Glasgow Math. J. **52A** (2010) 83–91. © Glasgow Mathematical Journal Trust 2010.
doi:10.1017/S0017089510000224.

MONOMORPHISMS AND KERNELS IN THE CATEGORY OF FIRM MODULES

JUAN GONZÁLEZ-FÉREZ and LEANDRO MARÍN
Department of Applied Mathematics, University of Murcia, 30071 Murcia, Spain

Abstract. In this paper we consider for a non-unital ring R, the category of firm R-modules for a non-unital ring R, i.e. the modules M such that the canonical morphism $\mu_M : R \otimes_R M \to M$ given by $r \otimes m \mapsto rm$ is an isomorphism. This category is a natural generalization of the usual category of unitary modules for a ring with identity and shares many properties with it. The only difference is that monomorphisms are not always kernels. It has been proved recently that this category is not Abelian in general by providing an example of a monomorphism that is not a kernel in a particular case. In this paper we study the lattices of monomorphisms and kernels, proving that the lattice of monomorphisms is a modular lattice and that the category of firm modules is Abelian if and only if the composition of two kernels is a kernel.

1. Introduction. Let R be an associative ring, possibly without identity. One of the problems that arises when we try to apply categorical methods to non-unital rings is to choose an appropriate category of modules. It is commonly accepted that the category of all modules is too big and does not properly reflect the structure of R, but the structure of the unital ring $R \times \mathbb{Z}$ (the Dorroh extension of R).

A category that seems to be a good generalization of the usual category of unitary modules for a unital ring is the category of firm modules $R - \mathsf{DMod}$. A module M is firm if the canonical morphism $\mu_M : R \otimes_R M \to M$ given by $r \otimes m \mapsto rm$ is an isomorphism.

This category is considered in Quillen (1997, unpublished data) and is left as an open question: Is this category always Abelian?. Many properties for the category of firm modules are proved in [**4**]. For example, it is proved to be a complete and cocomplete category with a generator. In fact the only property needed for Abelianness that is not proved is the normality. A category is said to be a normal category if every monomorphism is a kernel, therefore monomorphisms and kernels are the objects under consideration for the Abelianness problem.

Recently, [**3**] reported a negative answer. There is an example of a ring R and a monomorphism m in $R-\mathsf{DMod}$ such that m is not the kernel of h for any morphism h in $R-\mathsf{DMod}$. Despite this, we can define the concept of subobject of a firm module M in general by considering the equivalence classes of monomorphisms. In general, it is true that kernels are monomorphisms, therefore the kernels are a particular family of subobjects. In Section 2 we make the basic definitions and we also prove that the category of firm modules is Abelian if and only if the composition of kernels is a kernel.

In Section 3 we prove that the family of subobjects has a modular lattice structure with the usual order relation used in Abelian categories.

2. Monomorphisms, kernels and residues. We are going to use a notation similar to that in [**3**, **4**]. In particular, all modules will be left modules and morphisms will be written opposite to the scalars and, therefore, they will usually be written on the right. If $f : M \to N$ and $g : N \to K$ are morphisms, we will denote the composition gf in order to have the property $(m)(gf) = ((m)g)f$.

In what follows R will be an associative ring (possibly without identity) and $A = R \times \mathbb{Z}$ is the Dorroh extension of R (in $R \times \mathbb{Z}$ the sum is defined componentwise and the product is given by the formula $(r, z)(r', z') = (rr' + rz' + r'z, zz')$. Ring A is a ring with identity, $1_A = (0, 1)$, and we can identify the category of all left R-modules with the category of unitary A-modules, $A-\mathsf{Mod}$. This identification also satisfies that $\mathsf{Hom}_R(-,-) = \mathsf{Hom}_A(-,-)$ and $- \otimes_R - = - \otimes_A -$ because the elements of $\mathbb{Z}$ can be moved using linearity. This identification is standard, some details can be seen in [**1**].

We are going to use the following definitions:

DEFINITION 1. Let M be an R-module. We will say that M is

(1) *firm* if the canonical morphism $\mu_M : R \otimes_R M \to M$ given by $(r \otimes m)\mu_M = rm$ is an isomorphism.
(2) *unitary* if $RM = M$, i.e. μ_M is surjective.
(3) *vanishing* if the only unitary submodule in M is 0.

The sum of all unitary submodules of M is unitary. This is the biggest unitary submodule of M and will be denoted $\mathsf{U}(M)$ and it will be called the unitary part of M. (See [**3**, Definition 3] or [**4**, Section 2] for details).

With the previous definition U is an idempotent radical associated to the torsion theory given by the unitary and vanishing modules. In particular, $M/\mathsf{U}(M)$ is always a vanishing module.

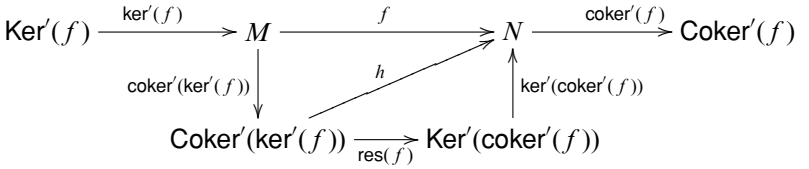
The full subcategory of $A-\mathsf{Mod}$ given by the firm modules will be denoted $R-\mathsf{DMod}$. The canonical inclusion $\mathsf{J} : R-\mathsf{DMod} \to A-\mathsf{Mod}$. This functor has a right adjoint $\mathsf{D} : A-\mathsf{Mod} \to R-\mathsf{DMod}$. The details of this construction can be seen in [**4**, Section 7]. The definition is as follows:

Let G be a generator of the category $R-\mathsf{DMod}$, $E = \mathsf{Hom}_R(G, G)$ and consider the functor $\mathsf{H} = \mathsf{Hom}_R(G, -) : R-\mathsf{DMod} \to E-\mathsf{Mod}$ and the natural morphism $\eta_M : G^{(\mathsf{H}(M))} \to M$ given by $((g_u)_{u\in\mathsf{H}(M)})\eta_M = \sum_{u\in\mathsf{H}(M)}(g_u)u$ for all $(g_u)_{u\in\mathsf{H}(M)} \in G^{(\mathsf{H}(M))}$. The functor $\mathsf{D}(M)$ is precisely $\mathsf{D}(M) = G^{(\mathsf{H}(M))}/\mathsf{U}(\mathsf{Ker}(\eta_M))$. The counit of the adjunction is $\nu_M : \mathsf{D}(M) \to M$ induced by η_M (because $\mathsf{U}(\mathsf{Ker}(\eta_M)) \subseteq \mathsf{Ker}(\eta_M)$). This construction satisfies that $\mathsf{Im}(\nu_M) = \mathsf{U}(M)$ (see [**4**, Proposition 17] for details).

The functor $R \otimes_R -$ commutes with colimits, so the colimit of firm modules is firm computed in $A-\mathsf{Mod}$ and it is also the colimit in the category $R-\mathsf{DMod}$. Nevertheless, limits of firm modules are not firm in general. In order to compute limits in $R-\mathsf{DMod}$ we have to compute them in $A-\mathsf{Mod}$ and then apply the functor D to put them back in $R-\mathsf{DMod}$. In particular, for any morphism $f : M \to N$ in $R-\mathsf{DMod}$, $\mathsf{Coker}(f) = \mathsf{Coker}'(f)$ and the induced morphism $\mathsf{coker}(f) : N \to \mathsf{Coker}(f)$ equals $\mathsf{coker}'(f)$, $\mathsf{Ker}'(f) = \mathsf{D}(\mathsf{Ker}(f))$ and $\mathsf{ker}'(f) : \mathsf{Ker}'(f) \to M$ is the composition $\nu_{\mathsf{Ker}(f)}\mathsf{ker}(f)$. (If we do not indicate anything, the constructions are made in $A-\mathsf{Mod}$ and we use the symbol $'$ to indicate that the constructions are made inside the subcategory $R-\mathsf{DMod}$).

https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

For any morphism $f : M \to N$ in $R-\mathsf{DMod}$ we can make the following decomposition:



The composition $\mathsf{ker}'(f)f = 0$, so we can find a unique h such that the upper triangle commutes. The composition $\mathsf{coker}'(\mathsf{ker}'(f))h\mathsf{coker}'(f) = f\mathsf{coker}'(f) = 0$ and $\mathsf{coker}'(\mathsf{ker}'(f)) = \mathsf{coker}(\mathsf{ker}'(f))$ is surjective, therefore $h\mathsf{coker}'(f) = 0$ and then we can find a unique morphism $\mathsf{res}(f)$ such that the lower triangle commutes. We call the morphism $\mathsf{res}(f)$ that appears in this diagram the residue of f. In Abelian categories, it is always an isomorphism.

PROPOSITION 2. *Let $f : M \to N$ be a morphism in $R-\mathsf{DMod}$, then*

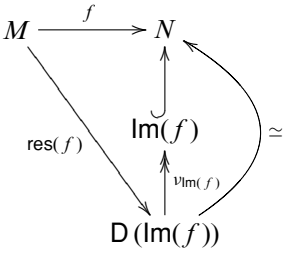
(1) *f is a monomorphism if and only if $\mathsf{Ker}(f)$ is vanishing.*
(2) *$\mathsf{res}(f)$ is always a monomorphism.*
(3) *If f is a monomorphism, then f is a kernel if and only if $\mathsf{res}(f)$ is an isomorphism.*
(4) *If f is a monomorphism, then f is an isomorphism if and only if $\mathsf{ker}'(\mathsf{coker}'(f))$ is an isomorphsim.*

Proof.

(1) This proof is given in [**4**, Proposition 14.5].
(2) As we have mentioned above, $\mathsf{Ker}'(f) = \mathsf{D}(\mathsf{Ker}(f))$ and the morphism $\mathsf{ker}'(f) = \nu_{\mathsf{Ker}(f)}\mathsf{ker}(f)$, but $\mathsf{Im}(\nu_{\mathsf{Ker}(f)}) = \mathsf{U}(\mathsf{Ker}(f))$, therefore

$$\mathsf{Coker}'(\mathsf{ker}'(f)) = \mathsf{Coker}(\mathsf{ker}'(f)) = M/\mathsf{Im}(\mathsf{ker}'(f)) = M/\mathsf{U}(\mathsf{Ker}(f))$$

The morphism $h : M/\mathsf{U}(\mathsf{Ker}(f)) \to N$ has kernel $\mathsf{Ker}(f)/\mathsf{U}(\mathsf{Ker}(f))$ that is vanishing, therefore h is a monomorphism.
(3) If f is a monomorphism, $\mathsf{Ker}'(f) = 0$ and $\mathsf{Coker}'(\mathsf{ker}'(f)) = M$. In this situation $f = h$. It is proved in [**3**, Proposition 2] that a morphism is a kernel if and only if it is the kernel of its cokernel, that is $h = f = \mathsf{ker}'(\mathsf{coker}'(f))$ and this is equivalent to $\mathsf{res}(f)$ isomorphism.
(4) If f is an isomorphism, $\mathsf{coker}'(f) = 0$ and $\mathsf{ker}'(0)$ is clearly an isomorphism. Conversely, suppose f is a monomorphism such that $\mathsf{ker}'(\mathsf{coker}'(f))$ is an isomorphism. If we compute explicitly $\mathsf{ker}'(\mathsf{coker}'(f))$ we have that $\mathsf{coker}'(f) = \mathsf{coker}(f) : N \to N/\mathsf{Im}(f)$ and $\mathsf{Ker}'(\mathsf{coker}'(f) = \mathsf{D}(\mathsf{Im}(f))$. We get the following diagram (in $A-\mathsf{Mod}$):



https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

The morphism $\nu_{\mathsf{Im}(f)}$ is surjective because $\mathsf{Im}(f)$ is unitary (it is a quotient module of M) and $\mathsf{Im}(\nu_{\mathsf{Im}(f)}) = \mathsf{U}(\mathsf{Im}(f)) = \mathsf{Im}(f)$. In $A-\mathsf{Mod}$, an epi+mono decomposition of an isomorphism should be trivial, therefore $\mathsf{D}(\mathsf{Im}(f)) = \mathsf{Im}(f) = N$. This proves that f is a monomorphism and an epimorphism in $R-\mathsf{DMod}$. Then using [**4**, Proposition 14.7] we get that f is an isomorphism. □

Using residues we can characterize the Abelianness of the category of firm modules.

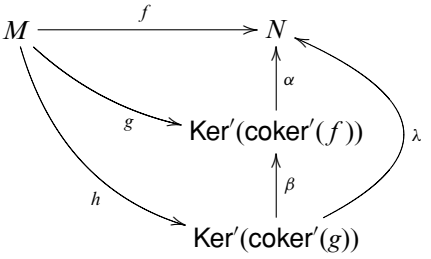
THEOREM 3. *The following conditions are equivalent:*

(1) *The category* $R-\mathsf{DMod}$ *is Abelian.*
(2) *For any morphism* $f : M \to N$ *in* $R-\mathsf{DMod}$, $\mathsf{res}(f)$ *is an isomorphism.*
(3) *For any monomorphism* $f : M \to N$ *in* $R-\mathsf{DMod}$, $\mathsf{res}(f)$ *is an isomorphism.*
(4) *The composition of two kernels in* $R-\mathsf{DMod}$ *is a kernel.*

Proof. Conditions ($1 \Rightarrow 2 \Rightarrow 3$) are trivial. If condition (3) holds, monomorphisms are kernels in $R-\mathsf{DMod}$, i.e. the category $R-\mathsf{DMod}$ is normal. This condition together with all the other ones given in [**4**, Proposition 14] prove that $R-\mathsf{DMod}$ is Abelian.

If condition (3) holds, monomorphisms and kernels are the same thing, therefore condition (4) is trivial because the composition of monomorphisms is always a monomorphism in all categories.

The only non-trivial part of the proof is ($4 \Rightarrow 3$). Let $g = \mathsf{res}(f)$ and $h = \mathsf{res}(g)$ and consider the following diagram:



The morphisms α and β are kernels, so using (4) we get that λ is a kernel, but using [**3**, Proposition 2], λ should be the kernel of $N \to N/\mathsf{Im}(\lambda)$. Applying the same proposition, we get that α should be the kernel of $N \to N/\mathsf{Im}(\alpha)$. Using the definition of α, we know that $\mathsf{Im}(\alpha) = \mathsf{U}(\mathsf{Im}(f)) = \mathsf{Im}(f)$. On the other hand $f = h\lambda$, therefore $\mathsf{Im}(f) \subseteq \mathsf{Im}(\lambda)$ and $\lambda = \beta\alpha$, therefore $\mathsf{Im}(\lambda) \subseteq \mathsf{Im}(\alpha) = \mathsf{Im}(f)$. This proves that $\mathsf{Im}(\lambda) = \mathsf{Im}(f) = \mathsf{Im}(\alpha)$ and therefore λ and α are the kernels of the same morphism. The uniqueness of the kernel proves that β should be an isomorphism and then g is an isomorphism (because of Proposition 2(4)). If g is an isomorphism, then $f = \alpha$ is a kernel. □

This property shows that in the general case in which the category of firm modules need not be Abelian, kernels do not behave very well. In the next section we are going to consider the subobjects based on monomorphisms. These subobjects form a modular lattices, so the behaviour is very similar to the case of unitary modules over unital rings.

https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

3. The lattice of subobjects is modular. Although the category of firm modules is not always Abelian, we are going to use the usual definition for subobjects in Abelian categories.

DEFINITION 4. Let N be a firm module and $l : L \to N$ and $m : M \to N$ be monomorphisms in $R-\mathsf{DMod}$. We say that l and m are equivalent if there exists an isomorphism $\alpha : L \to M$ such that $\alpha m = l$. This is an equivalence relation and an equivalence class is called a subobject of N. The family of subobjects of N is denoted $\mathcal{S}(N)$.

We can define an order relation in $\mathcal{S}(N)$ as follows: If $l : L \to N$ and $m : M \to N$ represent two subobjects of N, we say that the class of l is less or equal to the class of m if there exists a morphism $\alpha : L \to M$ such that $\alpha m = l$. It is straightforward to prove that this definition does not depend upon the election of the representatives and that α is a monomorphism.

In order to prove that this order relation defines a lattice, we have to prove the existence of the operators $\wedge$ and $\vee$.

PROPOSITION 5. *Let N be a firm module and $l : L \to N$, $m : M \to N$ represent two subobjects.*

(1) *If we define $f : L \coprod M \to N$ given by $(u, v)f = (u)l + (v)m$, then $\mathsf{Coker}'(\mathsf{ker}'(f)) = L \vee M$.*
(2) *If we consider the pullback diagram (in $R-\mathsf{DMod}$)*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & L \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle l} \\ M & \xrightarrow{m} & N \end{array}$$

then $P = L \wedge M$.
(3) *If we define $g : L \coprod M \to N$ given by $(u, v)g = (u)l - (v)m$, then $L \wedge M = \mathsf{Ker}'(g)$.*

Proof.

(1) In the proof of Proposition 2.2 we have proved that $\mathsf{Coker}'(\mathsf{ker}'(f))$ is the monomorphism $h : L \coprod M/\mathsf{U}(\mathsf{Ker}(f)) \to N$ induced by f, therefore h defines a subobject of N. Furthermore we can define $\alpha : L \to L \coprod M/\mathsf{U}(\mathsf{Ker}(f))$ by $(l)\alpha = (l, 0) + \mathsf{U}(\mathsf{Ker}(f))$ and $\beta : M \to L \coprod M/\mathsf{U}(\mathsf{Ker}(f))$ by $(m)\beta = (0, m) + \mathsf{U}(\mathsf{Ker}(f))$. These morphisms satisfy $\alpha h = l$ and $\beta h = m$. This proves that $\mathsf{Coker}'(\mathsf{ker}'(f))$ is equal to or bigger than L and M. Suppose now that $k : K \to N$ is a subobject of N equal to or bigger than N and M, then we can find morphisms $\overline{\alpha} : L \to K$ and $\overline{\beta} : M \to K$ such that $\overline{\alpha}k = l$ and $\overline{\beta}k = m$. Using these properties, we can define $g : L \coprod M \to K$ by $(l, m)g = (l)\overline{\alpha} + (m)\overline{\beta}$ and we have $gk = f$. The unitary submodule $(\mathsf{U}(\mathsf{Ker}(f)))g \subseteq K$ satisfies $(\mathsf{U}(\mathsf{Ker}(f)))gk = (\mathsf{U}(\mathsf{Ker}(f)))f = 0$, therefore $(\mathsf{U}(\mathsf{Ker}(f)))g \subseteq \mathsf{U}(\mathsf{Ker}(k))$, but k is a monomorphism, therefore $\mathsf{U}(\mathsf{Ker}(k)) = 0$ and then $(\mathsf{U}(\mathsf{Ker}(f)))g = 0$. So we can factor g through $\mathsf{U}(\mathsf{Ker}(f))$ and define $\overline{g} : L \coprod M/\mathsf{U}(\mathsf{Ker}(f)) \to K$ with $\overline{g}k = h$. This proves that K is equal to or bigger than $\mathsf{Coker}'(\mathsf{ker}'(f))$.
(2) First of all we are going to prove that $p : P \to N$ given by $p = \alpha l = \beta m$ is a monomorphism. Let $k : K \to P$ be such that $kp = 0$, then $k\alpha l = 0l = 0$ and

https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

$k\beta m = 0m = 0$, therefore using the uniqueness of the morphism $K \to P$ in the pullback diagram, we get $k = 0$.

On the other hand, suppose $w : W \to N$ is a subobject of N that is smaller than or equal to L and M, then we can find morphisms $\overline{\alpha} : W \to L$ and $\overline{\beta} : W \to M$ such that $\overline{\alpha} l = \overline{\beta} m = w$, so using the pullback structure, we can find $\gamma : W \to P$ such that $\gamma\alpha = \overline{\alpha}$ and $\gamma\beta = \overline{\beta}$. This proves that $W \leq P$.

(3) The pullback in $A-\mathsf{Mod}$ of l and m is precisely $\mathsf{Ker}(g)$, so bearing in mind that limits in $R-\mathsf{DMod}$ are computed by applying D to the limit computed in $A-\mathsf{Mod}$, the pullback of l and m in $R-\mathsf{DMod}$ is $\mathsf{D}(\mathsf{Ker}(g)) = \mathsf{Ker}'(g)$.

□

THEOREM 6. *The lattice $\mathcal{S}(N)$ is a modular lattice.*

Proof. Let L, M_1 and M_2 be subobjects of N such that $M_1 \leq M_2$, we have to prove that $(L \vee M_1) \wedge M_2 = (L \wedge M_2) \vee M_1$.

Is is clear that $L \wedge M_2 \leq (L \vee M_1) \wedge M_2$ and $M_1 \leq (L \vee M_1) \wedge M_2$ then $(L \wedge M_2) \vee M_1 \leq (L \vee M_1) \wedge M_2$ and we have a monomorphism $\gamma : (L \wedge M_2) \vee M_1 \to (L \vee M_1) \wedge M_2$. The problem is to prove that γ is in fact an isomorphism, or using that the category $R-\mathsf{DMod}$ is balanced (i.e. if a morphism is mono and epi then it is an isomorphism, see [**4**, Proposition 14.7]), we only have to prove that γ is surjective. In order to prove that, we are going to see that for every unitary support σ and any morphism $h : \langle\!\langle \sigma \rangle\!\rangle \to (L \vee M_1) \wedge M_2$ exists $\tau \supseteq \sigma$ and $g : \langle\!\langle \tau \rangle\!\rangle \to (L \wedge M_2) \vee M_1$ such that the following diagram is commutative

$$\begin{array}{ccc} (L \wedge M_2) \vee M_1 & \xrightarrow{\gamma} & (L \vee M_1) \wedge M_2 \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ \langle\!\langle \tau \rangle\!\rangle & \xrightarrow{\Phi_{\tau\sigma}} & \langle\!\langle \sigma \rangle\!\rangle \end{array}$$

This would prove that γ is surjective because for every $w \in (L \vee M_1) \wedge M_2$ we can find h such that $(\langle 1 \rangle_\sigma)h = w$ (see [**4**, Proposition 9]) and therefore $((\langle 1 \rangle_\tau)g)\gamma = ((\langle 1 \rangle_\tau)\Phi_{\tau\sigma})h = (\langle 1 \rangle_\sigma)h = w$.

Let $\lambda : L \to N$, $\nu : M_2 \to N$ and $\alpha : M_1 \to M_2$ be the monomorphisms that define the subobjects L and M_2 and the relation between M_1 and M_2. The monomorphism that defines the subobject M_1 is $\alpha\nu$.

In the coproduct $L \coprod M_1$ we will define $p_L, p_{M_1}, q_L, q_{M_1}$ the canonical projections and injections. Using Proposition 5.1, we can make the following decomposition

$$\begin{array}{ccc} L \coprod M_1 & \xrightarrow{p_L\lambda + p_{M_1}\alpha\nu} & N \\ & {\scriptstyle \epsilon}\searrow \quad \nearrow{\scriptstyle \mu} & \\ & L \vee M_1 & \end{array} \tag{1}$$

with ϵ an epimorphism (it is a cokernel) and μ a monomorphism.

https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

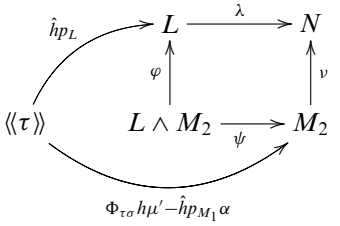
Using now Proposition 5.2 to build $(L \vee M_1) \wedge M_2$ we make the pullback diagram

$$\begin{array}{ccc} L \vee M_1 & \xrightarrow{\mu} & N \\ {\scriptstyle \nu'}\uparrow & & \uparrow{\scriptstyle \nu} \\ (L \vee M_1) \wedge M_2 & \xrightarrow{\mu'} & M_2 \end{array} \tag{2}$$

Let $h : \langle\!\langle \sigma \rangle\!\rangle \to (L \vee M_1) \wedge M_2$ be a morphism. The morphism ϵ is an epimorphism between firm modules, therefore it has a unitary kernel (see [**4**, Proposition 10]), so we can apply [**3**, Lemma 10] to $h\nu' : \langle\!\langle \sigma \rangle\!\rangle \to L \vee M_1$, and then we can find $\tau \geq \sigma$ and $\hat{h} : \langle\!\langle \tau \rangle\!\rangle \to L \coprod M_1$ making the following diagram commutative:

$$\begin{array}{ccc} L \coprod M_1 & \xrightarrow{\epsilon} & L \vee M_1 \\ {\scriptstyle \hat{h}}\uparrow & & \uparrow{\scriptstyle h\nu'} \\ \langle\!\langle \tau \rangle\!\rangle & \xrightarrow{\Phi_{\tau\sigma}} & \langle\!\langle \sigma \rangle\!\rangle \end{array} \tag{3}$$

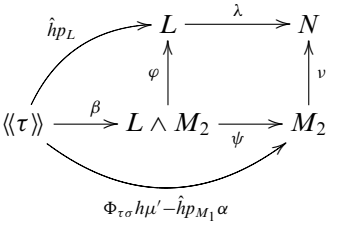
Consider now the pullback diagram that defines $L \wedge M_2$ in Proposition 5.2 and the morphisms $\langle\!\langle \tau \rangle\!\rangle \to L$ and $\langle\!\langle \tau \rangle\!\rangle \to M_2$



The outer square is commutative, because

$$\begin{gathered} (\Phi_{\tau\sigma} h\mu' - \hat{h}p_{M_1}\alpha)\nu = \Phi_{\tau\sigma} h\mu'\nu - \hat{h}p_{M_1}\alpha =^2 \\ \Phi_{\tau\sigma} h\nu'\mu - \hat{h}p_{M_1}\alpha =^1 \Phi_{\tau\sigma} h\nu'\mu - \hat{h}(\epsilon\mu - p_L\lambda) =^3 \\ \hat{h}\epsilon\mu - \hat{h}\epsilon\mu + \hat{h}p_L\lambda = \hat{h}p_L\lambda. \end{gathered}$$

Then we can apply the pullback property to define a unique morphism $\beta : \langle\!\langle \tau \rangle\!\rangle \to L \vee M_2$, making the following diagram commutative:



(4)

https://doi.org/10.1017/S0017089510000224 Published online by Cambridge University Press

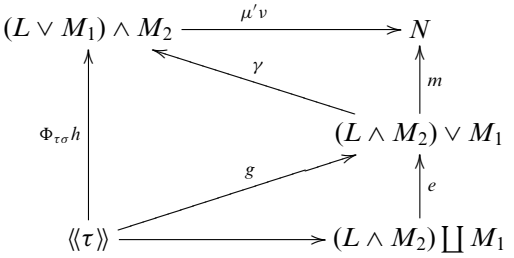
Consider now the relations that define $(L \wedge M_2) \vee M_1$. To do so, we have to construct the coproduct $(L \wedge M_2) \coprod M_1$ with the canonical injections $\iota_{L\wedge M_2}, \iota_{M_1}$ and projections $\pi_{L\wedge M_2}, \pi_{M_1}$. We apply Proposition 5.2 again and find an epimorphism e and a monomorphism m such that the following relation holds:

$$\begin{array}{ccccc} & & \overset{\pi_{L\wedge M_2}\varphi\lambda+\pi_{M_1}\alpha\nu}{\frown} & & \\ (L \wedge M_2) \coprod M_1 & \xrightarrow{e} & (L \wedge M_2) \vee M_1 & \xrightarrow{m} & M \end{array} \tag{5}$$

The morphism we are going to define from $\langle\!\langle \tau \rangle\!\rangle$ to $(L \wedge M_2) \vee M_1$ is precisely

$$g = (\beta\iota_{L\wedge M_2} + \hat{h}p_{M_1}\iota_{M_1})e$$

In order to check that this morphism satisfies the conditions we are looking for, we have to give the precise definition of $\gamma : (L \wedge M_2) \vee M_1 \to (L \vee M_1) \wedge M_2$. The morphism γ is the one that composed with the monomorphism $\mu'\nu : (L \vee M_1) \wedge M_2 \to N$ gives us the monomorphism $m : (L \wedge M_2) \vee M_1 \to N$, so we have the following diagram:



If we prove that the outer rectangle is commutative and bearing in mind that $\mu'\nu$ is a monomorphism, we have the result.

$$\begin{gathered}(\beta\iota_{L\wedge M_2} + \hat{h}p_{M_1}\iota_{M_1})em =^5 (\beta\iota_{L\wedge M_2} + \hat{h}p_{M_1}\iota_{M_1})(\pi_{L\wedge M_2}\varphi\lambda + \pi_{M_1}\alpha\nu) = \\ \beta\varphi\lambda + \hat{h}p_{M_1}\alpha\nu =^4 \hat{h}p_L\lambda + \hat{h}p_{M_1}\alpha\nu =^4 \\ \Phi_{\tau\sigma}h\mu'\nu - \hat{h}p_{M_1}\alpha\nu + \hat{h}p_{M_1}\alpha\nu = \Phi_{\tau\sigma}h\mu'\nu.\end{gathered}$$

□

In the previous proof, it is important to note that we are not, in general, in an Abelian category and also that we could not have a projective generator, both conditions would have made the proof simpler. Instead of these properties, in this category we use [**3**, Lemma 10], which is very helpful for firm modules.

ACKNOWLEDGEMENTS. The second author is partially suported by 'Fundación Séneca' (CARM) and the Spanish 'Ministerio de Educación y Ciencia'.

REFERENCES

1. D. Faith, Algebra I. Rings, modules and categories, in *Die Grundlehren der mathematischen Wissenschaften*, vol. 190. (Springer-Verlag, Berlin, Heidelberg, New York, 1973).

2. J. L. García and L. Marín, Morita theory for associative rings. *Commun. Algebra* **29**(12), (2001), 5735–5856.

3. J. González-Férez and L. Marín, The category of firm modules need not be abelian. *J. Algebra* **318**(1) (2007), 377–392.

4. L. Marín, The construction of a generator for *R*-DMod. in *Lecture notes in pure and applied mathematics* vol. 210 (New York, Marcel-Dekker, 2000), 287–296.

5. D. G. Quillen, Module theory over nonunital rings. 1997 unpublished notes.