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ON MINIMAL HAUSDORFF SPACES

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Abstract

In this paper, several characterizations of minimal Hausdorff spaces are given.

It is well-known that the set of Hausdorff topologies on a set X, $H(X)$, is partially ordered by inclusion and that a topological space (X, τ) is *minimal Hausdorff* provided τ is a minimal element in $H(X)$. A rather extensive survey of minimal topological spaces has been given by Berri, Porter, and Stephenson (1971). Several characterizations of minimal Hausdorff topological spaces have also been given there.

In the present paper, we utilize a recent characterization of minimal Hausdorff topological spaces by Herrington and Long (1975) and the concept of strongly-closed graph introduced by Herrington and Long (1975) to give several new characterizations of minimal Hausdorff topological spaces.

In the remainder of this paper, topological spaces will merely be called spaces and $\mathrm{cl}[K]$ will denote the closure of a subset K of a space. If $g: X \to X$ is a function, the set of fixed points of g, i.e., $\{x \in X: x = g(x)\}$, will be denoted by $F(g)$. The graph of a function $g: X \to Y$ will be denoted by $G(g)$.

1. Characterizations

A point x in a space X is in the *θ-closure of a set* $K \subset X$ (θ-$cl[K]$) if $\mathrm{cl}[V] \cap K \neq \varnothing$ for any V open about x (Velichko (1969)). A point x in a space is in the *θ-adherence of a filterbase* $\mathscr{W}$ on the space if $x \in \theta\text{-}\mathrm{cl}[F]$ for each $F \in \mathscr{W}$. In this case, we will sometimes say that *x is a θ-adherent point of* $\mathscr{W}$ using the notation $x \in \theta\text{-}adh\mathscr{W}$ from Velichko (1969). From Herrington and Long (1975) a Hausdorff space is *minimal Hausdorff* if and only if each filterbase on the space with at most one θ-adherent point is convergent. A

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function $g: X \to Y$ has a *strongly-closed graph* if for each $(x, y) \in (X \times Y) - G(g)$, there are open sets $V \subset X$ and $W \subset Y$ satisfying $(x, y) \in V \times W$ and $(V \times \mathrm{cl}[W]) \cap G(g) = \varnothing$. We let $\mathcal{S}$ denote a class of spaces containing as a subclass the class of Hausdorff completely normal fully normal spaces. The following result which we use in the sequel is a theorem of Herrington and Long (1975).

H-L) Let Y be a minimal Hausdorff space. If $g: X \to Y$ has a strongly-closed graph then g is continuous.

Before going to our main results we make an additional definition and give a preliminary theorem. If x_0 is a point in a space (X, τ) and $\mathcal{W}$ is a filterbase on X then $\{A \subset X: x_0 \in X - A \text{ or } F \cup \{x_0\} \subset A \text{ for some } F \in \mathcal{W}\}$ is a topology on X which we will call *the topology on X associated with x_0 and $\mathcal{W}$.* We will denote this topology by $\tau(x_0, \mathcal{W})$. We state the following readily established theorem without proof.

THEOREM 1. *If (X, τ) is a space, x_0 is a point in X and $\mathcal{W}$ is a filterbase on X which has empty intersection on $X - \{x_0\}$, then $(X, \tau(x_0, \mathcal{W}))$ is in class $\mathcal{S}$.*

We go now to our main results.

THEOREM 2. *A Hausdorff space (X, τ) is minimal Hausdorff if and only if for each topology τ^* on X for which (X, τ^*) is in class $\mathcal{S}$ and for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a closed graph, $F(g)$ is closed in X for each bijection $g: (X, \tau^*) \to (X, \tau)$ with a strongly-closed graph.*

PROOF. *Strong Necessity.* Let (X, τ) be minimal Hausdorff and let τ^* be any topology on X for which $i: (X, \tau^*) \to (X, \tau)$ has a closed graph. Let $g: (X, \tau^*) \to (X, \tau)$ be any function with a strongly-closed graph. By H-L above g is continuous. Thus, the restriction of the projection, $\pi_x: X \times Y \to X$, to $G(g)$ is a homeomorphism. Since $G(i)$ is closed, $G(i) \cap G(g)$ is a closed subset of $G(g)$ and therefore $F(g) = \pi_x(G(i) \cap G(g))$ is closed in X.

Sufficiency. Suppose $\mathcal{W}$ is a filterbase on X with at most one θ-adherent point x_0. We will assume that $\mathcal{W}$ does not converge to x_0 and thus obtain a contradiction. Since $\mathcal{W}$ does not converge to x_0, let $V_0 \in \tau$ with $x_0 \in V_0$ and $\mathcal{W}^* = \{F \cap (X - V_0): F \in \mathcal{W}\}$ a filterbase on X. Choose $y_0 \in X - V_0$. We show that $i: (X, \tau(y_0, \mathcal{W}^*)) \to (X, \tau)$ has a closed graph and that $g: (X, \tau(y_0, \mathcal{W}^*)) \to (X, \tau)$, defined by $g(x_0) = y_0$, $g(y_0) = x_0$ and $g(x) = x$ otherwise, has a strongly-closed graph. Since $(X, \tau(y_0, \mathcal{W}^*))$ is in class $\mathcal{S}$ by Theorem 1 and we see easily that $F(g) = X - \{x_0, y_0\}$ which is not closed in X, we obtain a contradiction:

a) *$i: (X, \tau(y_0, \mathcal{W}^*)) \to (X, \tau)$ has a closed graph.* Let $x, y \in X$ and $x \neq y$. If

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$x \neq y_0$, let $W \in \tau$ with $y \in W$ and $x \notin W$. Then $\{x\} \in \tau(y_0, \mathcal{W}^*)$ and $(\{x\} \times W) \cap G(i) = \emptyset$. If $x = y_0$ and $y = x_0$, then $y \neq y_0$; so there is a $W \in \tau$ with $y \in W$ and an $F \in \mathcal{W}$ satisfying $y_0 \notin W \cup F$. $(F \cap (X - V_0)) \cup \{y_0\} \in \tau(y_0, \mathcal{W}^*)$, $y \in W \cap V_0 \in \tau$ and $(((F \cap (X - V_0)) \cup \{y_0\}) \times (W \cap V_0)) \cap G(i) = \emptyset$. If $x = y_0$ and $y \neq x_0$, then $y \neq y_0$ and there is an $F \in \mathcal{W}$ and a $W \in \tau$ satisfying $y_0 \notin W$, $y \in W$, and $F \cap W = \emptyset$. Thus $((F \cup \{y_0\}) \times W) \cap G(i) = \emptyset$. This completes the demonstration that i has a closed graph.

b) *$g: (X, \tau(y_0, \mathcal{W}^*)) \to (X, \tau)$ has a strongly-closed graph.* Let $(x, y) \in (X \times X) - G(g)$. If $x \neq y_0$ choose a $W \in \tau$ with $y \in W$ and $g(x) \notin \mathrm{cl}[W]$; and $(\{x\} \times \mathrm{cl}[W]) \cap G(g) = \emptyset$. If $x = y_0$, then $y \neq x_0$; so there is a $W \in \tau$ and an $F \in \mathcal{W}$ with $y \in W$, $\mathrm{cl}[W] \cap F = \emptyset$, $x_0 \notin \mathrm{cl}[W]$, and $y_0 \notin F$; and $(((F \cap (X - V_0)) \cup \{y_0\}) \times \mathrm{cl}[W]) \cap G(g) = \emptyset$. This completes the demonstration that g has a strongly-closed graph;

The proof of the theorem is complete.

We may interchange the requirements of "closed graph" and "strongly-closed graph" in Theorem 2.

THEOREM 3. *A Hausdorff space (X, τ) is minimal Hausdorff if and only if for each topology τ^* on X for which (X, τ^*) is in class $\mathcal{S}$ and for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed in X for each bijection $g: (X, \tau^*) \to (X, \tau)$ with a closed graph.*

PROOF. *Strong Necessity.* Let (X, τ) be minimal Hausdorff and let τ^* be any topology on X for which $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph. Let $g: (X, \tau^*) \to (X, \tau)$ be any function with a closed graph. By H-L above, i is continuous and $G(g) \cap G(i)$ is closed in $G(i)$. So, as in the proof of the necessity of Theorem 2, $F(g)$ is closed in X.

Sufficiency. In the proof of the sufficiency of Theorem 2, use $\tau(x_0, \mathcal{W}^*)$. Using the same techniques as in the proof of the sufficiency of Theorem 2, we can show that $i: (X, \tau(x_0, \mathcal{W}^*)) \to (X, \tau)$ has a strongly-closed graph, that $g: (X, \tau(x_0, \mathcal{W}^*)) \to (X, \tau)$ has a closed graph, that $(X, \tau(x_0, \mathcal{W}^*))$ is in class $\mathcal{S}$, and we see easily that $F(g) = X - \{x_0, y_0\}$, which is not closed in X.

This contradiction completes the proof.

THEOREM 4. *A Hausdorff space (X, τ) is minimal Hausdorff if and only if for each topology τ^* on X for which (X, τ^*) is in class $\mathcal{S}$ and for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a closed graph, $F(g) = X$ whenever $g: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph and $F(g)$ is dense in X.*

PROOF. *Strong Necessity.* Let (X, τ) be minimal Hausdorff and let τ^* be any topology on X for which $i: (X, \tau^*) \to (X, \tau)$ has a closed graph; $F(g)$ is

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closed in X by the proof of the necessity of Theorem 2. So, if $F(g)$ is dense in X, we have $F(g)=X$.

Sufficiency. The proof may be carried out in a fashion similar to the proof of the sufficiency of Theorem 2 with the exception that g is defined by $g(x)=x$ if $x \neq y_0$ and $g(y_0)=x_0$. Then i has a closed graph and g has a strongly-closed graph, but $F(g)=X-\{y_0\}$ which is dense in X.

This contradiction completes the proof.

Theorem 5 below is related to Theorem 3 as Theorem 4 is related to Theorem 2. Thus, any statement of proof is omitted.

THEOREM 5. *A Hausdorff space (X,τ) is minimal Hausdorff if and only if for each topology τ^* on X for which (X,τ^*) is in class $\mathcal{S}$ and for which the identity function $i:(X,\tau^*)\to(X,\tau)$ has a strongly-closed graph, $F(g)=X$ whenever $g:(X,\tau^*)\to(X,\tau)$ has a closed graph and $F(g)$ is dense in X.*

2. Some examples

In this section, we give examples to indicate some of the limitations on the weakening of hypotheses in the theorems in this paper. By way of notation, we let N be the set of positive integers and for each $k \in N$, we let

$$O(k)=\left\{k+\frac{1}{2n}: n \in N\right\},\ E(k)=\left\{k+\frac{1}{2n-1}: n \in N\right\},$$

$$Z(k)=\{n \in N: n \geqq k\}$$

and $X=\{-1,0\}\cup\bigcup_N O(k)\cup\bigcup_N E(k)\cup N$.

EXAMPLE 1. *The requirement cannot be weakened to all functions having closed graphs in either of Theorems* 2, 3, 4, *or* 5. Let τ be the topology generated on X by the collection of all sets in the three collections, $\{\{-1\}\cup\bigcup_{k\geqq m}O(k): m \in N\}$, $\{\{0\}\cup\bigcup_{k\geqq m}E(k): m \in N\}$, and $\{V\cap(X-\{-1,0\}): V$ a usual open subset of the reals$\}$ as base. (X,τ) is easily shown to be minimal Hausdorff. We let $\mathcal{W}=\{Z(k): k \in N\}$. Let $i:(X,\tau(1,\mathcal{W}))\to(X,\tau)$ be the identity function; define $h,g:(X,\tau(1,\mathcal{W}))\to(X,\tau)$ by $h(x)=x$ if $x\neq 1$, and $h(1)=0$; and $g(0)=1$, $g(1)=0$ and $g(x)=x$ otherwise. Then i, g, and h all have closed graphs which are not strongly-closed. $F(h)=X-\{1\}$ which is dense in $(X,\tau(1,\mathcal{W}))$, and $F(g)=X-\{0,1\}$ which is not closed in $(X,\tau(1,\mathcal{W}))$.

EXAMPLE 2. *The requirement cannot be changed to all functions having strongly-closed graphs in either of Theorems* 2, 3, 4, *or* 5.

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Let $X = \{0\} \cup \bigcup_N E(k) \cup N$ with the relative topology τ from the space in Example 1. It is not difficult to show that (X, τ) is H-closed. (X, τ) is not minimal Hausdorff since the filterbase $\mathscr{W} = \{Z(k): k \in N\}$ possesses 0 as a unique θ-adherent point and is not convergent. Let τ^* be a topology on X for which $i:(X, \tau^*) \to (X, \tau)$ has a strongly-closed graph and let $g:(X, \tau^*) \to (X, \tau)$ have a strongly-closed graph. By Herrington and Long (1975), for each $x \in X$ and $W \in \tau$ with $g(x) \in W$, there is a $V \in \tau^*$ with $x \in V$ and $g(V) \subset \mathrm{cl}[W]$. We now show that $F(g)$ is closed. Let $v \in \mathrm{cl}[F(g)]$ and suppose $g(v) \neq v$. Since i has a strongly-closed graph, there is a $V \in \tau^*$ and a $W \in \tau$ with $(v, g(v)) \in V \times W$ and $(V \times \mathrm{cl}[W]) \cap G(i) = \varnothing$. Let $A \in \tau^*$ with $v \in A$ and $g(A) \subset \mathrm{cl}[W]$. Then $A \cap V \in \tau^*$, $v \in A \cap V$ and $(A \cap V) \cap F(g) = \varnothing$. This contradiction shows that $F(g)$ is closed. It follows also that $F(g) = X$ if $F(g)$ is dense in X.

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