A SELECTION THEOREM AND ITS APPLICATIONS

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In this paper, we first prove an improved version of the selection theorem of Yannelis-Prabhakar and next prove a fixed point theorem in a non-compact product space. As applications, an intersection theorem and two equilibrium existence theorems for a non-compact abstract economy are given.

1. INTRODUCTION

In convex analysis, the Fan-Browder fixed point theorem [2] is an essential tool in proving existence theorems of numerous nonlinear problems (for example see [2, 7, 13, 15]). Actually, the Fan-Browder fixed point theorem can be proved by constructing a continuous selection.

In [15], Yannelis-Prabhakar proved a continuous selection theorem and obtained a fixed point theorem in paracompact convex sets. Using this fixed point theorem, they obtained an equilibrium existence theorem for a compact abstract economy.

In this paper, we first give an improved version of the selection theorem of Yannelis-Prabhakar [15]. By applying this result, we prove a fixed point theorem in non-compact product spaces. As an application of our fixed point theorem, we first prove an intersection theorem which is closely related to a non-compact generalisation of Fan’s intersection theorem [6] due to Shih-Tan [12]. Next, two equilibrium existence theorems are obtained which are either closely related to or generalisations of those results of Borglin-Keiding [1], Shafer-Sonnenschein [11], Tarafdar [14] and Yannelis-Prabhakar [15].

We shall need the following notations and definitions. Let \( A \) be a non-empty set. We shall denote by \( 2^A \) the family of all subsets of \( A \). If \( A \) is a non-empty subset of a topological space \( X \), we shall denote by \( \text{cl}_X A \) the closure of \( A \) in \( X \). If \( A \) is a subset of a vector space, \( \text{co} A \) denotes the convex hull of \( A \). Let \( X, Y \) be topological spaces and \( \phi : X \rightarrow 2^Y \) be a correspondence.

(i) If \( A \subset X \), we shall denote the restriction of \( \phi \) to \( A \) by \( \phi|_A \), that is, \( \phi|_A : A \rightarrow 2^Y \) is the correspondence defined by \( \phi|_A(x) = \phi(x) \) for all \( x \in A \).
(ii) \( \phi \) is said to be upper semicontinuous if for each open subset \( V \) of \( Y \), the set \( \{ x \in X : \phi(x) \subseteq V \} \) is open in \( X \).

(iii) \( f : X \to Y \) is a continuous selection of \( \phi \) if \( f \) is continuous and \( f(x) \in \phi(x) \) for all \( x \in X \).

(iv) If \( Y \) is a vector space, the correspondence \( co\phi : X \to 2^Y \) is defined by \( (co\phi)(x) = co\phi(x) \) for all \( x \in X \).

2. SELECTION AND FIXED POINT THEOREMS

We shall first generalise a selection theorem of Yannelis-Prabhakar [15, Theorem 3.1] as follows:

**Theorem 1.** Let \( X \) be a non-empty paracompact Hausdorff topological space and \( Y \) be a non-empty convex subset of a topological vector space. Suppose \( S, T : X \to 2^Y \) are correspondences such that

1. For each \( x \in X \), \( coS(x) \subseteq T(x) \) and \( S(x) \neq \emptyset \),
2. For each \( y \in Y \), \( S^{-1}(y) \) is open in \( X \).

Then \( T \) has a continuous selection.

**Proof:** By (1), \( X = \bigcup_{y \in Y} S^{-1}(y) \). Since \( X \) is paracompact, by (2) and Lemma 1 of Michael [10], there exists an open locally finite refinement \( \mathcal{F} = \{ U_a : a \in A \} \) of the family \( \{ S^{-1}(y) : y \in Y \} \) where \( A \) is an index set and \( U_a \) is an open subset of \( X \). By Proposition 2 of Michael [10], there exists a family of continuous functions \( \{ g_a : a \in A \} \) such that \( g_a : X \to [0,1] \), \( g_a(x) = 0 \) for \( x \notin U_a \) and \( \sum_{a \in A} g_a(x) = 1 \) for all \( x \in X \). For each \( a \in A \), choose \( y_a \in Y \) such that \( U_a \subseteq S^{-1}(y_a) \). This can be done since \( \mathcal{F} \) is a refinement of \( \{ S^{-1}(y) : y \in Y \} \). Define \( f : X \to Y \) by

\[
\begin{align*}
f(x) = \sum_{a \in A} g_a(x) y_a \quad & \text{for each } x \in X.
\end{align*}
\]

From the local finiteness of \( \mathcal{F} \), it follows that for each \( x \in X \), at least one, and at most finitely many, \( g_a(x) \) is not zero, and \( f \) is a well-defined continuous function from \( X \) to \( Y \). Let \( x \in X \) and \( a \in A \) be such that \( g_a(x) \neq 0 \), then \( x \in U_a \subseteq S^{-1}(y_a) \) so that \( y_a \in S(x) \). By (1) and the definition of \( f \), we have \( f(x) \in coS(x) \subseteq T(x) \) for each \( x \in X \). This completes the proof.

If \( S = T \), Theorem 1 reduces to Theorem 3.1 of Yannelis-Prabhakar [15].

We shall need the following lemma.

**Lemma 1.** Let \( D \) be a non-empty compact subset of a topological vector space \( E \). Then \( coD \) is \( \sigma \)-compact and hence is paracompact.
PROOF: The proof that \( \text{co} D \) is \( \sigma \)-compact can be found in [9, p.49]. For completeness, we shall include the simple proof here. For each \( n \in \mathbb{N} \), let \( S_n = \{ (\lambda_1, \ldots, \lambda_n) : \lambda_1, \ldots, \lambda_n \geq 0 \text{ with } \sum_{i=1}^n \lambda_i = 1 \} \) and define \( f_n : S_n \times \prod_{i=1}^n D \to E \) by

\[
f_n(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) = \sum_{i=1}^n \lambda_i x_i.
\]

Then \( f_n \) is continuous. Since \( S_n \times \prod_{i=1}^n D \) is compact, \( f_n(S_n \times \prod_{i=1}^n D) \) is compact. But then \( \text{co} D = \bigcup_{n=1}^{\infty} f_n \left( S_n \times \prod_{i=1}^n D \right) \) is \( \sigma \)-compact. It follows that \( \text{co} D \) is Lindelöf. Since \( \text{co} D \) is regular, \( \text{co} D \) is paracompact by Corollary 33.15 in [3, p.341]. This completes the proof.

We remark here that the topological vector space \( E \) in the above lemma is not assumed to be Hausdorff.

We shall prove the following fixed point theorem.

**Theorem 2.** Let \( \{X_i\}_{i \in I} \) be a family of non-empty convex sets, each in a locally convex Hausdorff topological vector space \( E_i \), where \( I \) is an index set. For each \( i \in I \), let \( D_i \) be a non-empty compact subset of \( X_i \) and \( S_i, T_i : X = \prod_{i \in I} X_i \to 2^{D_i} \) be such that for each \( i \in I \),

1. for each \( x \in X \), \( \text{co} S_i(x) \subseteq T_i(x) \) and \( S_i(x) \neq \emptyset \),
2. for each \( y_i \in D_i \), \( S_i^{-1}(y_i) \) is open in \( X \).

Then there exists a point \( \hat{x} \in D = \prod_{i \in I} D_i \) such that \( \hat{x} \in T(\hat{x}) = \prod_{i \in I} T_i(\hat{x}) \), that is, \( \hat{x}_i \in T_i(\hat{x}) \) for all \( i \in I \), where \( \hat{x}_i \) is the projection of \( \hat{x} \) onto \( X_i \) for each \( i \in I \).

**Proof:** Since \( D = \prod_{i \in I} D_i \) is compact in \( X \), it follows from Lemma 1 that \( \text{co} D \) is paracompact in \( X \). For each \( i \in I \), let \( S_i^*, T_i^* \) be the restrictions of \( S_i, T_i \) on \( \text{co} D \), then we have

1. for each \( x \in \text{co} D \), \( \text{co} S_i^*(x) \subseteq \text{co} T_i^*(x) \) and \( \text{co} S_i^*(x) \neq \emptyset \),
2. for each \( y_i \in D_i \),

\[
(S_i^*)^{-1}(y_i) = \{ x \in \text{co} D : y_i \in S_i(x) \} = \{ x \in \text{co} D : y_i \in S_i(x) \} = \text{co} D \cap S_i^{-1}(y_i)
\]

is open in \( \text{co} D \).

By Theorem 1, for each \( i \in I \), \( T_i^* \) has a continuous selection \( f_i : \text{co} D \to D_i \) such that \( f_i(x) \in T_i^*(x) = T_i(x) \) for each \( x \in \text{co} D \).
Define \( f : \co D \to D \) and \( T : \co D \to 2^D \) by

\[
f(x) = \prod_{i \in I} f_i(x) \quad \text{and} \quad T(x) = \prod_{i \in I} T_i(x) \quad \text{for each} \ x \in \co D.
\]

Then \( f \) is clearly continuous. By Theorem 4.5.1 of Smart [13], there exists \( \widehat{x} \in D \) such that \( \widehat{x} = f(\widehat{x}) \in T(\widehat{x}) \). This completes the proof.

Theorem 2 generalises Theorem 3.2 of Yannelis-Prabhakar [15] in several ways:

(i) \( I \) need not be a singleton set,
(ii) \( X_i \) need not be paracompact, and
(iii) \( S_i \) and \( T_i \) need not be identical.

3. APPLICATIONS

Let \( X_1, \ldots, X_n (n \geq 2) \) be topological spaces and \( X = \prod_{i=1}^{n} X_i \). Let \( i \in \{1, \ldots, n\} \) be arbitrarily fixed. Let \( \widehat{X}_i = \prod_{j \neq i} X_j \) and \( \pi_i : X \to X_i \) and \( \widehat{\pi}_i : X \to \widehat{X}_i \) be the projections. If \( x \in X \), we can write \( \pi_i(x) = x_i \) and \( \widehat{\pi}_i(x) = \widehat{x}_i \). Let \( A \) be a subset of \( X \), \( x_i \in X_i \) and \( \widehat{x}_i \in \widehat{X}_i \). Then \( [x_i, \widehat{x}_i] \) denotes the point \( x \in X \) such that \( \pi_i(x) = x_i \) and \( \widehat{\pi}_i(x) = \widehat{x}_i \) and we define \( A(\pi_i) = \{ \widehat{y}_i \in \widehat{X}_i : [x_i, \widehat{y}_i] \in A \} \) and \( A(\widehat{x}_i) = \{ y_i \in X_i : [y_i, \widehat{x}_i] \in A \} \). If \( A_1 \subset X_1 \) and \( \widehat{A}_1 \subset \widehat{X}_1 \), \( A_1 \otimes \widehat{A}_1 \) denotes the set \( \{ [y_i, \widehat{y}_i] \in X : y_i \in A_1 \text{ and } \widehat{y}_i \in \widehat{A}_1 \} \).

We shall give an application of a fixed point theorem to an intersection theorem as follows:

**THEOREM 3.** Let \( \{X_i\}_{i \in I} \) be a family of non-empty convex sets, each in a locally convex Hausdorff topological vector space \( E_i \). For each \( i \in I \), let \( D_i \) be a non-empty compact subset of \( X_i \). Suppose that \( \{A_i\}_{i \in I}, \{B_i\}_{i \in I} \) are two families of subsets of \( X = \prod_{i \in I} X_i \) having the following properties:

1. For each \( i \in I \) and \( x_i \in D_i \), the set \( B_i(x_i) \) is open in \( \widehat{X}_i \),
2. For each \( i \in I \), and \( \widehat{y}_i \in \widehat{X}_i \), the set \( B_i(\widehat{y}_i) \cap D_i = \{ x_i \in D_i : [x_i, \widehat{y}_i] \in B_i \} \neq \emptyset \) and \( \co (B_i(\widehat{y}_i) \cap D_i) \subset A_i(\widehat{y}_i) \cap D_i \).

Then we have \( \bigcap_{i \in I} A_i \neq \emptyset \).

**Proof:** Define \( S_i, T_i : X \to 2^{D_i} \) as follows:

\[
S_i(y) = B_i(\widehat{y}_i) \cap D_i, \quad T_i(y) = A_i(\widehat{y}_i) \cap D_i, \quad \text{for each} \ y \in X.
\]
Then by (2), for each \( i \in I \) and \( y \in X, \co S_i(y) \subseteq T_i(y) \) and \( S_i(y) \neq \emptyset \). By (1), for each \( i \in I \) and \( x_i \in D_i \),

\[
S_i^{-1}(x_i) = \{ y \in X : x_i \in S_i(y) \} = \{ y \in X : x_i \in B_i(y_i) \cap D_i \} = \{ y \in X : [x_i, y_i] \in B_i \}
\]

is open in \( X \).

By Theorem 2, there exists \( x \in D = \prod_{i \in I} D_i \) such that \( x \in T(x) = \prod_{i \in I} T_i(x) \), that is, \( x_i \in A_i(\tilde{x}_i) \) for all \( i \in I \) and hence \( x = [x_i, \tilde{x}_i] \in \bigcap_{i \in I} A_i \). Therefore \( \bigcap_{i \in I} A_i \neq \emptyset \). This completes the proof.

We remark that Theorem 3 is closely related to but not comparable to Theorem 2 of Shih-Tan [12] which was a non-compact generalisation of Fan’s intersection theorem [6] (in our case, the space \( E_i \) is required to be locally convex).

Next we shall give two equilibrium existence theorems for a non-compact abstract economy with an infinite number of commodities and an infinite number of agents. We first give some definitions in equilibrium theory. Let the set \( I \) of agents be any (possibly uncountable) set. An abstract economy \( \Gamma = (X_i, A_i, B_i, P_i)_{i \in I} \) is defined as a family of ordered quadruples \( (X_i, A_i, B_i, P_i) \) where \( A_i, B_i : \prod X_j \to 2^{X_i} \) are constraint correspondences and \( P_i : \prod X_j \to 2^{X_i} \) is a preference correspondence. An equilibrium for \( \Gamma \) is a point \( \tilde{x} \in X = \prod_{i \in I} X_i \) such that for each \( i \in I \), \( \tilde{x}_i \in \cl_{X_i} B_i(\tilde{x}) \) and \( A_i(\tilde{x}) \cap P_i(\tilde{x}) = \emptyset \). When \( A_i = B_i \) for each \( i \in I \), our definitions of an abstract economy and an equilibrium coincide with the standard definitions, for example in Borglin-Keiding [1, p.315] or in Yannelis-Prabhakar [15, p.242].

We shall first show that by applying Himmelberg’s fixed point theorem [8, Theorem 2] instead of Ky Fan’s fixed point theorem [5], the proof of Theorem 6.1 of Yannelis-Prabhakar [15] can be used to prove its non-compact case.

**Theorem 4.** Let \( \Gamma = (X_i, A_i, B_i, P_i)_{i \in I} \) be an abstract economy such that for each \( i \in I \),

1. \( X_i \) is a non-empty convex subset of a locally convex Hausdorff topological vector space \( E_i \) and \( D_i \) is a non-empty compact subset of \( X_i \),
2. for each \( x \in X = \prod X_i, A_i(x) \) is non-empty, \( A_i(x) \subseteq B_i(x) \subseteq D_i \) and \( B_i(x) \) is convex,
3. the correspondence \( \cl B_i : X \to 2^{X_i} \) defined by \( (\cl B_i)(x) = \cl_{X_i} B_i(x) \) for each \( x \in X \), is upper semicontinuous,
(4) for each $y \in D_i$, $A_i^{-1}(y)$ is open in $X$,
(5) for each $y \in X_i$, $P_i^{-1}(y)$ is open in $X$,
(6) for each $z \in X$, $x_i \notin \text{co}P_i(x)$,
(7) the set $\{x \in X : \text{co}A_i(x) \cap \text{co}P_i(x) \neq \emptyset \}$ is paracompact.

Then $\Gamma$ has an equilibrium $\hat{x} \in X$, that is, for each $i \in I$,

$$\hat{x}_i \in \text{cl}_X B_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$ 

**Proof:** We first fix $i \in I$. Define $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = \text{co}A_i(x) \cap \text{co}P_i(x) \quad \text{for each } x \in X.$$

By (4), (5) and Lemma 5.1 of Yannelis-Prabhakar [15], it is easy to see that for each $y \in X_i$, $\phi_i^{-1}(y)$ is open in $X$. Let $U_i = \{x \in X : \phi_i(x) \neq \emptyset \}$. Since $U_i = \bigcup_{y \in X_i} \phi_i^{-1}(y)$, $U_i$ is open in $X$. By (7), $U_i$ is paracompact. Note that $\phi_i|U_i : U_i \to 2^{X_i}$ has the following properties:

(i) for each $x \in U_i$, $\phi_i|U_i(x)$ is non-empty and convex,
(ii) for each $y \in X_i$, $(\phi_i|U_i)^{-1}(y) = \phi_i^{-1}(y) \cap U_i$ is open in $U_i$.

By Theorem 3.1 of Yannelis-Prabhakar [15] (which is the case $S = T$ in our Theorem 1), there exists a continuous selection $f_i : U_i \to 2^{X_i}$ such that $f_i(x) \in \phi_i|U_i(x)$ for all $x \in U_i$. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} 
\{f_i(x)\}, & \text{if } x \in U_i, \\
\text{cl}_X B_i(x), & \text{if } x \notin U_i.
\end{cases}$$

By (3) and Lemma 6.1 of Yannelis-Prabhakar [15], $F_i : X \to 2^{X_i}$ is upper semicontinuous on $X$. Clearly for each $x \in X$, $F_i(x)$ is a non-empty closed convex subset of $D_i$ by (2). Finally we define $F : X \to 2^X$ by

$$F(x) = \prod_{i \in I} F_i(x) \quad \text{for each } x \in X.$$ 

It follows from Lemma 3 of Fan [5] that $F$ is upper semicontinuous on $X$. Obviously for each $x \in X$, $F(x)$ is a closed convex subset of $D = \prod_{i \in I} D_i$. By Tychonoff’s product theorem (for example see Dugundji [4, p.224]), $D$ is a compact subset of $X$. Hence by Theorem 2 of Himmelberg [8], there exists a point $\hat{x} \in D$ such that $\hat{x} \in F(\hat{x})$. If $\hat{x} \in U_i$ for some $i \in I$, then $\hat{x}_i = f_i(\hat{x}) \in \text{co}A_i(\hat{x}) \cap \text{co}P_i(\hat{x}) \subset \text{co}P_i(\hat{x})$ which contradicts (6). Thus for each $i \in I$, we must have $\hat{x} \notin U_i$ so that $\hat{x}_i \in \text{cl}_X B_i(\hat{x})$ and
co $A_i(\tilde{x}) \cap co P_i(\tilde{x}) = \emptyset$. Consequently, $\tilde{x}$ is an equilibrium for $\Gamma$. This completes the proof. \[\square\]

As we have seen in the proof, we can obtain a stronger separation result, that is, for each $i \in I$, $co A_i(\tilde{x}) \cap co P_i(\tilde{x}) = \emptyset$.

Theorem 4 generalises Theorem 6.1 of Yannelis-Prabhakar [15] in the following ways:

(i) for each $i \in I$, the space $E_i$ need not be metrisable,
(ii) for each $i \in I$, the set $X_i$ need not be compact, and
(iii) the set $I$ of agents need not be countable.

**Theorem 5.** Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions hold:

1. $X_i$ is a non-empty convex subset of a locally convex Hausdorff topological vector space $E_i$ and $D_i$ be a non-empty compact subset of $X_i$,
2. for each $x \in X$, $A_i(x)$ is non-empty and $co A_i(x) \subset B_i(x) \subset D_i$,
3. for each $y_i \in D_i$, the set $[(co P_i)^{-1}(y_i) \cup F_i] \cap A_i^{-1}(y_i)$ is open in $X$, where $F_i = \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$,
4. for each $x \in X$, $x_i \notin co P_i(x)$.

Then $\Gamma$ has an equilibrium.

**Proof:** For each $i \in I$, let $G_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ and for each $x \in X$, let $I(x) = \{i \in I : A_i(x) \cap P_i(x) \neq \emptyset\}$. For each $i \in I$, we define the correspondences $S_i, T_i : X = \prod_{i \in I} X_i \rightarrow 2^{D_i}$ by

$$S_i(x) = \begin{cases} co P_i(x) \cap A_i(x), & \text{if } i \in I(x), \\ A_i(x), & \text{if } i \notin I(x), \end{cases}$$

$$T_i(x) = \begin{cases} co P_i(x) \cap B_i(x), & \text{if } i \in I(x), \\ B_i(x), & \text{if } i \notin I(x). \end{cases}$$

Then we have the following properties:

(i) for each $i \in I$ and $x \in X$, $co S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$,
(ii) for each $i \in I$ and $y_i \in D_i$,

$$S_i^{-1}(y_i) = \{[(co P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)] \cap G_i \} \cup [A_i^{-1}(y_i) \cap F_i]$$

$$= [(co P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)] \cup [A_i^{-1}(y_i) \cap F_i]$$

$$= [(co P_i)^{-1}(y_i) \cup F_i] \cap A_i^{-1}(y_i)$$

is open in $X$ by (3).
By Theorem 2, there exists $\tilde{x} \in D$ such that $\tilde{x}_i \in T_i(\tilde{x})$ for all $i \in I$. By (4) and the definition of $T_i$, we have $\tilde{x}_i \in B_i(\tilde{x})$ and $A_i(\tilde{x}) \cap P_i(\tilde{x}) = \emptyset$ for all $i \in I$. This completes the proof. □

Finally we remark that Theorems 4 and 5 are closely related to those results of Shafer-Sonnenschein [11, p.347], Borglin-Keiding [1, p.315] and Tarafdar [14, Theorem 3.1].

REFERENCES