DIFFERENTIAL EQUATIONS OF NON-INTEGER ORDER

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Introduction. In §1, we define a differential-integral operator, which for positive real indices is commonly known as the Liouville-Riemann generalized integral. For positive integer indices, we obtain an iterated integral. For negative real indices we obtain the Riemann-Holmgren **(5; 9)** generalized derivative, which for negative integer indices gives the ordinary derivative of order corresponding to the negative of such an integer. Following M. Riesz **(10)** we extend these ideas to include complex indices. An equation involving this operator where the real part of the index is negative will be called a differential equation of non-integer order. It is to be noted that the distinction between a differential equation and an integral equation disappears when the index is not an integer, although for rational indices such an equation may be transformed into an ordinary differential equation. The Riemann-Holmgren form of the definition itself involves both differentiation and integration of the ordinary kind and contributes to the breaking down of this distinction.

A classical example of a differential equation of non-integer order is the inverse of the Abel integral equation **(1**, p. 8**)**; that is, consider the solution as the "differential" equation, then the integral equation becomes the solution of this equation. An example of such an equation was discussed by Post **(8)** and Davis **(3)** related such equations to Volterra integral equations. In this paper we are concerned with equations of irrational and even complex order. For the fundamental equation (A) of §2 we note that the only solution which is continuous at the "lower limit" a of our operator is the trivial one and we find that it is of interest to allow a singularity at a. In §2 we show that for the index α real and between 0 and 1 the solutions of (A) have many of the same properties as e^{-x}, which is the principal solution for $\alpha = 1$. In §4 we use properties established in §2 to add to the discussions by Mittag-Leffler **(7)** and Wiman **(12)** on the behavior of the complex entire function $E_\alpha(z)$ for $0 < \alpha < 1$ on the real negative z-axis. Then for $1 < \alpha$ we apply theorems of Mittag-Leffler and Wiman to establish the behavior of our solutions for this range of the index α.

The operator with non-integer positive real indices makes its appearance in solutions of partial differential equations, for example, the Euler-Poisson equation **(2**, p. 54**)** which plays an important role in the theory of partial differential equations of mixed types as developed by Tricomi **(11)**. This

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operator is the one-dimensional case of the n-dimensional operator of M. Riesz **(10)**. Because of this and since Holmgren **(5)** published the idea of generalized differentiation before Riemann's work **(9**, pp. 331-344**)** appeared in print we call this operator the one-dimensional Holmgren-Riesz Transform.

1. Properties of the Holmgren-Riesz Transform. Let a and b be real numbers, $a < b$; $L(a, b)$ be the class of all complex functions of a real variable x which are summable (Lebesgue) on $a \leqslant x \leqslant b$; $\alpha = \alpha_1 + i\alpha_2$ be a complex number; the real part $\mathrm{R}\alpha = \alpha_1$; for A real and positive, $A^\alpha = A^{\mathrm{R}\alpha}$ $(\cos\alpha_2 \ln A + i \sin \alpha_2 \ln A)$ and $||\alpha|| = \max(|\alpha_1|, |\alpha_2|)$, i.e. $||\alpha||$ is not less than the two non-negative numbers, $|\alpha_1|, |\alpha_2|$.

If $f(t)$ is a function which is defined a.e. on $a \leqslant t \leqslant b$ then the one-dimensional Holmgren-Riesz Transform of index α will be represented by the notation: $I(\alpha; a, b \,|\, f)$.

Definition 1.1. If $0 < \mathrm{R}\alpha$, then

$$I(\alpha;\ a, b|f) = \int_a^b f(t)\ \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt,$$

provided that this integral (Lebesgue) exists.

An extension of Definition 1.1 is:

Definition 1.2. If $\mathrm{R}\alpha \leqslant 0$; n is the smallest positive integer $> -\mathrm{R}\alpha$; then

$$I(\alpha;\ a, b|f) = D_x^n\, I(n + \alpha;\ a, x|f)$$

at $x = b$, provided that $I(n + \alpha; a, x \,|\, f)$ and its first $(n - 1)$ derivatives exist in a segment, $|b - x| < h$, and the nth derivative exists at $x = b$.

Example 1.1. For complex β, $\mathrm{R}\beta > -1$ and $x > a$:

$$I\left(\alpha;\ a, x\,\middle|\, \frac{(t-a)^\beta}{\Gamma(\beta+1)}\right) = \begin{cases} \dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, & \mathrm{R}(\alpha+\beta) \neq \text{negative integer}, \\[2ex] 0 \quad , & \mathrm{R}(\alpha+\beta) = \text{negative integer}. \end{cases}$$

The following is an extension of a theorem due to Hardy **(4)** which was stated only for real numbers α and β.

THEOREM 1.1. *If* $0 < \mathrm{R}\alpha$, $\mathrm{R}\alpha \leqslant \mathrm{R}\beta$; $f(x)$ *belongs to* $L(a, b)$; *and* $I(\mathrm{R}\alpha; a, b \,|\, ||f||)$ *exists then* $I(\beta; a, b \,|\, f)$ *exists.*

Proof. Let $F(x) = \max\,[||f(x)||, ||f(x)||\ (b - x)^{\mathrm{R}\alpha-1}]$, then $F(x)$ is in $L(a, b)$. Also, $f(x)\ (b - x)^{\beta-1}$ is measurable on $a \leqslant x \leqslant b$ and the absolute value of each component is not greater than $F(x)$. Hence, $f(x)\ (b - x)^{\beta-1}$ is in $L(a, b)$.

THEOREM 1.2. *Let* $0 < \mathrm{R}\alpha$; *and* β *real;* $0 \leqslant \beta \leqslant \mathrm{R}\alpha$; $0 < M$; $a < x_0 < b$ *and* $f(x)$ *belong to* $L(a, b)$. *Then*

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(a) *If* $a < d < x_0$ *and* $||f(x)|| \leqslant M(x_0 - x)^{-\beta}$ *on* $d \leqslant x \leqslant x_0$ *then* $I(\alpha; a, x \mid f)$ *exists on* $d < x \leqslant x_0$ *and has left-hand continuity at* $x = x_0$;

(b) *If* $x_0 < d \leqslant b$; $||f(x)|| \leqslant M(x - x_0)^{-\beta}$ *for* $x_0 < x < d$ *and* $I(\mathrm{R}\alpha; a, x_0 \mid f)$ *exists, then* $I(\alpha; a, x \mid f)$ *exists on* $x_0 \leqslant x \leqslant d$ *and has right-hand continuity at* $x = x_0$.

Proof. Part (a). Note that $I(\alpha; a, x \mid f)$ exists on $d \leqslant x \leqslant x_0$ and

$$I(\alpha;\ a, x_0|f) - I(\alpha;\ a, x|f)$$
$$= \int_a^x f(t)\,dt \int_x^{x_0} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds + \int_x^{x_0} f(t)\,\frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)}\,dt$$

on $d \leqslant x < x_0$. Write the first integral:

$$\int_a^x = \int_a^d + \int_d^x$$

and it is clear that each of the three integrals on the right-hand side approaches zero as x approaches x_0. Part (b) is proved in a similar manner with the additional observation that $I(\alpha; a, x_0 \mid f)$ exists since $I(\mathrm{R}\alpha; a, x_0 \mid f)$ exists. That this does not follow from the other hypotheses of part (b) is illustrated by the following:

Example 1.2. Let $0 < \alpha < 1$ and $f(x) = (1 - x)^{-\alpha}$ for $0 \leqslant x < 1$, $f(x) = 0$ for $1 \leqslant x$. Then $f(x)$ belongs to $L(0, 1)$ and $I(\alpha; 0, 1 \mid f)$ does not exist. Furthermore, for $1 < x$,

$$I(\alpha;\ 0, x|f) > -\frac{\ln(x-1)}{\Gamma(\alpha)}$$

which increases without bound as x approaches 1.

From Theorem 1.2 we have, immediately:

COROLLARY 1.2.1. *If* $\mathrm{R}\alpha > 0$ *and* $f(x)$ *is continuous on* $a \leqslant x \leqslant b$ then $I(\alpha; a, x \mid f)$ *exists and is continuous with respect to* x *on* $a \leqslant x \leqslant b$.

The next property is an extension of another theorem due to Hardy **(4)** which was stated for real summable $f(x)$, real α and $\beta = 0$. Riesz **(10)** has discussed this theorem and its corollaries for continuous $f(x)$ and complex α and β.

THEOREM 1.3. *If* $\mathrm{R}\alpha > 0$; $\mathrm{R}\beta \geqslant 0$ *and* $f(x)$ *belongs to* $L(a, b)$ *then:*

(a) $I(\alpha; a, x \mid f)$ *exists* $\begin{cases} \textit{everywhere on } a \leqslant x \leqslant b, \textit{for } \mathrm{R}\alpha \geqslant 1 \\ \quad\textit{a.e.} \quad \textit{on } a \leqslant x \leqslant b, \textit{for } \mathrm{R}\alpha < 1 \end{cases}$

(b) $I(\beta + 1; a, b \mid I(\alpha; a, t \mid f)) = I(\alpha + \beta + 1; a, b \mid f)$.

Proof. If $\mathrm{R}\alpha \geqslant 1$, then by Theorem 1.1, $I(\alpha; a, x \mid f)$ exists everywhere on $a \leqslant x \leqslant b$. For the general case, $\mathrm{R}\alpha > 0$, we will follow the argument suggested by Hardy **(4**, p. 146**)** for his restricted case. Since $f(x)$ is in $L(a, b)$ then so is

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$||f(x)||$. Let $g_n(x) = \min [||f||, n]$, $a \leqslant x \leqslant b$; $K_n(x) = \min [x^{R\alpha-1}, n]$, $0 < x \leqslant b - a$; $K_n(0) = n$. Then $g_n(t)K_n(x - t)(b - x)^{R\beta}$ is a summable function of x and t over the triangle T: $a \leqslant x \leqslant b$, $a \leqslant t \leqslant x$; since it is the product of bounded summable functions. Then, by Fubini's Theorem **(6)** we have that:

$$\begin{aligned}\int_T g_n(t)\, K_n(x-t)(b-x)^{R\beta} dT &= \int_a^b dx \int_a^x g_n(t)\, K_n(x-t)(b-x)^{R\beta} dt \\ &= \int_a^b dt \int_t^b g_n(t)\, K_n(x-t)(b-x)^{R\beta} dx \\ &\leqslant \int_a^b g_n(t)\, dt \int_t^b (x-t)^{R\alpha-1}(b-x)^{R\beta} dx \\ &= B(R\alpha, R\beta+1) \int_a^b ||f(t)||(b-t)^{R\alpha+R\beta} dt.\end{aligned}$$

Since $g_n(t)\, K_n(x - t)(b - x)^{R\beta}$ is a non-decreasing sequence of summable functions over T, then

$$\int_T ||f(t)||(x-t)^{R\alpha-1}(b-x)^{R\beta} dx dt$$

exists. Then since $f(t)\,(x - t)^{\alpha-1}\,(b - x)^{\beta}$ is a measurable function of x and t over T, each of whose components is bounded by $||f(t)||\,(x - t)^{R\alpha-1}\,(b - x)^{R\beta}$, we see that each of the following exist and

$$\begin{aligned}\int_T f(t)(x-t)^{\alpha-1}(b-x)^{\beta} dx dt &= \int_a^b dx \int_a^x f(t)(x-t)^{\alpha-1}(b-x)^{\beta} dt \\ &= \int_a^b f(t) dt \int_t^b (x-t)^{\alpha-1}(b-x)^{\beta} dx,\end{aligned}$$

from which (b) follows easily.

COROLLARY 1.3.1. *If* $R\alpha > 0$, $R\beta > 0$ *and* $f(x)$ *belongs to* $L(a, b)$ *then on* $a \leqslant x \leqslant b$:

(a) $\displaystyle\int_a^x I(\alpha;\, a, t|f)\, dt = I(\alpha + 1;\, a, x|f)$;

(b) *If* $R\alpha > 1$ or $\alpha = 1$; $I(\alpha; a, x\,|\,f)$ *is absolutely continuous in* x;

(c) $D_x I(\alpha + 1; a, x\,|\,f) = I(\alpha; a, x\,|\,f) \begin{cases} \textit{everywhere, if } R\alpha > 1, & \alpha = 1, \\ \textit{a.e.} \quad \textit{if } R\alpha \leqslant 1, & \alpha \neq 1; \end{cases}$

(d) $I(\alpha + \beta; a, x\,|\,f) = I(\beta; a, x\,|\,I(\alpha; a, t\,|\,f)) \begin{cases} \textit{everywhere, if } R(\alpha+\beta) > 1, \\ \qquad \alpha + \beta = 1, \\ \textit{a.e.} \quad \textit{if } R(\alpha+\beta) \leqslant 1, \\ \qquad \alpha + \beta \neq 1. \end{cases}$

THEOREM 1.4. *If* $f(x)$ *is absolutely continuous on* $a \leqslant x \leqslant b$ *and* $R\alpha > 0$ *then*

$$I(\alpha;\, a, x|f) = \frac{f(a)(x-a)^{\alpha}}{\Gamma(\alpha+1)} + I(\alpha + 1, a, x|f').$$

Proof. Use integration by parts **(6)**.

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COROLLARY 1.4.1. *If* $R\alpha > 0$*;* $f(x)$ *is* $L(a, b)$*, then*

$$I\left(\alpha;\ a, x \,\middle|\, \int_a^t f\right) = I(\alpha + 1;\ a, x|f).$$

COROLLARY 1.4.2. *If* $f(x)$ *is absolutely continuous on* $a \leqslant x \leqslant b$ *and* $R\alpha > 0$ *then* $I(\alpha; a, x \,|\, f)$ *is absolutely continuous on* $a \leqslant x \leqslant b$.

COROLLARY 1.4.3. *If* n *is a positive integer;* $f(x)$ *is of class* $C^{(n)}$ *on* $a < x \leqslant b$ *and belongs to* $L(a, b)$ *and* $R\alpha > 0$ *then* $I(\alpha; a, x \,|\, f)$ *is of class* $C^{(n)}$ *on* $a < x \leqslant b$ *and belongs to* $L(a, b)$.

Proof. Let $n = 1$. $I(\alpha; a, x \,|\, f)$ belongs to $L(a, b)$ from Theorem 1.3. Let $a < x_0 < b$ then

$$I(\alpha;\ a, x|f) = \int_a^{x_0} f(t)\ \frac{(x - t)^{\alpha-1}}{\Gamma(\alpha)}\, dt + \frac{f(x_0)\,(x - x_0)^\alpha}{\Gamma(\alpha + 1)} + I(\alpha + 1;\ x_0, x|f'); \qquad x > x_0.$$

Since $f'(x)$ is continuous on $x_0 \leqslant x \leqslant b$ then $I(\alpha; x_0, x \,|\, f')$ is continuous and $D_x\, I(\alpha; a, x \,|\, f)$ is continuous on $x_0 \leqslant x \leqslant b$. Since this is true for any such x_0, the conclusion follows for $n = 1$. By induction the theorem can be established for any positive integer n.

THEOREM 1.5. *If* $R\alpha > 0$ *and* $f(x)$ *is in* $L(a, b)$ *then*

$$I(-\alpha; a, x \,|\, I(\alpha; a, t \,|\, f)) = f(x), \text{ a.e. on } a \leqslant x \leqslant b.$$

Proof. Let n be the smallest integer $> R\alpha$, then applying Definition 1.2 and Corollary 1.3.1 we have:

$$D_x^n\, I(n - \alpha;\ a, x|I(\alpha;\ a, t|f)) = D_x^n\, I(n;\ a, x|f) = f(x), \text{ a.e. on } a \leqslant x \leqslant b.$$

THEOREM 1.6. *If* $R\alpha > 0$*;* n *is the smallest integer* $> R\alpha$*;* $f(x)$ *is in* $L(a, b)$ *and* $I(1 - \alpha; a, x \,|\, f)$ *exists and is absolutely continuous on* $a \leqslant x \leqslant b$*, then* $I(i - \alpha; a, a^+ \,|\, f) = K_i$ *exists for* $i = 1, 2 \ldots n$*;* $I(-\alpha; a, x \,|\, f)$ *exists a.e. on* $a \leqslant x \leqslant b$*, is in* $L(a, b)$ *and*

$$I(\alpha;\ a, x|I(-\alpha;\ a, t|f)) = f(x) - \sum_{p=1}^{n} \frac{K_p(x - a)^{\alpha-p}}{\Gamma(\alpha - p + 1)}, \text{ a.e. on } a \leqslant x \leqslant b.$$

Furthermore, the equality holds everywhere on $a < x \leqslant b$*, if, in addition,* $f(x)$ *is continuous on* $a < x \leqslant b$.

Proof. Let $g(x) = I(-\alpha; a, x \,|\, f)$ a.e. on $a \leqslant x \leqslant b$. Since $I(1-\alpha; a, x \,|\, f)$ is absolutely continuous on $a < x \leqslant b$ then $I(1 - \alpha; a, a^+ \,|\, f)$ exists and $I(1 - \alpha; a, x \,|\, f) = K_1 + I(1, a, x \,|\, g)$ on $a < x \leqslant b$. If $n > 1$, then by continuing this process we have

$$I(n - \alpha;\ a, x|f) = \sum_{p=1}^{n} \frac{K_p(x - a)^{n-p}}{\Gamma(n - p + 1)} + I(n;\ a, x|g) \text{ on } a < x \leqslant b.$$

Then

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$$I(n;\ a, x|f) = I(\alpha;\ a, x|I(n-\alpha;\ a, t|f)) = \sum_{p=1}^{n} \frac{K_p(x-a)^{n-p+\alpha}}{\Gamma(n-p+\alpha+1)} + I(n+\alpha;\ a, x|g)$$

and

$$(x) = D_x^n\, I(n;\ a, x|f) = \sum_{p=1}^{n} \frac{K_p(x-a)^{\alpha-p}}{\Gamma(\alpha-p+1)} + I(\alpha;\ a, x|g) \text{ a.e. on } a \leqslant x \leqslant b.$$

We solve for $I(\alpha; a, x \mid g)$ and obtain the desired equality.

If $0 < \mathrm{R}\alpha < 1$ and $f'(x)$ exists on $a \leqslant x \leqslant b$ and is continuous at $x = a$ then, by Theorem 1.4 and Corollary 1.2.1, $K_1 = 0$ and

$$I(\alpha; a, x \mid I(-\alpha, a, t \mid f)) = f(x)$$

on $a < x \leqslant b$. For $K_1 \neq 0$, note the following:

Example 1.3. Let $0 < \mathrm{R}\alpha < 1$ and

$$f(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \qquad x > a,$$

then $K_1 = 1$ and

$$I(\alpha;\ a, x|I(-\alpha;\ a, t|f)) = f(x) - K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} = 0, \qquad x > a.$$

THEOREM 1.7. *If* $\mathrm{R}\alpha > 0$; λ *is a complex number*; $f(x)$ *is in* $L(a, b)$; $a < x_0 < b$ *and* $I(\mathrm{R}\alpha; a, x_0 \mid f)$ *exists then*

(a) $\quad ||I(\alpha;\ a, x_0|f)|| \leqslant A \cdot I(\mathrm{R}\alpha;\ a, x_0|\ ||f||)$, *where* $A = \dfrac{\Gamma(\mathrm{R}\alpha)}{|\Gamma(\alpha)|}$;

(b) $\quad \sum_{p=0}^{\infty} \lambda^p\ I(p\alpha;\ a, x|f)$ *converges absolutely and uniformly a.e. on* $a \leqslant x \leqslant b$.

Proof. Since $f(t)\ (x_0 - t)^{\mathrm{R}\alpha-1}$ is in $L(a, x_0)$ then so are $f(t)\ (x_0 - t)^{\alpha-1}$ and $||f(t)||\ (x_0 - t)^{\mathrm{R}\alpha-1}$; hence $I(\alpha; a, x_0 \mid f)$ and $I(\mathrm{R}\alpha; a, x_0 \mid ||f||)$ exist. Furthermore, by considering separately the real and imaginary components the inequality (a) follows. For part (b) let m be the smallest positive integer such that $m\alpha > 1$. Then, $||I(m\alpha; a, x \mid f)||$ is continuous and, hence, bounded (say by M) on $a \leqslant x \leqslant b$ and for $n > m$ we have that

$$I(n\alpha; a, x \mid f) = I(\overline{n-m}\alpha; a, x \mid I(m\alpha; a, t \mid f))$$

and

$$||I(n\alpha;\ a, x|f)|| \leqslant A^{n-m} \cdot I(\mathrm{R}(n-m)\alpha;\ a, x|M) = \frac{MA^{n-m}(x-a)^{\mathrm{R}(n-m)\alpha}}{\Gamma(\mathrm{R}(n-m)\alpha+1)}.$$

The conclusions of part (b) follow easily with the use of the following inequality:

LEMMA. *If* X *and* β *are real, positive numbers, then* $\Gamma(\beta+1) > X^\beta e^{-X}$.

Proof.

$$\Gamma(\beta+1) = \int_0^\infty e^{-x} x^\beta\, dx > \int_X^\infty e^{-x} x^\beta\, dx > X^\beta e^{-X}.$$

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This list of properties of the Transform will be concluded with the following theorem which makes use of the discussion of modes of convergence by McShane (**6**, pp. 160-168).

THEOREM 1.8. *If* $\mathrm{R}\alpha > 0$; $f_1(x), f_2(x), \ldots$ *is a sequence of functions in* $L(a, b)$ *which converges almost everywhere on* $a \leqslant x \leqslant b$ *to a function* $f(x)$ *in* $L(a, b)$ *and there exists a non-negative real function* $g(x)$ *in* $L(a, b)$ *such that* $||f_n(x)|| \leqslant g(x)$ *for all n and all x on* $a \leqslant x \leqslant b$ *then*

$$\lim_{n\to\infty} I(\alpha;\ a, x|f_n) = I(\alpha;\ a, x|f)$$

almost everywhere on $a \leqslant x \leqslant b$ *and, hence, almost uniformly.*

Proof. Let x be a number such that $a < x < b$ and $I(\mathrm{R}\alpha; a, x \mid g)$ exists. Let $h_n(t)$ be one of the (real or imaginary) components of $f_n(t)(x-t)^{\alpha-1}$ and $h(t)$ be the corresponding component of $f(t)(x-t)^{\alpha-1}$ on $a \leqslant t < x$. Then

$$|h_n(t)| \leqslant 2g(t)\,(x-t)^{\mathrm{R}\alpha-1}$$

and since $h_n(t)$ is measurable and $I(\mathrm{R}\alpha; a, x \mid g)$ exists then $\int_a^x h_n(t)\,dt$ exists and, similarly, $\int_a^x h(t)\,dt$ exists. Furthermore $h_n(t)$ converges to $h(t)$ almost everywhere on $a \leqslant t < x$ and, hence (**6**, p. 168),

$$\lim_{n\to\infty} \int_a^x h_n(t)\,dt = \int_a^x h(t)$$

and it follows that

$$\lim_{n\to\infty} I(\alpha;\ a, x|f_n) = I(a;\ a, x|f).$$

Since the above discussion applies to the interval $a \leqslant x \leqslant b$ except at most a subset of measure zero, the convergence holds almost everywhere. Finally, the transforms are all in $L(a, b)$ which ensures that the convergence is almost uniform on $a \leqslant x \leqslant b$ (**6**, p. 164).

2. Linear differential equations of non-integer order. We shall be concerned with the following linear integral-differential equation for $\mathrm{R}\alpha > 0$, any complex number λ and $h(x)$ in $L(a, b)$:

(A) $$I(-\alpha; a, x|y) + \lambda y = h(x).$$

Because of Theorem 1.6 on inverse operations, we shall impose boundary conditions of the type:

(B) $I(i-\alpha; a, a^+|y) = K_i$; $\quad i = 1,2,\ldots,n$; where $n-1 \leqslant \mathrm{R}\alpha < n$.

Definition 2.1. A function $f(x)$ is said to be an *L-solution* of (A) provided that it belongs to $L(a, b)$; $I(1-\alpha; a, x|f)$ exists and is absolutely continuous on $a < x \leqslant b$ and equation (A) is satisfied by $y = f(x)$ a.e. on $a \leqslant x \leqslant b$. $f(x)$ is said to be a *unique* solution of (A) and (B) provided that any other solution $g(x)$ differs from $f(x)$ only on a null sub-set of $a \leqslant x \leqslant b$.

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Definition 2.2. A function $f(x)$ is said to be an *R-solution* of (A) provided that it is an L-solution which satisfies (A) on $a < x \leqslant b$.

Suppose that $\lambda, K_1, K_2, \ldots K_n, \alpha$ are complex numbers; $h(x)$ is in $L(a, b)$; n is the positive integer such that $n - 1 \leqslant R\alpha < n$; $\alpha \neq n - 1$ and $f(x)$ is an L-solution of (A) and (B). Then by Theorem 1.6:

$$f(x) = \sum_{p=1}^{n} \frac{K_p(x-a)^{\alpha-p}}{\Gamma(\alpha - p + 1)} + I(\alpha;\ a, x|h) - \lambda I(\alpha;\ a, x|f).$$

By successive substitutions it follows that for any positive integer m and a.e. on $a \leqslant x \leqslant b$:

$$f(x) = \sum_{q=1}^{m} \sum_{p=1}^{n} (-\lambda)^{q-1} \frac{K_p(x-a)^{q\alpha-p}}{\Gamma(q\alpha - p + 1)} + \sum_{q=1}^{m} (-\lambda)^{q-1} I(q\alpha;\ a, x|h) + (-\lambda)^m I(m\alpha;\ a, x|f).$$

Then, using Theorem 1.7 we have $f(x) = g(x)$ a.e. on $a \leqslant x \leqslant b$ where

$$g(x) = \sum_{p=1}^{n} K_p \sum_{q=1}^{\infty} (-\lambda)^{q-1} \frac{(x-a)^{q\alpha-p}}{\Gamma(q\alpha - p + 1)} + \sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\ a, x|h), \quad \text{on } a \leqslant x \leqslant b.$$

This establishes that if there is an L-solution it must be equal to $g(x)$ a.e. on $a < x \leqslant b$. Therefore all that remains to be done in order to show that there is a unique L-solution is to show that $g(x)$ is an L-solution.

Each of these series converges uniformly and absolutely a.e. on $a \leqslant x \leqslant b$, for all values of λ, which allows the interchange of order of the operations which follow.

By use of Theorems 1.5, 1.8 and Example 1.1 we have: $g(x)$ is in $L(a, b)$ and

$$I(-\alpha;\ a, x|g) = \sum_{q=2}^{\infty} \sum_{p=1}^{n} (-\lambda)^{q-1} \frac{K_p(x-a)^{(q-1)\alpha-p}}{\Gamma((q-1)\alpha - p + 1)} + h(x) - \lambda \sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\ a, x|h),$$

which reduces to $I(-\alpha; a, x|g) = h(x) - g(x)$, a.e. on $a \leqslant x \leqslant b$. Furthermore by computing $I(i - \alpha; a, x|g)$, we see that

$$I(i - \alpha; a, a^+|g) = K_i, \qquad i = 1, 2, \ldots, n.$$

Thus, $g(x)$ is a unique L-solution of (A) and (B).

Let

$$U_p(x;\ \lambda) = \sum_{q=1}^{\infty} \frac{(-\lambda)^{q-1} x^{q\alpha-p}}{\Gamma(q\alpha - p + 1)}$$

for $x > 0$. Then $U_p(x - a; \lambda)$ is an R-solution of (A) and (B) where $K_i = 1$ for $i = p$ and $K_i = 0$ for $i \neq p$. Furthermore

$$\sum_{q=1}^{\infty} (-\lambda)^{q-1} I(q\alpha;\ a, x|h) = \int_a^x h(t)\, U_1(x - t;\ \lambda)\, dt.$$

If $\alpha = n - 1$, an integer, the case is that of ordinary linear differential equations.

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THEOREM 2.1. *If* $R\alpha > 0$; *n is the smallest positive integer* $> R\alpha$; λ *is a complex number;* $K_1, K_2, \ldots, K_n$ *is a complex number sequence; and* $h(x)$ *is in* $L(a, b)$ *then*

$$f(x) = \sum_{p=1}^{n} K_p\, U_p(x - a;\ \lambda) + \int_a^x h(t)\, U_1(x - t;\ \lambda)\, dt$$

is the unique L-solution of (A) *and* (B) *on* $a \leqslant x \leqslant b$.

COROLLARY 2.1.1. *If, in addition to the hypothesis of Theorem* 2.1, $h(x)$ *is continuous on* $a < x \leqslant b$ *then* $f(x)$ *is a unique R-solution of* (A) *and* (B) *on* $a < x \leqslant b$.

3. Behavior of solutions of the homogeneous equation where $0 < \alpha < 1$. Let α be a real number between 0 and 1 and let $Y(x)$ be the unique R-solution of (A) and (B) on $a < x \leqslant b$ where $K_1 = 1$, $\lambda = 1$, and $a = 0$ and $h(x) = 0$ for $x > 0$:

$$Y(x) = U_1(x;\ 1) = \sum_{q=1}^{\infty} \frac{(-1)^{q-1} x^{q\alpha-1}}{\Gamma(q^\alpha)}, \qquad x > 0.$$

Since $I(1 - \alpha; 0, 0^+ | Y) = 1$; it is clear that for some $\bar{x} > 0$: $Y(x) > 0$ on $0 < x < \bar{x}$. Suppose that $Y(x)$ has a zero on $0 < x < b$ and let x_0 be the smallest such zero.

Then $Y(x) > 0$ for $0 < x < x_0$ and $Y(x_0) = 0$. Recall that

$$D_x\, I(1 - \alpha; 0, x | Y) + Y(x) = 0,$$

Then,

$$I(1 - \alpha; 0, x | Y) + I(1, 0, x | Y) = 1,$$

$$I(1 - \alpha;\ x_0, x_0^{+} | Y) + I(1, 0, x | Y) = 1 - \int_0^{x_0} Y(t)\, \frac{(x - t)^{-\alpha}}{\Gamma(1 - \alpha)}\, dt, \qquad x > x_0,$$

and $I(1 - \alpha; x_0, x_0^{+} | Y) = 0$, since $Y(x)$ is continuous at $x = x_0$. Let $0 < d < x_0$, then since $Y(x_0) = 0$ we have the integration by parts:

$$h(x) = -\int_0^d \frac{Y(t)(x - t)^{-\alpha-1}}{\Gamma(-\alpha)} dt - \frac{Y(d)(x - d)^{-\alpha}}{\Gamma(1 - \alpha)} - \int_d^{x_0} \frac{Y'(t)(x - t)^{-\alpha}}{\Gamma(1 - \alpha)} dt, \quad x > x_0,$$

and, hence, $h(x)$ is continuous on $x_0 < x \leqslant b$ and is in $L(x_0, b)$.

Now, from Corollary 2.1.1 it follows that

$$Y(x) = I(1 - \alpha;\ x_0, x_0^{+} | Y) \cdot Y(x - x_0) + \int_{x_0}^x h(t)\ Y(x - t)\, dt,\ \ x_0 < x \leqslant b.$$

Note that $h(x) > 0$ for $x > x_0$ and $Y(x - t) > 0$ for $x_0 < t < x < 2x_0$. Therefore $Y(x) > 0$ for $x_0 < x < 2x_0$ and $Y'(x_0) = 0$. However,

$$Y(x_0)\, \frac{(x - x_0)^{1-\alpha}}{\Gamma(2 - \alpha)} + I(2 - \alpha;\ x_0, x | Y') + I(1;\ 0, x | Y)$$

$$= 1 - \int_0^{x_0} Y(t)\, \frac{(x - t)^{-\alpha}}{\Gamma(1 - \alpha)}\, dt, \qquad x > x_0;$$

$$I(-\alpha;\ x_0, x | Y') + Y'(x) = -\int_0^{x_0} \frac{Y(t)(x - t)^{-2-\alpha}}{\Gamma(-\alpha - 1)}\, dt = h_1(x) < 0, \qquad x > x_0,$$

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and since $Y(x_0) = Y'(x_0) = 0$, $h_1(x)$ is in $L(x_0, b)$; and since $I(1-\alpha; x_0, x_0{}^+| Y') = 0$,

$$Y'(x) = \int_{x_0}^{x} h_1(t)\, Y(x-t)\, dt < 0, \qquad x_0 < x < 2x_0,$$

which contradicts $Y(x) > 0$ for $x > x_0$ and $Y(x_0) = 0$.

These results may be summarized as follows:

THEOREM 3.1. *If $0 < \alpha < 1$; $K_1 = 1$; $\lambda = 1$ and $a = 0$ then the unique R-solution of (A) and (B),*

$$Y(x) = \sum_{q=1}^{\infty} \frac{(-1)^{q-1} x^{q\alpha - 1}}{\Gamma(q\alpha)},$$

is positive for all $x > 0$.

COROLLARY 3.1.1. *If $0 < \alpha < 1$ then $U_1(x-a; 1) = Y(x-a) > 0$ for $x > a$, and any R-solution of (A) for $h(x) = 0$ has a zero on $x > a$ only if it is identically zero for $x > a$.*

We notice that if $\alpha = 1$, the corresponding solution is e^{-x} which is positive for $x > 0$. Also, $Y(x)$ satisfies the properties satisfied by $e^{-x} (\alpha = 1)$ given by the following theorem.

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, we have*

(a) $\lim_{x\to\infty} Y(x) = 0,$

(b) $\int_{x_0}^{\infty} Y(t)\, dt = I(1-\alpha, 0, x_0| Y)$, *for* $x_0 > 0$, *and* $\int_0^{\infty} Y = 1,$

(c) if $0 < \beta < 1$, $\lim_{x\to\infty} I(\beta, 0, x| Y) = 0.$

Proof. Recall that $Y(x) + I(\alpha; 0, x| Y) = x^{\alpha-1}/\Gamma(\alpha)$ for $x > 0$. Since $0 < Y(x) < x^{\alpha-1}/\Gamma(\alpha)$, then part (a) follows and $I(\alpha; 0, x| Y) \to 0$ as $x \to \infty$. Now $D_x I(1-\alpha; 0, x| Y) = -Y(x) < 0$ and $I(1-\alpha; 0, x| Y) > 0$. Let

$$\lim_{x\to\infty} I(1-\alpha;\ 0, x| Y) = d \geqslant 0.$$

Suppose that $d > 0$, then there exists a number $X > 0$ such that for $x > X$, $I(1-\alpha; 0, x| Y) > \frac{1}{2}d$ and also

$$I(1;\ 0, x| Y) > I(\alpha;\ X, x| I(1-\alpha;\ 0, t| Y)) > \frac{d(x-X)^{\alpha}}{2\Gamma(\alpha+1)} \to \infty \qquad \text{as } x \to \infty.$$

But $I(1-\alpha; 0, x| Y) + I(1; 0, x| Y) = 1$, so that we have a contradiction. Hence $d = 0$. Thus part (b) follows.

To prove part (c), let $0 < \beta < \alpha$. For $x > 1$, then

$$I(\beta;\ 0, x| Y) = \int_0^{x-1} Y(t)\, \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\, dt + \int_{x-1}^{x} Y(t)\, \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\, dt.$$

Since $x - t > 1$ and $(x-t)^{\beta-1} < (x-t)^{\alpha-1}$ for $0 < t < x-1$, then $I(\beta; 0, x| Y) < \Gamma(\alpha)/\Gamma(\beta)$ and

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$$I(\alpha;\ 0, x|Y) + \frac{(x-1)^{\alpha-1}}{\Gamma(\alpha)\,\Gamma(\beta+1)} \to 0 \qquad \text{as } x \to \infty.$$

Similarly, it follows that if $0 < \beta < 1 - \alpha$, then $I(\beta; 0, x| Y) \to 0$ as $x \to \infty$. Furthermore

$$I(\alpha + \delta;\ 0, x|Y) + I(\delta;\ 0, x|Y) = \frac{x^{\alpha+\delta} - 1}{\Gamma(\alpha+\delta)}$$

from which we see that $I(\alpha + \delta; 0, x| Y) \to 0$ as $x \to \infty$ for $\alpha + \delta < 1$; and $\delta \leqslant \max(\alpha, 1 - \alpha)$. If

$$\beta > \max(\alpha, 1-\alpha) \geqslant \tfrac{1}{2} \text{ and } \delta = \beta - \alpha$$

then $0 < \delta < \frac{1}{2} \leqslant \max(\alpha, 1 - \alpha)$. Hence, the result is proved for all $0 < \beta < 1$.

To complete this discussion we call attention to the above properties for $\alpha = 1$ and the corresponding $Y(x) = e^{-x}$, together with a few properties of e^{zt}, where z is any number.

THEOREM 3.3. *If z is a complex number;* $0 < \beta < 1$ *and*

(a) *if* $\mathrm{R}z < 0$, *then* $\lim_{x\to\infty} I(\beta;\ 0, x|e^{zt}) = 0$;

(b) *if* $\mathrm{R}z \geqslant 0$, *and* $z \neq 0$ *then* $\lim_{x\to\infty} \left[I(\beta;\ 0, x|e^{zt}) - \frac{e^{zx}}{z^{\beta}} \right] = 0.$

Proof. We have

$$I(\beta;\ 0, x|e^{zt}) = \int_0^x e^{zt} \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} dt = e^{zx} \int_0^x e^{-zu} \frac{u^{\beta-1}}{\Gamma(\beta)} du.$$

For $\mathrm{R}z \geqslant 0$, $z \neq 0$ we recognize the Laplace Transform

$$\int_0^\infty e^{-zu} u^{\beta-1} du = \Gamma(\beta) z^{-\beta}$$

from which part (b) follows immediately. If $Rz < 0$, then

$$||I(\beta;\ 0, x|e^{zt})|| \leqslant e^{\mathrm{R}zx} \int_0^x e^{-\mathrm{R}zu} \frac{u^{\beta-1}}{\Gamma(\beta)} du.$$

Let $\epsilon > 0$ and x_1 be a number such that

$$\frac{x_1^{\beta-1}}{\Gamma(\beta)} < -\mathrm{R}z \cdot \tfrac{1}{2}\epsilon, \quad \text{and} \quad e^{\mathrm{R}zx} \int_0^x e^{-\mathrm{R}zu} \frac{u^{\beta-1}}{\Gamma(\beta)} du < \tfrac{1}{2}\epsilon.$$

Then $||I(\beta; 0, x|e^{zt})|| < \epsilon$, for $x > x_1$.

4. Behavior of the entire function $E_\alpha(z)$ on the real axis and its relation to the behavior of $Y(x)$. At the beginning of this century extensive studies were made of the entire function:

$$E_\alpha(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha + 1)}; \qquad \mathrm{R}\alpha > 0.$$

The following identities exist between the function $Y(x)$ of §3 and $E_\alpha(z)$:

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THEOREM 4.1. *For the above mentioned* $E_\alpha(z)$ *and* $Y(x)$ *and* $R\alpha > 0$ *and* $x > 0$*:*

(a) $I(1-\alpha;\, 0, x|Y) = E_\alpha(-x^\alpha)$,

(b) $\phi(x) = \int_0^x Y(t)\,dt = 1 - E_\alpha(-x^\alpha)$,

(c) $Y(x) = \alpha x^{\alpha-1} E'_\alpha(-x^\alpha)$.

Mittag-Leffler **(7)** proved that for $0 < \alpha < 2$; $|E_\alpha(z)| \to 0$ as $x \to \infty$ and $\frac{1}{2}\alpha\pi < \arg z < 2\pi - \frac{1}{2}\alpha\pi$; such a domain includes the negative real axis, thus

$$\lim_{x\to\infty} E_\alpha(-x) = 0, \qquad 0 < \alpha < 2.$$

By applying Theorem 3.2 (c) and Theorem 4.1 (a) we have another proof of this latter fact for $0 < \alpha < 1$. Furthermore, using Theorem 4.1 (c) we see that $E'_\alpha(x) > 0$ for $x < 0$. From the series form of $E_\alpha(x)$ we observe that $E'_\alpha(x) > 0$ and $E_\alpha(x) > 0$ for $x \geqslant 0$ and $0 < \alpha < 1$. Now, from Theorem 3.1, $Y(x) > 0$ and it follows immediately that $E_\alpha(-x^\alpha) > 0$ for $0 < \alpha < 1$ and $x > 0$. Also, from Theorem 4.1 (b) and the fact that $\phi(x) > 0$ we see that $E_\alpha(-x^\alpha) < 1$ for $x > 0$. These results are summarized in the following:

THEOREM 4.2. *For* $0 < \alpha < 1$, $E_\alpha(z)$ *has no zeros on the real axis;* $0 < E_\alpha(x) < 1$ *for* $x < 0$ *and* $E'_\alpha(x) > 0$ *for the whole real axis.*

Wiman **(12)** proved that for $0 < \alpha < 1$, the zeros of $E_\alpha(z)$ in the upper (or lower) half z-plane approach the line $\arg z = \frac{1}{2}\alpha\pi$ (or $-\frac{1}{2}\alpha\pi$) as the modulus of the zero increases without bound. However, his discussion will not supply the fact that there are no zeros on the negative real axis.

Now, consider $1 < \alpha < 2$ and reverse the roles of $Y(x)$ and $E_\alpha(x)$, that is, use $E_\alpha(x)$ to complete the picture of $Y(x)$. In addition to the previously mentioned result of Mittag-Leffler, we will make use of the fact due to Wiman **(12)** that for large x, $E_\alpha(-x) < 0$. Thus, using Theorem 4.1 (b) we see that for large x, $\phi(x) > 1$ and $\lim \phi(x) = 1$ as $x \to \infty$. Hence, for large x, $Y(x) < 0$ and $\lim Y(x) = 0$ as $x \to \infty$. Also, from Theorem 4.1 (c) and the fact that $Y(0) = 0$ it follows that $Y(x)$ has at least as many zeros on the non-negative x-axis as $E_\alpha(x)$ has on the negative x-axis. Summarizing these results together with one which is an immediate application of a result of Wiman we have:

THEOREM 4.3. *For* $1 < \alpha < 2$:

(a) $\lim Y(x) = 0$ *as* $x \to \infty$ *and* $Y(x) < 0$ *for* x *large*;

(b) $Y(x)$ *has a finite number of zeros for* $x \geqslant 0$ *and if for each* α, $N(\alpha)$ *is this number of zeros*

$$\lim_{\alpha\to 2} N(\alpha) = \infty.$$

By direct application of Theorem 4.1 (c) to another result of Wiman we have:

THEOREM 4.4. *For* $2 < \alpha$, $Y(x)$ *has infinitely many zeros, i.e.,* $Y(x)$ *is oscillatory on* $x \geqslant 0$.

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