all $x, y \in C_d$. In addition $\nu_d(\mathrm{d}y) := \lambda(\mathrm{d}y)\mathbf{1}_{\{C_d\}}(y)/\int_{C_d}\lambda(\mathrm{d}y) \in \mathcal{P}_v$.

(A5) With $\underline{d}$ as in (A1), and $\tilde{\epsilon}_d^-$ as in (A4) for each $d \in [\underline{d}, \infty)$ there exist $\tilde{\epsilon}_d^+ \in [\tilde{\epsilon}_d^-, \infty)$ such that, $\sup_{n\geq 1} G_{n-1}(x)H_n(x, y) \leq \tilde{\epsilon}_d^+$ for all $x, y \in C_d$

(A6) It holds that $\sup_{n\geq 0}\sup_{x\in\mathcal{X}} G_n(x) < +\infty$.

Assumptions (A1), (A2) and (A4)–(A6) are Assumptions (H1)–(H5) of [12], except slightly modified to our density notation. Assumption (A3) appears to be needed under our analysis, but can be verified in practice. Under the other assumptions of this paper, α as in (A3) could be verified if $H_n \in \mathcal{L}_{\bar{v}^{\beta_1}}$ and $(\inf_{x\in C_d} G_{n-1}(x)H_n(x, y))^{-1} \in \mathcal{L}_{v^{\beta_2}}$ with $\beta_1, \beta_2 > 0$ and $\alpha = \beta_1 + \beta_2$. A discussion of the unmodified assumptions and comparison to the assumptions of [6] can be found in [12].

3. Central limit theorem

The asymptotic variance in the CLT for the forward only smoothing (respectively FFBS) is, under some conditions, given by [3, Theorem 3.1] (see also [7]):

$$\sigma_n^2(F_n) := \sum_{p=0}^n \eta_p\left(\left[h_{p,n}\left\{P_{p,n}(F_n) - \frac{\eta_p(D_{p,n}(F_n))}{\eta_p(D_{p,n}(1))}\right\}\right]^2\right) \tag{4}$$

for the predictor. The operators are for $0 \leq p \leq n$,

$$h_{p,n}(x_p) = \frac{Q_{p,n}(1)(x_p)}{\eta_p(Q_{p,n}(1))}, \qquad P_{p,n}(F_n)(x_p) = \frac{D_{p,n}(F_n)(x_p)}{D_{p,n}(1)(x_p)},$$

$$D_{p,n}(F_n)(x_p) = \int \mathcal{M}_p(x_p, \mathrm{d}x_{0:p-1})\mathcal{Q}_{p,n}(x_p, \mathrm{d}x_{p+1:n})F_n(x_{0:n})$$

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and

$$\mathcal{Q}_{p,n}(x_p, \mathrm{d}x_{p+1:n}) = \prod_{q=p}^{n-1} Q_{q+1}(x_q, \mathrm{d}x_{q+1}). \tag{5}$$

With the conventions that $D_{0,n} = \mathcal{Q}_{0,n}$ and $D_{n,n} = \mathcal{M}_n$. Note that if $\mathbb{Q}_n(F_n) = 0$, $\sigma_n^2(F_n) = \sum_{p=0}^{n} \eta_p(h_{p,n}^2 P_{p,n}(F_n)^2)$. We obtain the CLT under weaker assumptions than considered by [3] and [7], but only for bounded functions; we note that (A1) and (A3) need not be time-uniform, but in order to connect with the next section, we make them time-uniform. Indeed, we can pose (A1) as $Q_n(v) \leq c(n)v^{1-\delta}$. We use $\mathbb{E}\{\cdot\}$ to denote expectation with respect to the particle system.

Theorem 1. *Assume that (A1)–(A3) hold. Suppose that for each* $n \geq 0$, $1/G_n \in \mathcal{L}_{v^{\delta/2}}$, *with* δ *as in (A1), then for any* $n \geq 0$, $F_n \in \mathbb{B}_\mathrm{b}(\mathcal{X}^{n+1})$, $\sqrt{N}[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma_n^2(F_n))$.

Proof. By translation, we can assume that $\mathbb{Q}_n(F_n) = 0$. For notational convenience, we introduce the rescaled quantity

$$\hat{D}_{p,n}(F_n) = \frac{D_{p,n}(F_n)}{\eta_p Q_{p,n}(1)}$$

and its empirical analogue

$$\hat{D}_{p,n}^N(F_n) = \frac{D_{p,n}^N(F_n)}{\eta_p Q_{p,n}(1)}$$

for $D_{p,n}^N(F_n) = \int \mathcal{M}_p^N(x_p, \mathrm{d}x_{0:\,p-1})\mathcal{Q}_{p,n}(x_p, \mathrm{d}x_{p+1:n})F_n(x_{0:\,n})$. From De Moral *et al.* [3, p. 965] and Equation (5.3) of[3, p. 962], it follows that

$$\sqrt{N}[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n) = \sqrt{N} \sum_{p=0}^{n} \frac{\bar{\gamma}_p^N(1)}{\bar{\gamma}_n^N(1)} [\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{D}_{p,n}^N(F_n)),$$

where we have set $\bar{\gamma}_p^N(1) = \gamma_p^N(1)/\gamma_p(1)$. Since the quantity $\bar{\gamma}_p^N(1)$ converges to one in probability (e.g. Proposition 1), from Slutsky's lemma we show that we can ignore the $\bar{\gamma}_p^N(1)/\bar{\gamma}_n^N(1)$ term for proving the CLT. The proof consists of exploiting the decomposition for $\hat{F}_{p,n}^N = \hat{D}_{p,n}^N(F_n) - \hat{D}_{p,n}(F_n)$,

$$\begin{aligned}&\sum_{p=0}^{n} \sqrt{N}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{D}_{p,n}^N(F_n)) \\ &\quad = \sqrt{N}\Bigg(\sum_{p=0}^{n}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{F}_{p,n}^N) + \sum_{p=0}^{n}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{D}_{p,n}(F_n))\Bigg).\end{aligned}$$

and prove that the first term on the right-hand side converges to 0 in probability while the second term converges in law towards a centred Gaussian distribution with variance $\sigma_n^2(F_n)$.

As $G_p \in \mathbb{B}_\mathrm{b}(\mathcal{X})$ (for each $p \geq 0$), $F_n \in \mathbb{B}_\mathrm{b}(\mathcal{X})$, and $\hat{D}_{p,n}(F_n) \in \mathbb{B}_\mathrm{b}(\mathcal{X})$ for $0 \leq p \leq n$; by Corollary 9.3.1 of [1], the sequence $\sqrt{N}([\eta_0^N - \eta_0](\hat{D}_{0,n}(F_n)), \ldots, [\eta_n^N - \Phi_n(\eta_{n-1}^N)](\hat{D}_{n,n}(F_n)))$ converges in law towards a centred Gaussian vector with covariance matrix

$$\mathrm{diag}(\mathrm{var}_{\eta_0}(\hat{D}_{0,n}(F_n)), \ldots, \mathrm{var}_{\eta_n}(\hat{D}_{n,n}(F_n))).$$

It follows that $\sum_{p=0}^{n}\sqrt{N}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{D}_{p,n}(F_n))$ converges in law towards a centred Gaussian distribution with variance $\sum_{j=0}^{n}\mathrm{var}_{\eta_j}(\hat{D}_{j,n}(F_n))$; this is just another way of writing $\sigma_n^2(F_n)$.

In the last part of the proof we show that the term $\sum_{p=0}^{n}\sqrt{N}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{F}_{p,n}^N)$ converges to 0 in probability; this quantity has zero expectation and by the first inequality of [3, p. 965]

$$\mathbb{E}\bigg\{\bigg(\sum_{p=0}^{n}\sqrt{N}[\eta_p^N - \Phi_p(\eta_{p-1}^N)](\hat{F}_{p,n}^N)\bigg)^2\bigg\} \le c\sum_{p=0}^{n}\mathbb{E}\{\Phi_p(\eta_{p-1}^N)([\hat{F}_{p,n}^N]^2)\}$$

(where $c < +\infty$ does not depend on N). It thus remains to verify that for any index $0 \le p \le n$ the quantity $\mathbb{E}\{\Phi_p(\eta_{p-1}^N)([\hat{F}_{p,n}^N]^2)\}$ converges to 0 as $N \to \infty$. We use the decomposition $\Phi_p(\eta_{p-1}^N)([\hat{F}_{p,n}^N]^2) = \eta_p([\hat{F}_{p,n}^N]^2) + [\Phi_p(\eta_{p-1}^N) - \Phi_p(\eta_{p-1})]([\hat{F}_{p,n}^N]^2)$ and treat each term separately. As $G_p \in \mathbb{B}_b(\mathcal{X})$ (for each $p \ge 0$), $F_n \in \mathbb{B}_b(\mathcal{X})$, and $\hat{F}_{p,n}^N(F_n) \in \mathbb{B}_b(\mathcal{X})$ for $0 \le p \le n$; it follows from the dominated convergence theorem, Fubini's theorem, and Lemma 1 that $\mathbb{E}\{\eta_p([\hat{F}_{p,n}^N]^2)\}$ converges to 0. For dealing with the second term, as $\hat{F}_{p,n}^N(F_n) \in \mathbb{B}_b(\mathcal{X})$, we note that $\mathbb{E}\{|[\Phi_p(\eta_{p-1}^N) - \Phi_p(\eta_{p-1})]([\hat{F}_{p,n}^N]^2)|\}$ is upper-bounded by (c does not depend on N)

$$c\int_{\mathcal{X}}\mathbb{E}\bigg|\frac{\eta_{p-1}^N(G_{p-1}H_p(\cdot,x_p))}{\eta_{p-1}^N(G_{p-1})} - \frac{\eta_{p-1}(G_{p-1}H_p(\cdot,x_p))}{\eta_{p-1}(G_{p-1})}\bigg|\lambda(\mathrm{d}x_p) \tag{6}$$

By assumption (A3) and the boundedness of G_{p-1} for every fixed $x_p \in \mathcal{X}$ Proposition 1 applies to the function $G_{p-1}H_p(\cdot,x_p)$ and G_{p-1}; it follows that for every fixed $x_p \in \mathcal{X}$ the function

$$\bigg|\frac{\eta_{p-1}^N(G_{p-1}H_p(\cdot,x_p))}{\eta_{p-1}^N(G_{p-1})} - \frac{\eta_{p-1}(G_{p-1}H_p(\cdot,x_p))}{\eta_{p-1}(G_{p-1})}\bigg|\frac{1}{\eta_{p-1}(G_{p-1}H_p(\cdot,x_p))} \xrightarrow{\mathrm{D}} 0. \tag{7}$$

From Lemma 3, we show that for λ-almost everywhere (a.e.) with fixed $x_p \in \mathcal{X}$ (7) is also uniformly integrable; consequently for λ-a.e. with fixed $x_p \in \mathcal{X}$ (7) converges in expectation to zero. In addition, by Lemma 3, (6) is upper-bounded by

$$c\int_{\mathcal{X}} v(x_p)^{2\alpha}\eta_{p-1}(G_{p-1}H_p(\cdot,x_p))\lambda(\mathrm{d}x_p).$$

Application of Fubini's theorem and repeated use of Corollary 1 allows us to show that $\int_{\mathcal{X}} v(x_p)^{2\alpha}\eta_{p-1}(G_{p-1}H_p(\cdot,x_p))\lambda(\mathrm{d}x_p) \le c$, where $c < +\infty$ depends on p but not N. Thus, by the dominated convergence theorem, we have shown that the term in (6) goes to zero, which completes the proof.

4. Control of the asymptotic variance

We now consider the asymptotic variance when $F_n(x_{0:n}) = \sum_{p=0}^{n} f_p(x_p)$, $f_p\colon \mathcal{X} \to \mathbb{R}$. Contrary to Theorem 1 we will not assume that the f_p are bounded; let $\|f\|_{v^\alpha} = \sup_{p\ge 0}\|f_p\|_{v^\alpha}$.

Theorem 2. *Assume that (A1)–(A6) hold. Then if $\|f\|_{v^\alpha} < +\infty$, $\alpha \in (0, \frac{1}{6})$ there exists a $c < +\infty$ which only depends upon the constants in (A1) and (A3)–(A6) such that for any $n \ge 1$: $\sigma^2(F_n) \le c\|f\|_{v^\alpha}(n+1)$.*

Proof. Consider the term $h_{p,n}(x)\{P_{p,n}(F_n)(x) - \eta_p(D_{p,n}(F_n))/\eta_p(D_{p,n}(1))\}$ in (4). We have the simple calculation

$$P_{p,n}(F_n)(x) - \frac{\eta_p(D_{p,n}(F_n))}{\eta_p(D_{p,n}(1))} = \frac{(\delta_x \otimes \eta_p - \eta_p \otimes \delta_x)(\bar{D}_{p,n}(F_n \otimes 1))}{\eta_p(D_{p,n}(1))D_{p,n}(1)(x)},$$

where $\bar{D}_{p,n} = D_{p,n} \otimes D_{p,n}$ and the $\bar{\bullet}$ notation is used to denote operators/functions on the product space. Let $\bar{\eta}_{p,x} := (\delta_x \otimes \eta_p - \eta_p \otimes \delta_x)$ then using the additive nature of the functional F_n, we derive

$$\begin{aligned}
&\bar{\eta}_{p,x}\bar{D}_{p,n}(F_n \otimes 1) \\
&\quad = \sum_{q=0}^{p-1} \bar{\eta}_{p,x}(\bar{Q}_{p,n}(1)\bar{M}_{p:q}(f_q \otimes 1)) + \sum_{q=p}^{n} \bar{\eta}_{p,x_p}(\bar{Q}_{p,q}((f_q \otimes 1)\bar{Q}_{q,n}(1))),
\end{aligned}$$

where $M_{p:q} = M_{p,\eta_{p-1}}, \dots, M_{q+1,\eta_q}$.

We consider first for $p \geq 1$,

$$\begin{aligned}
&\frac{h_{p,n}(x)}{\eta_p(D_{p,n}(1))D_{p,n}(1)(x)} \sum_{q=0}^{p-1} \bar{\eta}_{p,x}(\bar{Q}_{p,n}(1)\bar{M}_{p:q}(f_q \otimes 1)) \\
&\qquad = \frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))} \sum_{q=0}^{p-1} \eta_p(Q_{p,n}(1)[M_{p:q}(f_q)(x) - M_{p:q}(f_q)]). \qquad (8)
\end{aligned}$$

By Proposition 3 the right-hand side is upper-bounded by $c\|f\|_{v^\alpha} v(x)^{2\alpha}$. Then, we consider (which covers the $p = 0$ case)

$$\begin{aligned}
&\frac{h_{p,n}(x)}{\eta_p(D_{p,n}(1))D_{p,n}(1)(x)} \sum_{q=p}^{n} \bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q \otimes 1)\bar{Q}_{q,n}(1))) \\
&\qquad = \frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \sum_{q=p}^{n} \bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q \otimes 1)\bar{Q}_{q,n}(1))). \qquad (9)
\end{aligned}$$

By Proposition 2, the right-hand side is upper-bounded by $c\|f\|_{v^\alpha} v(x)^{3\alpha}$. Thus, we have proved that $\sigma_n^2(F_n) \leq c\|f\|_{v^\alpha} \sum_{p=0}^{n} \eta_p(v^{6\alpha})$. We conclude by noting that $\alpha \in (0, \frac{1}{6})$ and using Proposition 1 of [12].

5. An example

An example where our assumptions can hold, is that of [12, Section 3.2], with some minor modifications. Let $\mathcal{X} = \mathbb{R}^{d_x}$ with $n \geq 0$, $X_{n+1} = X_n + W_n$. Let W_n be independent and identically distributed (i.i.d.) $\sim \mathcal{N}_{d_x}(0, I_{d_x})$ with I_{d_x} the $d_x \times d_x$ identity matrix. We can take $V(x) = 1 + x^\top x/2(1 + \delta_0)$, $\delta_0 > 1$. The observation model is taken as $Y_n | X_n = x \sim \mathcal{N}_{d_y}(H(x), \sigma^2 I_{d_y})$ where $H : \mathcal{X} \to \mathbb{R}^{d_y}$; that is, $G_n(x)$ is the d_y-dimensional Gaussian density with mean $H(x)$ covariance I_{d_y} and is evaluated point-wise at the observed y_n. It is assumed that the actual observations lie on a space $\mathcal{Y}_\star \subset \mathbb{R}^{d_y}$, with $\mathcal{Y}_\star$ compact. If H is bounded such that $\lim_{r\to\infty} \sup_{|x|\geq r}[(\frac{1}{2})x^\top x(1 + \delta_1)/(\delta_0(1 + \delta_0)) + (\frac{1}{2})\sigma_y^2 \sup_{y\in\mathcal{Y}_\star} |y| \sup_{|\lambda|=1} \lambda^\top H(x) - (H(x)^\top H(x))/(2\sigma_y^2)] < 0$ with $\delta_1 \in (0, 1)$ then we can verify all of the assumptions,

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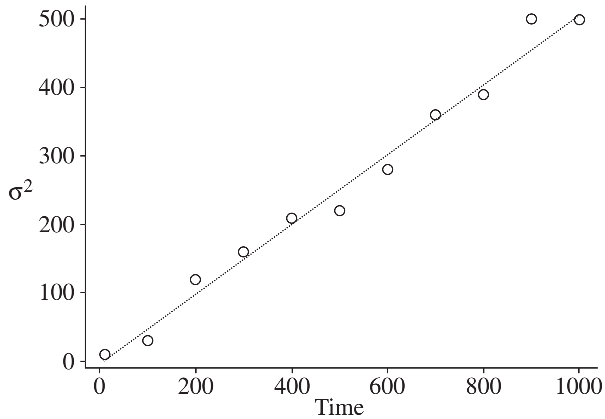

FIGURE 1: Estimate of the asymptotic variance. The function is $F_n(x_{0:n}) = \sum_{p=0}^{n} x_p$ and an SMC algorithm is computed with $N = 10\,000$, 100 times for each time point. The y axis is rescaled for presentation purposes and the dotted line is a least squares estimate of a linear regression of σ^2 against time.

including $1/G_{n-1} \in \mathcal{L}_{v^{\delta/2}}$ using the work of [12], apart from (A3). This holds if for $\alpha \in (0, \frac{1}{2}) \inf_{y\in\mathcal{X}}((\inf_{x\in C_d} H_n(x, y))v(y)^{\alpha}) > 0$. This is because $\eta_{n-1}(C_d)$ can be shown to be lower-bounded uniformly in n (see the proof of [12, Lemma 8]) and G_{n-1} is (uniform in n) upper and lower-bounded if $\mathcal{Y}_\star$ is compact (which it is). Simple calculations show that $\inf_{y\in\mathcal{X}}((\inf_{x\in C_d} H_n(x, y))v(y)^{\alpha}) > 0$, can hold if $\sigma_y^2 > 4$ and then taking $1 < \delta_0$ small enough.

Another observation model (with the above hidden Markov chain and $v(x)$) for which we can verify the assumptions of this paper can be found in [12, Section 3.1.1]. Here, we set $\mathcal{Y}_\star = \mathcal{Y} = \{0, 1\}^{d_x}$ and write $\mathcal{B}(p)$ as the Bernoulli distribution with success probability p, the observation model is $Y_n | X_n = x \sim \mathcal{B}(p(x^1)) \otimes \cdots \otimes \mathcal{B}(p(x^{d_x}))$ where $p(x) = 1/(1 + \mathrm{e}^{-x})$. It is easily shown that $1/G_{n-1} \in \mathcal{L}_{v^{\delta/2}}$ and all the other assumptions apart from (A3) easily follow. Assumption (A3) will follow by the above calculations and the fact that (treating G_n as a function of the observations also) $G_n(x; y) \leq 1$ and $\inf_{(y,x)\in\mathcal{Y}\times C_d} G_n(x; y) > 0$. To illustrate the linear growth bound for this example ($d_x = 1$), we generate 1000 data points from the model and estimate the asymptotic variance, using an SMC algorithm for the function $F_n(x_{0:n}) = \sum_{p=0}^{n} x_p$. The results, shown in Figure 1, demonstrate the expected linear in time increase.

Appendix A. Technical results for the central limit theorem

Note that $\mathcal{F}_n^N$ is the natural filtration of the particles at time n.

Lemma 1. *Assume that (A1)–(A3) hold. Suppose that for each* $n \geq 0$, $1/G_n \in \mathcal{L}_{v^{\delta}}$, *with* δ *as in (A1). Let* $p > 0$ *then for* λ*-a.e.* $x_p \in \mathcal{X}$ *and any* $F \in \mathbb{B}_{\mathrm{b}}(\mathcal{X}^{n+1})$, $[D_{p,n}^N - D_{p,n}](F)(x_p) \xrightarrow{\mathrm{D}} 0$.

Proof. By [3, Lemma 6.1], we have

$$[D_{p,n}^N - D_{p,n}](F)(x_p) = \sum_{q=0}^{p} [\mathcal{M}_{p,q,\eta_q^N} - \mathcal{M}_{p,q,\Phi_q(\eta_{q-1}^N)}](S_{p,q,n}^N(F))(x_p), \tag{10}$$

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where for $\mu \in \mathcal{P}$, $0 \le q < p$. It holds that

$$\mathcal{M}_{p,q,\mu}(x_p, \mathrm{d}x_{q:p-1}) = \frac{\mu(\mathrm{d}x_q)\mathcal{Q}_{q,p-1}(x_q, \mathrm{d}x_{q+1:p-1})Q_p(x_{p-1}, x_p)}{\mu Q_{q,p-1}(Q_p(\cdot, x_p))},$$

where $\mathcal{Q}_{q,p-1}$ is defined in (5) and

$$S^N_{p,q,n}(F)(x_{q:p}) = \int_{\mathcal{X}^{q+n-p}} \mathcal{Q}_{p,n}(x_p, \mathrm{d}x_{p+1:n})\mathcal{M}^N_q(x_q, \mathrm{d}x_{0:\,q-1})F(x_{0:\,n});$$

see (3) for a definition of $\mathcal{M}^N_q$. We note that

$$\sup_{x_{q:p} \in \mathcal{X}^{p-q+1}} |S^N_{p,q,n}(F)(x_{q:p})| \le c\|F\|_\infty, \tag{11}$$

where c is a finite constant that may depend on p, n but not on N. We will show that each summand on the right-hand side of (10) will converge to 0 in probability.

It is first remarked that by (A1), (A3), and Proposition 1,

$$\eta^N_q Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right) \xrightarrow{\mathrm{D}} \eta_q Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)$$

and

$$\Phi_q(\eta^N_{q-1})\left[Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)\right] \xrightarrow{\mathrm{D}} \eta_q Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)$$

so it is enough to show that

$$[\eta^N_q - \Phi_q(\eta^N_{q-1})]\left[\mathcal{Q}_{q,p-1}\left(\frac{Q_p(\cdot, x_p)S^N_{p,q,n}(F)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)\right] \xrightarrow{\mathrm{D}} 0. \tag{12}$$

We have via the Jensen and the (conditional) Marcinkiewicz–Zygmund inequalities that

$$\begin{aligned}\mathbb{E}\left\{\left|[\eta^N_q - \Phi_q(\eta^N_{q-1})]\left[\mathcal{Q}_{q,p-1}\left(\frac{Q_p(\cdot, x_p)S^N_{p,q,n}(F)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)\right]\right|\right\} \\ \le \frac{c}{\sqrt{N}}\mathbb{E}\left\{\left|\mathcal{Q}_{q,p-1}\left(\frac{Q_p(\cdot, x_p)S^N_{p,q,n}(F)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)(x^1_q)\right|^2\right\}^{1/2}.\end{aligned}$$

From (11) it follows that

$$\begin{aligned}\mathbb{E}\left\{\left|\mathcal{Q}_{q,p-1}\left(\frac{Q_p(\cdot, x_p)S^N_{p,q,n}(F)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)(X^1_q)\right|^2\right\}^{1/2} \\ \le c\|F\|_\infty \mathbb{E}\left\{Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)(X^1_q)^2\right\}^{1/2}.\end{aligned}$$

Then by assumption (A3) and repeated application of Corollary 1, we have

$$\mathbb{E}\left\{Q_{q,p-1}\left(\frac{Q_p(\cdot, x_p)}{\eta_{p-1}(Q_p(\cdot, x_p))}\right)(X^1_q)^2\right\}^{1/2} \le cv(x_p)^\alpha \mathbb{E}\{v(X^1_q)^{2\alpha}\}^{1/2}.$$

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For $\mathbb{E}\{v(X_q^1)^{2\alpha}\}$, Jensen's inequality and application of Lemma 2 yields

$$\mathbb{E}\left\{\left|[\eta_q^N - \Phi_q(\eta_{q-1}^N)]\left[\mathcal{Q}_{q,p-1}\left(\frac{Q_p(\cdot,x_p)S_{p,q,n}^N(F)}{\eta_{p-1}(Q_p(\cdot,x_p))}\right)\right]\right|\right\} \le \frac{c}{\sqrt{N}}v(x_p)^\alpha.$$

Thus, we have shown that (12) holds, from which we conclude the proof.

Lemma 2. *Assume that (A1) and (A2) hold. Suppose that for each* $n \ge 0$, $1/G_n \in \mathcal{L}_{v^\delta}$ *with* δ *as in (A1) then for any* $n \ge 0$ *there exists a* $c < +\infty$ *such that for any* $N \ge 2$, $\mathbb{E}\{v(X_n^1)\} \le c$.

Proof. We proceed via induction. The $n = 0$ case follows as $\eta_0 \in \mathcal{P}_v$. Thus, we assume for $n-1$ and consider n, $\mathbb{E}\{v(X_n^1)\} = \mathbb{E}\{\eta_{n-1}^N(Q_n(v))/\eta_{n-1}^N(G_{n-1})\}$. Now, consider

$$\eta_{n-1}^N(G_{n-1}) \ge \left\|\frac{1}{G_{n-1}}\right\|_{v^\delta}^{-1}\eta_{n-1}^N\left(\frac{1}{v^\delta}\right) \ge \left\|\frac{1}{G_{n-1}}\right\|_{v^\delta}^{-1}\frac{1}{\eta_{n-1}^N(v^\delta)}. \tag{13}$$

So, we have that $\mathbb{E}\{v(X_n^1)\} \le \|1/G_{n-1}\|_{v^\delta}\mathbb{E}\{\eta_{n-1}^N(Q_n(v))\eta_{n-1}^N(v^\delta)\}$. Now, via the multiplicative drift $Q_n(v) \le cv^{1-\delta}$, so

$$\begin{aligned}\mathbb{E}\{\eta_{n-1}^N(Q_n(v))\eta_{n-1}^N(v^\delta)\} &\le c\mathbb{E}\left\{\frac{1}{N^2}\left(\sum_i v(X_{n-1}^i) + \sum_{i\ne j} v(X_{n-1}^i)^{1-\delta}v(X_{n-1}^j)^\delta\right)\right\}\\ &= c\left(\mathbb{E}\{v(X_{n-1}^1)\} + \frac{N-1}{N}\mathbb{E}\{v(X_{n-1}^1)^{1-\delta}v(X_{n-1}^2)^\delta\}\right)\\ &\le 2c\mathbb{E}\{v(X_{n-1}^1)\},\end{aligned}$$

where we have applied Hölder's inequality to obtain the last line; the induction hypothesis completes the proof.

Lemma 3. *Assume that (A1)–(A3) hold. Suppose that for each* $n \ge 0$, $1/G_n \in \mathcal{L}_{v^{\delta/2}}$ *with* δ *as in (A1), then there exists a* $1 \ge \kappa > 0$ *such that for any* $n \ge 1$ *there exists a* $c < +\infty$ *such that for* λ*-a.e.* $x_n \in \mathcal{X}$,

$$\begin{aligned}&\mathbb{E}\left\{\left|\left[\frac{\eta_{n-1}^N(G_{n-1}H_n(\cdot,x_n))}{\eta_{n-1}^N(G_{n-1})} - \frac{\eta_{n-1}(G_{n-1}H_p(\cdot,x_n))}{\eta_{n-1}(G_{n-1})}\right]\frac{1}{\eta_{n-1}(G_{n-1}H_n(\cdot,x_n))}\right|^{1+\kappa}\right\}\\ &\quad\le cv(x_n)^{(1+\kappa)\alpha},\end{aligned}$$

where α *is as in (A3).*

Proof. We have

$$\begin{aligned}&\mathbb{E}\left\{\left|\left[\frac{\eta_{n-1}^N(G_{n-1}H_n(\cdot,x_n))}{\eta_{n-1}^N(G_{n-1})} - \frac{\eta_{n-1}(G_{n-1}H_p(\cdot,x_n))}{\eta_{n-1}(G_{n-1})}\right]\frac{1}{\eta_{n-1}(G_{n-1}H_n(\cdot,x_n))}\right|^{1+\kappa}\right\}\\ &\quad\le c\left(\frac{1}{\eta_{n-1}(G_{n-1})^{1+\kappa}} + \mathbb{E}\left\{\left|\frac{\eta_{n-1}^N(G_{n-1}H_n(\cdot,x_n))}{\eta_{n-1}^N(G_{n-1})\eta_{n-1}(G_{n-1}H_n(\cdot,x_n))}\right|^{1+\kappa}\right\}\right).\end{aligned}$$

Then by the application of assumption (A3), we obtain

$$\mathbb{E}\left\{\left|\frac{\eta_{n-1}^N(G_{n-1}H_n(\cdot,x_n))}{\eta_{n-1}^N(G_{n-1})\eta_{n-1}(G_{n-1}H_n(\cdot,x_n))}\right|^{1+\kappa}\right\} \le cv(x_n)^{(1+\kappa)\alpha}\mathbb{E}\left\{\left|\frac{\eta_{n-1}^N(v^\alpha)}{\eta_{n-1}^N(G_{n-1})}\right|^{1+\kappa}\right\}. \tag{14}$$

We will now show that (see the right-hand side of (14)) $\mathbb{E}\{|\eta_{n-1}^N(v^\alpha)/\eta_{n-1}^N(G_{n-1})|^{1+\kappa}\} \le c$ for some $1 \ge \kappa > 0$. From the proof of (13) in Lemma 2, we can show in a similar manner that $\mathbb{E}\{|\eta_{n-1}^N(v^\alpha)/\eta_{n-1}^N(G_{n-1})|^{1+\kappa}\} \le c\mathbb{E}\{\{\eta_{n-1}^N(v^\alpha)\eta_{n-1}^N(v^{\delta/2})\}^{1+\kappa}\}$. From Minkowski's inequality, we have $\mathbb{E}\{\{\eta_{n-1}^N(v^\alpha)\eta_{n-1}^N(v^{\delta/2})\}^{1+\kappa}\}$ is upper-bounded by

$$\begin{aligned}
&\frac{1}{N^{2(1+\kappa)}}\Bigg(\mathbb{E}\Bigg\{\Bigg\{\sum_i v(X_{n-1}^i)^{\alpha+\delta/2}\Bigg\}^{1+\kappa}\Bigg\}^{1/(1+\kappa)} \\
&\qquad + \mathbb{E}\Bigg\{\Bigg\{\sum_{i\neq j} v(X_{n-1}^i)^\alpha v(X_{n-1}^j)^{\delta/2}\Bigg\}^{1+\kappa}\Bigg\}^{1/(1+\kappa)}\Bigg)^{1+\kappa} \\
&\le \frac{1}{N^{2(1+\kappa)}}(N\mathbb{E}\{v(X_{n-1}^1)^{(\alpha+\delta/2)(1+\kappa)}\}^{1/(1+\kappa)} \\
&\qquad + N(N-1)\mathbb{E}\{v(X_{n-1}^1)^{\alpha(1+\kappa)}v(X_{n-1}^2)^{\delta(1+\kappa)/2}\}^{1/(1+\kappa)})^{1+\kappa}.
\end{aligned}$$

For $0 < \kappa < \min\{1/\delta - 1, 1/(2\alpha) - 1, 1/(\alpha+\delta/2) - 1\}$, we will show that the two expectations in the above equation are upper-bounded by a constant. For $\mathbb{E}\{v(X_{n-1}^1)^{(\alpha+\delta/2)(1+\kappa)}\}^{1/(1+\kappa)}$, we can apply Jensen's inequality followed by Lemma 2. For $\mathbb{E}\{v(X_{n-1}^1)^{\alpha(1+\kappa)}v(X_{n-1}^2)^{\delta(1+\kappa)/2}\}^{1/(1+\kappa)}$, we can apply the Cauchy–Schwarz inequality to obtain the upper-bound

$$\mathbb{E}\{v(X_{n-1}^1)^{2\alpha(1+\kappa)}\}^{1/2(1+\kappa)}\mathbb{E}\{v(X_{n-1}^2)^{\delta(1+\kappa)}\}^{1/2(1+\kappa)};$$

the two terms are controlled via Jensen's inequality followed by Lemma 2. Hence, we can deduce that $\mathbb{E}\{|\eta_{n-1}^N(v^{1/2})/\eta_{n-1}^N(G_{n-1})|^{1+\kappa}\} \le c$ for some $\kappa > 0$ which concludes the proof of the lemma.

Proposition 1. *Assume that (A1) and (A2) hold. Suppose that for each* $n \ge 0$, $1/G_n \in \mathcal{L}_{v^\delta}$, *with* δ *as in (A1) then for any* $\varrho > 0$, $f \in \mathcal{L}_{v^{1/(1+\varrho)}}$, $n \ge 0$, $\eta_n^N(f) \xrightarrow{\text{D}} \eta_n(f)$.

Proof. The result is proved by induction. The $n = 0$ case follows by the weak law of large numbers for i.i.d. random variables; $\eta_0 \in \mathcal{P}_v$. Thus, the result is assumed for $n-1$ and we consider n. We have

$$[\eta_n^N - \eta_n](f) = [\eta_n^N - \Phi_n(\eta_{n-1}^N)](f) + [\Phi_n(\eta_{n-1}^N) - \eta_n](f). \tag{15}$$

First, we deal with the second term on the right-hand side of (15). We have the standard decomposition

$$\begin{aligned}
&[\Phi_n(\eta_{n-1}^N) - \eta_n](f) \\
&\quad = \left[\frac{1}{\eta_{n-1}^N(G_{n-1})} - \frac{1}{\eta_{n-1}(G_{n-1})}\right]\eta_{n-1}^N(Q_n(f)) + \frac{1}{\eta_{n-1}(G_{n-1})}[\eta_{n-1}^N - \eta_{n-1}](Q_n(f)).
\end{aligned}$$

By Corollary 1, $Q_n(f) \in \mathcal{L}_{v^{1/(1+\varrho)}}$ (recall that for any $n \ge 0$, $\|G_n\|_\infty < +\infty$), so by the induction hypothesis it follows that $[\Phi_n(\eta_{n-1}^N) - \eta_n](f) \xrightarrow{\text{D}} 0$.

We now deal with the first term on the right-hand side of (15). We can use [4, Theorem A.1], which can be applied using Lemma 2. We have to verify Equations (25) and (26) of [4]. In the notation of this paper, they read:

(i) $\sup_N \mathbb{P}(\Phi_n(\eta_{n-1}^N)(|f|) \ge \kappa) \to 0$ as $\kappa \to \infty$.

(ii) $(1/N)\sum_{i=1}^N \mathbb{E}\{|f(x_n^i)|\,\mathbf{1}_{\{|f(x_n^i)|/N \ge \epsilon\}} \mid \mathcal{F}_{n-1}^N\} \xrightarrow{\text{D}} 0$ for any $\epsilon > 0$.

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We see that (i) follows from $[\Phi_n(\eta_{n-1}^N) - \eta_n](f) \xrightarrow{\mathrm{D}} 0$. For (ii), set $0 < \kappa \le \varrho \wedge \delta/(1-\delta)$, we easily see that $(1/N)\sum_{i=1}^N \mathbb{E}\{|f(x_n^i)| \mathbf{1}_{\{|f(x_n^i)|/N \ge \epsilon\}} \mid \mathcal{F}_{n-1}^N\} \le \Phi_n(\eta_{n-1}^N)(|f|^{1+\kappa})/(\epsilon N)^\kappa$. As $Q_n(|f|^{1+\kappa}) \in \mathcal{L}_{v^{1/(1+\varrho)}}$ by construction, it follows that $\Phi_n(\eta_{n-1}^N)(|f|^{1+\kappa})/(\epsilon N)^\kappa \xrightarrow{\mathrm{D}} 0$ which completes the proof.

Appendix B. Proofs for the asymptotic variance

We provide the proofs which are used for Theorem 2. This is broken into three sections: controlling the forward (9) and backward (8) part of the asymptotic variance and the technical results used to achieve this. We write $\overline{\mathbb{E}}_{\mu\otimes\eta}$ as the expectation with respect to the inhomogeneous Markov chain $\{\bar{X}_p\}_{p\ge 0}$ on $\overline{\mathcal{X}} := \mathcal{X}^2$ with initial distribution $\mu \otimes \eta$ and transition $H_p(x_{p-1}, x_p)H_p(y_{p-1}, y_p)\lambda(\mathrm{d}x_p) \otimes \lambda(\mathrm{d}y_p)$. Also, $\bar{M}_{p,q}^d := \sum_{k=p}^{q-1} \mathbf{1}_{\{\bar{C}_d\}}(\bar{X}_k)\,\mathbf{1}_{\{\bar{C}_d\}}(\bar{X}_{k+1})$.

B.1. Controlling the forward part

Proposition 2. *Assume that (A1), (A2), and (A4)–(A6) hold. Then if $\|f\|_{v^\alpha} < +\infty$, $\alpha \in (0, \frac{1}{3})$ there exist a $c < +\infty$ and $\rho \in (0,1)$ which depend only upon the constants in (A1) and (A4)–(A6) such that for any $x \in \mathcal{X}$,*

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \sum_{q=p}^{n} \bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q \otimes 1)\bar{Q}_{q,n}(1))) \le c\|f\|_{v^\alpha} v(x)^{3\alpha}\left\{1 + \frac{\rho(1-\rho^{n-p})}{1-\rho}\right\}. \tag{16}$$

Proof. We break up our proof into three parts where we deal with controlling the summands on the left-hand side of (16).

(i) The $q = p$ case of (16). We have

$$\bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q \otimes 1)\bar{Q}_{q,n}(1))) = \bar{\eta}_{p,x}((f_p \otimes 1)\bar{Q}_{p,n}(1)).$$

Then as $f_p \in \mathcal{L}_{v^\alpha}$, we have

$$\delta_x \otimes \eta_p((f_q \otimes 1)\bar{Q}_{p,n}(1)) \le \|f\|_{v^\alpha} v(x)^\alpha Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1)). \tag{17}$$

Thus, by using a similar argument to (17),

$$\bar{\eta}_{p,x}((f_p \otimes 1)\bar{Q}_{p,n}(1)) \le c\|f\|_{v^\alpha} Q_{p,n}(1)(x)[v(x)^\alpha \eta_p(Q_{p,n}(1)) + \eta_p(v^\alpha Q_{p,n}(1))].$$

Hence, we have

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \bar{\eta}_{p,x}((f_p \otimes 1)\bar{Q}_{p,n}(1)) \le c\|f\|_{v^\alpha} \frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))}[v(x)^\alpha \eta_p(Q_{p,n}(1)) + \eta_p(v^\alpha Q_{p,n}(1))]. \tag{18}$$

Now for the first term on the right-hand side of (18) we have

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))} v(x)^\alpha \eta_p(Q_{p,n}(1)) \le c v(x)^{2\alpha},$$

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where we have used Propositions 1 and 2 of [12] and Lemma 7, namely,

$$\sup_{n\geq 1}\sup_{0\leq p\leq n}\|h_{p.n}\|_{v^\alpha} < +\infty.$$

For the second term on the right-hand side of (18) we have, for any $r \in [\underline{d}, \infty)$, $h_{p,n}(x)/\ \eta_p(Q_{p,n}(1))\ \eta_p(v^\alpha Q_{p,n}(1)) = h_{p,n}(x)\eta_p(v^\alpha h_{p,n}) \leq cv(x)^\alpha \eta_p(v^{2\alpha})$, where we use $\sup_{n\geq 1}\sup_{0\leq p\leq n}\|h_{p.n}\|_{v^\alpha} < +\infty$. By Proposition 1 of [12] it follows that $\sup_{p\geq 0}\|\eta_p(v^{2\alpha})\|_{v^\alpha} < +\infty$, thus, $h_{p,n}(x)/\eta_p(Q_{p,n}(1))\eta_p(v^\alpha Q_{p,n}(1)) \leq cv(x)^\alpha$. Thus, for the $q = p$ case, we have established that

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)}\bar{\eta}_{p,x}((f_p\otimes 1)\bar{Q}_{p,n}(1)) \leq c\|f\|_{v^\alpha}v(x)^{2\alpha}. \tag{19}$$

(ii) The $q = n$ case of (16). We have $\bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q\otimes 1)\bar{Q}_{q,n}(1))) = \bar{\eta}_{p,x}(\bar{Q}_{p,n}((f_p\otimes 1)))$. Then, we can apply the proof of Theorem 1 of [12] to show that there exists a $\rho \in (0, 1)$ (which depends upon the constants in (A1)–(A6)),

$$\frac{\bar{\eta}_{p,x}(\bar{Q}_{p,n}((f_p\otimes 1)))}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \leq c\|f\|_{v^\alpha}\frac{v_{p,n,\alpha}(x)}{\|h_{p,n}\|_{v^\alpha}}\mu(v^\alpha)\rho^{n-p},$$

where $v_{p,n,\alpha}(x) = v(x)^\alpha\|h_{p,n}\|_{v^\alpha}/h_{p,n}(x)$. Thus, we have established that for $q = n$,

$$h_{p,n}(x)\frac{\bar{\eta}_{p,x}(\bar{Q}_{p,n}((f_p\otimes 1)))}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \leq c\|f\|_{v^\alpha}v(x)^\alpha\mu(v^\alpha)\rho^{n-p}. \tag{20}$$

(iii) The $p < q < n$ case of (16). Using almost the same calculations as Theorem 1 of [12] (which relies on the proofs of [6]), we have for arbitrary $d, \beta \in (0, 1)$,

$$\begin{aligned}
&\bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q\otimes 1)\bar{Q}_{q,n}(1))) \\
&\quad\leq 2\|f\|_{v^\alpha}\Bigg[\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\Bigg\{\prod_{s=p}^{q-1}\bar{G}_q(\bar{X}_s)\bar{v}(\bar{X}_q)^\alpha\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}\geq\beta(q-p)\}}\,\rho_d^{\bar{M}^d_{p,q}}\Bigg\} \\
&\qquad+\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\Bigg\{\prod_{s=p}^{q-1}\bar{G}_q(\bar{X}_s)\bar{v}(\bar{X}_q)^\alpha\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\,\rho_d^{\bar{M}^d_{p,q}}\Bigg\}\Bigg],
\end{aligned}\tag{21}$$

where $\rho_d = 1 - (\epsilon_d^-/\epsilon_d^+)^2$. We begin by considering the first term on the right-hand side of (21), when multiplied by the term outside the summation on the left-hand side of (16). As in Theorem 1 of [12] as $\rho_d < 1$, we have

$$\begin{aligned}
&\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \\
&\qquad\times\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\Bigg\{\prod_{s=p}^{q-1}\bar{G}_q(\bar{X}_s)\bar{v}(\bar{X}_q)^\alpha\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}\geq\beta(q-p)\}}\,\rho_d^{\bar{M}^d_{p,q}}\Bigg\} \\
&\quad\leq\rho_d^{\beta(q-p)}h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]}.
\end{aligned}$$

Then, we can apply Lemma 4 to show that

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)} \times \overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\left\{\prod_{s=p}^{q-1}\bar{G}_q(\bar{X}_s)\bar{v}(\bar{X}_q)^\alpha \bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}\geq\beta(q-p)\}}\,\rho_d^{\bar{M}^d_{p,q}}\right\}$$
$$\leq c\|f\|_{v^\alpha}\rho_d^{\beta(q-p)}v(x)^{3\alpha}.$$

Now consider the second term on the right-hand side of (21), when multiplied by the term outside the summation on the left-hand side of (16). We have

$$\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^\alpha\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))}$$
$$\leq c(d,\alpha,\beta)\mu(v^{3\alpha})v(x)^{3\alpha}\exp\{-(q-p)c(d,\alpha,\beta)]\},$$

where we note that d was arbitrary and we have applied Lemma 5. Then, we can make d larger so that we have for $p<q<n$,

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)}\bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q\otimes 1)\bar{Q}_{q,n}(1)))\leq c\|f\|_{v^\alpha}\rho^{q-p}v(x)^{3\alpha}, \quad (22)$$

where $\rho\in(0,1)$ depends upon the constants in (A1)–(A6) as well as α.

Then combining (19), (20), and (22), we have proved that for any $x\in\mathcal{X}$,

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))Q_{p,n}(1)(x)}\sum_{q=p}^{n}\bar{\eta}_{p,x}(\bar{Q}_{p,q}((f_q\otimes 1)\bar{Q}_{q,n}(1)))$$
$$\leq c_\mu\|f\|_{v^\alpha}v(x)^{3\alpha}\left[1+\sum_{q=p+1}^{n}\rho^{q-p}\right]$$

from which we conclude the proof.

B.2. Controlling the backward part

Proposition 3. *Assume that (A1)–(A6) hold. Then if $\|f\|_{v^\alpha}<+\infty$ for $\alpha\in(0,\frac{1}{2})$ there exists a $c<+\infty$ which depends only upon the constants in (A1) and (A3)–(A6), such that for any $x\in\mathcal{X}$, $p\geq 1$,*

$$\frac{h_{p,n}(x)}{\eta_p(Q_{p,n}(1))}\sum_{q=0}^{p-1}\eta_p(Q_{p,n}(1)[M_{p:q}(f_q)(x)-M_{p:q}(f_q)])\leq c\|f\|_{v^\alpha}v(x)^{2\alpha}. \quad (23)$$

Proof. Consider the summand in (23),

$$\eta_p(Q_{p,n}(1)[M_{p:q}(f_q)(x)-M_{p:q}(f_q)])$$
$$=\|f\|_{v^\alpha}\eta_p\left([Q_{p,n}(1)v(x)^\alpha v^\alpha]\left[\frac{[M_{p:q}(f_q/\|f\|_{v^\alpha})(x)-M_{p:q}(f_q/\|f\|_{v^\alpha})]}{v(x)^\alpha v^\alpha}\right]\right).$$

Then applying Lemma 6, we have

$$\eta_p(Q_{p,n}(1)[M_{p:q}(f_q)(x) - M_{p:q}(f_q)]) \leq c\|f\|_{v^\alpha}\rho^{(p-q-1)}\eta_p(Q_{p,n}(1)v^\alpha)v(x)^\alpha.$$

Thus, (23) is upper-bounded by $c\|f\|_{v^\alpha}h_{p,n}(x)\eta_p(h_{p,n}v^\alpha)v(x)^\alpha$. Then, we have

$$\begin{aligned} c\|f\|_{v^\alpha}h_{p,n}(x)\eta_p(h_{p,n}v^\alpha)v(x)^\alpha &\leq c\|f\|_{v^\alpha}\Big[\sup_{n\geq 1}\sup_{0\leq p\leq n}\|h_{p,n}\|_{v^\alpha}\Big]^2 v(x)^\alpha\eta_p(v^{2\alpha})v(x)^\alpha \\ &\leq c\|f\|_{v^\alpha}\Big[\sup_{n\geq 1}\sup_{0\leq p\leq n}\|h_{p,n}\|_{v^\alpha}\Big]^2 \sup_{p\geq 0}\|\eta_p\|_{v^{2\alpha}}v(x)^{2\alpha}. \end{aligned}$$

From Propositions 1 and 2 of [12] it follows that

$$\Big[\sup_{n\geq 1}\sup_{0\leq p\leq n}\|h_{p,n}\|_{v^\alpha}\Big]^2 \sup_{p\geq 0}\|\eta_p\|_{v^{2\alpha}} < +\infty$$

from which we conclude the proof.

B.3. Technical results

Lemma 4. *Assume that (A1), (A2), and (A4)–(A6) hold. Then for any* $\alpha \in (0, \frac{1}{2})$ *there exists a* $c < +\infty$ *depending only on the constants in (A1) and (A3)–(A6), such that for any* $n \geq 1$, $0 \leq p < q < n$, $x \in \mathcal{X}$:

$$h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \leq cv(x)^{3\alpha}.$$

Proof. Note that throughout c denotes a generic finite constant that may depend upon α, but whose value may change upon each appearance. Define the Markov semi-group $T_{p,q}(x, \mathrm{d}y) = Q_{p,q}(x, \mathrm{d}y)/Q_{p,q}(1)(x)$. Then, we have

$$\begin{aligned} &h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \\ &\quad = h_{p,n}(x)\frac{T_{p,q}(v^\alpha h_{q,n})(x)}{T_{p,q}(h_{q,n})(x)}\frac{\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})]}{\eta_p[h_{p,q}T_{p,q}(h_{q,n})]}. \end{aligned} \tag{24}$$

We will consider the right-hand side of (24). First, the term

$$\frac{h_{p,n}(x)}{T_{p,q}(h_{q,n})(x)} = \frac{Q_{p,n}(1)(x)}{\prod_{s=p}^{n-1}\lambda_s}\frac{Q_{p,q}(1)(x)\prod_{s=q}^{n-1}\lambda_s}{Q_{p,n}(1)(x)},$$

where $\lambda_s = \eta_s(G_s)$ and we have used, recursively, Lemma 1 of [12]. Then by cancelling, it clearly follows that $h_{p,n}(x)/T_{p,q}(h_{q,n})(x) = h_{p,q}(x)$. Hence, combining our calculations together and returning to (24), we have established that

$$\begin{aligned} &h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \\ &\quad = h_{p,q}(x)T_{p,q}(v^\alpha h_{q,n})(x)\frac{\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})]}{\eta_p[h_{p,q}T_{p,q}(h_{q,n})]}. \end{aligned} \tag{25}$$

We now focus on the term $1/\eta_p[h_{p,q}T_{p,q}(h_{q,n})]$ in (25). We note that for any $x \in \mathcal{X}$,

$$h_{p,q}(x)T_{p,q}(h_{q,n})(x) = \frac{Q_{p,q}(1)(x)}{\prod_{s=p}^{q-1}\lambda_s} \frac{Q_{p,n}(1)(x)}{Q_{p,q}(1)(x)\prod_{s=q}^{n-1}\lambda_s} = h_{p,n}(x).$$

By Lemma 10 of [12] for any arbitrary $d \in [\underline{d}, \infty)$ it follows that

$$\inf_{n\geq 1}\inf_{0\leq p\leq n}\inf_{x\in C_d} h_{p,n}(x) > 0$$

and so for any d as stated, and by using the above calculation, it follows that

$$\eta_p[h_{p,q}T_{p,q}(h_{q,n})] \geq \eta_p[\mathbf{1}_{\{C_d\}}h_{p,n}] \geq \eta_p(C_d)\Big[\inf_{n\geq 1}\inf_{0\leq p\leq n}\inf_{x\in C_d} h_{p,n}(x)\Big].$$

Now, by using the proof of Lemma 8 of [12], we have for large enough d that there is a finite $c > 0$ such that

$$\inf_{p\geq 0}\eta_p(C_d)[\inf_{n\geq 1}\inf_{0\leq p\leq n}\inf_{x\in C_d} h_{p,n}(x)] \geq c.$$

Thus, returning to (25), we have

$$\begin{aligned} h_{p,n}(x)&\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \\ &\leq ch_{p,q}(x)T_{p,q}(v^\alpha h_{q,n})(x)\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})]. \end{aligned} \tag{26}$$

Now, using the above arguments, we have $\sup_{n\geq 1}\sup_{1\leq q\leq n}\|h_{q,n}\|_{v^\alpha} < +\infty$, so, we have for any $x \in \mathcal{X}$ $T_{p,q}(v^\alpha h_{q,n})(x) \leq cT_{p,q}(v^{2\alpha})(x)$, where c does not depend upon p, q, n. Then using Theorem 1 of [12], we arrive at Equation (61) of [12], thus,

$$T_{p,q}(v^\alpha h_{q,n})(x) \leq c\frac{v_{p,q,2\alpha}(x)}{\|h_{p,q}\|_{v^{2\alpha}}}, \tag{27}$$

where $v_{p,q,2\alpha}(x) = v(x)^{2\alpha}\|h_{p,q}\|_{v^{2\alpha}}/h_{p,q}(x)$ and we are invoking Lemma 7. Hence, returning to (26), we have

$$h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \leq cv(x)^{3\alpha}\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})]. \tag{28}$$

We now turn to $\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})]$ on the right-hand side of (28). By using (27), we have $\eta_p[h_{p,q}T_{p,q}(v^\alpha h_{q,n})] \leq c\eta_p(v^{2\alpha})$, where c depends on α only. Using Proposition 1 of [12] (noting again Lemma 7 and that $\alpha \in (0, \frac{1}{2})$), we can thus conclude that

$$h_{p,n}(x)\frac{Q_{p,q}(v^\alpha Q_{q,n}(1))(x)}{Q_{p,q}(Q_{q,n}(1))(x)}\frac{\eta_p[Q_{p,q}(v^\alpha Q_{q,n}(1))]}{\eta_p[Q_{p,q}(Q_{q,n}(1))]} \leq cv(x)^{3\alpha},$$

which completes the proof.

Lemma 5. *Assume that (A1), (A2), and (A4)–(A6) hold. Then there exists a $d \in [\underline{d}, \infty)$ (which can be arbitrarily large) such that for any $\alpha \in (0, \frac{1}{3})$, $\beta \in (0, 1)$ there exists a $0 < c(d, \alpha, \beta) < +\infty$ such that for any, $n \geq 1$, $0 \leq p < q < n$, $x \in \mathcal{X}$,*

$$\begin{aligned} &\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^\alpha\bar{Q}_{q,n}(1)(\bar{X}_q)\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))} \\ &\leq c(d,\alpha,\beta)\mu(v^{3\alpha})v(x)^{3\alpha}\exp\{-(q-p)c(d,\alpha,\beta)\}. \end{aligned}$$

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Proof. Throughout c denotes a generic finite and positive constant that depends upon α, β, d, but whose value may change upon each appearance. The dependencies of c are omitted in the proof in order to simplify the notation.

We can rewrite the above equation as

$$\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))}$$
$$= \frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{h}_{q,n}(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,q}(h_{q,n})(x)\eta_p(Q_{p,q}(h_{q,n}))}. \tag{29}$$

Now consider the $h_{p,n}(x)/Q_{p,q}(h_{q,n})(x)$ term in (29). We have

$$\frac{h_{p,n}(x)}{Q_{p,q}(h_{q,n})(x)} = \frac{Q_{p,n}(1)(x)\prod_{s=q}^{n-1}\lambda_s}{\prod_{s=p}^{n-1}\lambda_s Q_{p,n}(1)(x)} = \frac{1}{\prod_{s=p}^{q-1}\lambda_s}.$$

Now, using Propositions 1 and 2 of [12], it follows that $\underline{\lambda} := \inf_{s\ge 0}\lambda_s > 0$ and, thus, by the above calculation it follows that $h_{p,n}(x)/Q_{p,q}(h_{q,n})(x) \le 1/\underline{\lambda}^{q-p}$. This leaves us with

$$\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))}$$
$$\le \frac{\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{h}_{q,n}(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{\underline{\lambda}^{q-p}\eta_p(Q_{p,q}(h_{q,n}))}. \tag{30}$$

We next consider the $1/\eta_p(Q_{p,q}(h_{q,n}))$ term on the right-hand side of (30). Pick a fixed $r \in [\underline{d}, d)$. Then by repeatedly applying (A4), we obtain

$$\begin{aligned}\eta_p(Q_{p,q}(h_{q,n})) &\ge \eta_p(Q_{p,q}(C_r))\inf_{n\ge 1}\inf_{0\le q\le n}\inf_{x\in C_r} h_{q,n}(x)\\ &\ge \eta_p(C_r)(\epsilon_r^-\nu_r(C_r))^{q-p}\inf_{n\ge 1}\inf_{0\le q\le n}\inf_{x\in C_r} h_{q,n}(x).\end{aligned}$$

Now, by Lemma 10 of [12] it follows that $\inf_{n\ge 1}\inf_{0\le q\le n}\inf_{x\in C_r} h_{q,n}(x) > 0$. For r and, hence, large enough d it follows that $\inf_{p\ge 0}\eta_p(C_r) > 0$ by the proof of Lemma 8 of [12]. Now fix r from here on in. Thus, we have shown that for r, large enough d,

$$\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))}$$
$$\le c\,\frac{\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\bar{h}_{q,n}(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{(\underline{\lambda}\epsilon_r^-\nu_r(C_r))^{q-p}}. \tag{31}$$

Now to complete the proof, we note that as $h_{q,n} \in \mathcal{L}_{v^{\alpha}}$ and $\sup_{n\ge 1}\sup_{0\le q\le n}\|h_{q,n}\|_{v^{\alpha}} < +\infty$, by Propositions 1 and 2 of [12] and Lemma 7, the upper-bound of the right-hand side of (31) is equal to

$$c\,\frac{\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{3\alpha}\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{(\underline{\lambda}\epsilon_r^-\nu_r(C_r))^{q-p}}.$$

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Then by the proof of Theorem 1 of [12], we note that

$$\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\left\{\left[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)\right]\bar{v}(\bar{X}_q)^{3\alpha}\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\right\}$$
$$\leq c\mu(v^{3\alpha})v(x)^{3\alpha}\exp\left\{-\frac{d\delta(q-p)(1-\beta)}{2}+\frac{3d\delta}{2}\right\}.$$

Hence, we have proved that for r, large enough d,

$$\frac{h_{p,n}(x)\overline{\mathbb{E}}_{\delta_x\otimes\eta_p}\{[\prod_{s=p}^{q-1}\bar{G}_s(\bar{X}_s)]\bar{v}(\bar{X}_q)^{\alpha}\,\bar{Q}_{q,n}(1)(\bar{X}_q)\,\mathbf{1}_{\{\bar{M}^d_{p,q}<\beta(q-p)\}}\}}{Q_{p,n}(1)(x)\eta_p(Q_{p,n}(1))}$$
$$\leq c\mu(v^{3\alpha})v(x)^{3\alpha}\exp\left\{-(q-p)\left[\frac{d\delta(1-\beta)}{2}+\log(\underline{\lambda})+\log(\epsilon_r^{-1}\mu_r(C_r))\right]+\frac{3d\delta}{2}\right\}.$$

On noting that r is fixed, we can increase d to ensure that the result holds.

Lemma 6. *Assume that (A1)–(A6) hold. Then for any $\alpha\in(0,\frac{1}{2})$, $p\geq 1$, $q\in\{0,\dots,p-1\}$ there exists a $c<+\infty$ which depends only upon the constants in (A1) and (A3)–(A6) such that*

$$\sup_{(x,z)\in\overline{\mathcal{X}}}\ \sup_{|f|\leq v^{\alpha}}\frac{|M_{p:q}(f)(x)-M_{p:q}(f)(z)|}{\bar{v}(x,z)^{\alpha}}\leq c\rho^{(p-q-1)}.$$

Proof. We start by using Lemma 4.3 of [3], which provides the neat reversal equation:

$$M_{p:q}(f)(x)=\frac{\eta_q(fQ_{q,p-1}[Q_p(\cdot,x)])}{\eta_q(Q_{q,p-1}[Q_p(\cdot,x)])}\quad\text{for all } x\in\mathcal{X}. \tag{32}$$

First, we focus on the $q\in\{0,\dots,p-2\}$ case. We note that using a similar proof to Lemma 1 of [12] it follows that for any $\varphi\colon\mathcal{X}\to\mathbb{R}$,

$$\eta_q(Q_{q,p-1}(\varphi))=\left(\prod_{s=q}^{p-2}\lambda_s\right)\eta_{p-1}(\varphi). \tag{33}$$

Using the representation (32) and the identity (33), we have

$$\frac{M_{p:q}(f)(x)-M_{p:q}(f)(z)}{\bar{v}(x,z)^{\alpha}}$$
$$=\frac{(\eta_q\otimes\eta_q)(f\{Q_{q,p-1}[Q_p(\cdot,x)]Q_{q,p-1}[Q_p(\cdot,z)]-Q_{q,p-1}[Q_p(\cdot,z)]Q_{q,p-1}[Q_p(\cdot,x)]\})}{(\prod_{s=q}^{p-2}\lambda_s)^2\eta_{p-1}[Q_p(\cdot,x)]\eta_{p-1}[Q_p(\cdot,z)]\bar{v}(x,z)^{\alpha}}. \tag{34}$$

Consider the argument of the function that is operated on by $(\eta_q\otimes\eta_q)$, when excluding f on the right-hand side of (34). This can be written as $(\delta_s\otimes\delta_t-\delta_t\otimes\delta_s)(\bar{Q}_{q,p-1}(Q_p(\cdot,x)\otimes Q_p(\cdot,z)))$. Then by (A3) as $Q_p(y,x)/\eta_{p-1}[Q_p(\cdot,x)]\in\mathcal{L}_{\bar{v}^{\alpha}}$, and via decompositions and calculations in [6] (see, e.g. the proof of Theorem 1 of [12]), we obtain

$$\frac{(\delta_s\otimes\delta_t-\delta_t\otimes\delta_s)(\bar{Q}_{q,p-1}(Q_p(\cdot,x)\otimes Q_p(\cdot,z)))}{\eta_{p-1}[Q_p(\cdot,x)]\eta_{p-1}[Q_p(\cdot,z)]}\leq c(\delta_s\otimes\delta_t)\bar{R}_{q,p-1}(\bar{v}^{\alpha})\bar{v}(x,z)^{\alpha},$$

where c depends on $\sup_{p\geq 1}\|Q_p/\eta_{p-1}[Q_p]\|_{\bar{v}^{\alpha}}$ and

$$\bar{R}_r(\bar{x},\mathrm{d}\bar{y})=\bar{Q}_r(\bar{x},\mathrm{d}\bar{y})-\mathbf{1}_{\{\bar{C}_d\}}(\bar{x})(\epsilon_d^{-})^2\nu_d\otimes\nu_d(\mathrm{d}\bar{y})$$

with $\bar{x} = (x_1, x_2) \in \overline{\mathcal{X}}$, $\bar{y} = (y_1, y_2) \in \overline{\mathcal{X}}$, and $\bar{R}_{q,p-1} = \bar{R}_{q+1} \dots \bar{R}_{p-1}$. Using Theorem 1 of [12], we have

$$(\delta_s \otimes \delta_t)\bar{R}_{q,p-1}(\bar{v}^\alpha) \leq c\rho_d^{\beta(p-q-1)} \bar{Q}_{q,p-1}(\bar{v}^\alpha)(s,t) + c\exp\left\{-(p-q-1)\left[\frac{\delta d(1-\beta)}{2} - 2b_{\underline{d}}\right] + \frac{3\delta d}{2}\right\}\bar{v}(s,t)^\alpha,$$

where c does not depend upon d, and $d \geq \underline{d}$, $\beta \in (0,1)$ are arbitrary and $\rho_d = (1 - (\epsilon_d^- / \epsilon_d^+)^2)$. Thus, returning to (34), we have established that

$$\begin{aligned}\frac{M_{p:q}(f)(x) - M_{p:q}(f)(z)}{\bar{v}(x,z)^\alpha} &\leq c\left(\prod_{s=q}^{p-2} \lambda_s\right)^{-2} (\eta_q \otimes \eta_q)\Bigg(v^\alpha\Bigg\{\rho_d^{\beta(p-q-1)}\bar{Q}_{q,p-1}(\bar{v}^\alpha) \\ &\quad + \exp\left\{-(p-q-1)\left[\frac{\delta d(1-\beta)}{2} - 2b_{\underline{d}}\right] + \frac{3\delta d}{2}\right\}\bar{v}^\alpha\Bigg\}\Bigg).\end{aligned} \tag{35}$$

We split the right-hand side of (35) into the sum of two expressions:

$$c\left(\prod_{s=q}^{p-2} \lambda_s\right)^{-2} (\eta_q \otimes \eta_q)(v^\alpha \rho_d^{\beta(p-q-1)} \bar{Q}_{q,p-1}(\bar{v}^\alpha)) \tag{36}$$

and

$$c\left(\prod_{s=q}^{p-2} \lambda_s\right)^{-2} (\eta_q \otimes \eta_q)\left(v^\alpha \exp\left\{-(p-q-1)\left[\frac{\delta d(1-\beta)}{2} - 2b_{\underline{d}}\right] + \frac{3\delta d}{2}\right\}\bar{v}^\alpha\right). \tag{37}$$

We start with (36) and rewrite it as

$$c\rho_d^{\beta(p-q-1)} \frac{\eta_q(v^\alpha Q_{q,p-1}(v^\alpha))}{\prod_{s=q}^{p-2}\lambda_s} \frac{\eta_q(Q_{q,p-1}(v^\alpha))}{\prod_{s=q}^{p-2}\lambda_s}.$$

By Theorem 1 of [12], we have the upper-bound,

$$c\rho_d^{\beta(p-q-1)} \eta_q(v^\alpha[h_{q,p-1}\eta_{p-1}(v^\alpha) + \tilde{\rho}^{\beta(p-q-1)}\mu(v^\alpha)c_\mu v^\alpha])\eta_q([h_{q,p-1}\eta_{p-1}(v^\alpha) + \tilde{\rho}^{\beta(p-q-1)}\mu(v^\alpha)c_\mu v^\alpha]),$$

where $c < \infty$, and $\tilde{\rho} \in (0,1)$ that does not depend on d. As $\sup_{q\geq 1}\sup_{1\leq p\leq q+1}\|h_{q,p-1}\|_{v^\alpha} < +\infty$ by Proposition 2 of [12] and by Proposition 1 of [12], we have that $\sup_{p\geq 1}\|\eta_{p-1}(v^\alpha)\|_{v^\alpha} < +\infty$, it follows that $c\rho_d^{\beta(p-q-1)}\eta_q(v^{2\alpha})\eta_q(v^\alpha)$ is the upper-bound on (36), where again c does not depend on d. Noting that $\alpha \in (0, \frac{1}{2})$ and applying Jensen's inequality and Proposition 1 of [12], we have the upper-bound, $c\rho_d^{\beta(p-q-1)}$ for c independent of d.

Now, turning to (37), by Proposition 2 of [12] it follows that $\inf_{p\geq 0}\lambda_p = \underline{\lambda} > 0$, and, by the above argument $\sup_{p\geq 1}\|\eta_{p-1}(v^{2\alpha})\|_{v^\alpha} < +\infty$, hence, we have

$$c\exp\left\{-(p-q-1)\left[\left(\tfrac{1}{2}\right)\delta d(1-\beta) - 2b_{\underline{d}} + 2\log(\underline{\lambda})\right] + \left(\tfrac{3}{2}\right)\delta d\right\},$$

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the upper-bound on (37). Thus, combining this upper-bound with that of $c\rho_{d}^{\beta(p-q-1)}$ on (36) and recalling that the sum of these terms are upper-bounded on the left-hand side of (35), we have established that

$$\frac{M_{p:q}(f)(x) - M_{p:q}(f)(z)}{v(x)^{\alpha}v(z)^{\alpha}} \leq c\left[\rho_{d}^{\beta(p-q-1)} + \exp\left\{-(p-q-1)\left[\frac{\delta d(1-\beta)}{2} - 2b_{\underline{d}} + 2\log(\underline{\lambda})\right] + \frac{3\delta d}{2}\right\}\right],$$

where $q \in \{0, \ldots, p-2\}$, c does not depend upon d and $d > \underline{d}$ is arbitrary. As d is arbitrary, we can conclude that for large enough d, there is a $\rho \in (0, 1)$ such that for any $q \in \{0, \ldots, p-2\}$,

$$\sup_{(x,z)\in\overline{\mathcal{X}}} \sup_{|f|\leq v^{\alpha}} \frac{|M_{p:q}(f)(x) - M_{p:q}(f)(z)|}{v(x)^{\alpha}v(z)^{\alpha}} \leq c\rho^{(p-q-1)},$$

with $c < +\infty$.

For the $q = p-1$ case, we have by definition of the backward kernel,

$$\frac{M_{p,\eta_{p-1}}(f)(x) - M_{p,\eta_{p-1}}(f)(z)}{v(x)^{\alpha}v(z)^{\alpha}} = \frac{\eta_{p-1}(fQ_p(\cdot,x))}{\eta_{p-1}(Q_p(\cdot,x))v(x)^{\alpha}v(z)^{\alpha}} - \frac{\eta_{p-1}(fQ_p(\cdot,z))}{\eta_{p-1}(Q_p(\cdot,z))v(x)^{\alpha}v(z)^{\alpha}}.$$

By (A3) as $Q_p(y,x)/\eta_{p-1}[Q_p(\cdot,x)] \in \mathcal{L}_{\bar{v}^{\alpha}}$ and as $v \geq 1$, we have

$$\frac{M_{p,\eta_{p-1}}(f)(x) - M_{p,\eta_{p-1}}(f)(z)}{v(x)^{\alpha}v(z)^{\alpha}} \leq c\eta_{p-1}(v^{2\alpha}).$$

Using $\alpha \in (0, \frac{1}{2})$ and Proposition 1 of [12], we conclude the proof.

Appendix C. Additional technical results

The following result is Lemma 3 of [12] and is included as it is frequently referred to in the text. A resulting corollary is also given.

Lemma 7. *Assume that (A1), (A2), and (A4)–(A6) hold with v the drift function in (A1), (A2), (A4), and (A5). Then for any $\alpha \in (0, 1)$ the statements of (A1), (A2), (A4), and (A5) also hold for the drift function v^{α} and with α-dependent constants*

Corollary 1. *Assume that (A1) holds and that for every $n \geq 0$, $\|G_n\|_{\infty} < +\infty$. Then for any $\alpha \in (0, 1)$, $n \geq 1$ there exist constants $\delta(n) \in (0, 1)$ and $\underline{d}(n) \geq 1$ with the following properties. For each $d(n) \in (\underline{d}(n), +\infty)$ there exists a $b_d(n) < +\infty$ such that for all $x \in \mathcal{X}$ $Q_n(\mathrm{e}^{\alpha V})(x) \leq \exp((1-\delta(n))V(x) + b_d(n)\,\mathbf{1}_{\{C_d(n)\}}(x))$, where $C_d(n) = \{x \in \mathcal{X}\colon V(x) \leq d(n)\}$.*

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