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UNIQUENESS OF ENTIRE FUNCTIONS SHARING A VALUE WITH LINEAR DIFFERENTIAL POLYNOMIALS

INDRAJIT LAHIRI✉ and RAJIB MUKHERJEE

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Abstract

We study the uniqueness of entire functions sharing a nonzero finite value with linear differential polynomials and improve a result of P. Li.

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1. Introduction, definitions and results

Let f be a nonconstant entire function in the open complex plane $\mathbb{C}$. We denote by $\overline{E}(a; f)$ the set of distinct a-points of f. We also respectively denote by $\overline{E}_{1)}(a; f)$ and $\overline{E}_{(2}(a; f)$ the sets of distinct simple and multiple a-points of f.

In 1986 Jank *et al.* [2] proved a uniqueness theorem for entire functions sharing a single value with two derivatives. Their results can be stated as follows.

Theorem A [2]. *Let f be a nonconstant entire function and a be a nonzero finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

In fact, in Theorem A f and $f^{(1)}$ share the value a, counting multiplicities. Considering $f = e^{\omega z} + \omega - 1$, where $\omega^{n-1} = 1$, $\omega \neq 1$ and $m \geq 3$ is an integer, and $a = \omega$, we can verify that the second derivative in Theorem A cannot, in general, be replaced by the mth derivative for $m \geq 3$ (see [7]).

In 1995 Zhong [7] generalised Theorem A and proved the following theorem.

Theorem B [7]. *Let f be a nonconstant entire function and $a \neq 0$ be a finite number. If f and $f^{(1)}$ share the value a, counting multiplicities, and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n \geq 1$, then $f \equiv f^{(n)}$.*

For $A \subset \mathbb{C}$, we denote by $N_A(r, a; f)$ $(\overline{N}_A(r, a; f))$ the counting function (reduced counting function) of those a-points of f which belong to A.

Recently, Theorem B was improved in the following manner.

Theorem C [3]. *Let f be a nonconstant entire function and a, b be two nonzero finite constants. Suppose further that $A = \overline{E}(a; f)\backslash\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)})\backslash\{\overline{E}(a; f^{(n)}) \cap \overline{E}(b; f^{(n+1)})\}$ for $n \geq 1$. If each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity and $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, then $f = \lambda e^{bz/a} + (ab - a^2)/b$ or $f = \lambda e^{bz/a} + a$, where $\lambda \neq 0$ is a constant.*

Throughout the paper, we denote by L a nonconstant linear differential polynomial in f of the form

$$L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)},$$

where $a_1, a_2, \ldots, a_n, a_n \neq 0$, are constants.

In 1999 Li [4] improved and extended Theorem B by considering a linear differential polynomial. He proved the following theorem.

Theorem D [4]. *Let f be a nonconstant entire function and $a \neq 0$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^1)$, then $f \equiv f^{(1)} \equiv L$.*

For other results on linear differential polynomials, one may see [5, 6].

In this paper, we improve Theorem D in the following manner.

Theorem 1.1. *Let f be a nonconstant entire function and $a \neq 0$ be a finite number. Suppose further that:*

(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, *where* $A = \overline{E}(a; f)\backslash\overline{E}(a; f^{(1)})$ *and* $B = \overline{E}(r, f^{(1)})\backslash(\overline{E}(a; L) \cap \overline{E}(a; L^{(1)}))$;
(ii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})$; *and*
(iii) $\overline{E}_{(2}(a; f) \cap \overline{E}(0; L^{(1)}) = \emptyset$.

Then $L = \alpha e^z$ and $f = \alpha e^z$ or $f = a + \alpha e^z$, where $\alpha \neq 0$ is a constant.

Putting $A = B = \emptyset$ in Theorem 1.1, we get the following result.

Corollary 1.2. *Let f be a nonconstant entire function and $a \neq 0$ be a finite number. If $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $L = \alpha e^z$ and $f = \alpha e^z$ or $f = a + \alpha e^z$, where $\alpha \neq 0$ is a constant.*

For standard definitions and notation in value distribution theory, we refer the reader to [1]. However, we require the following definitions.

Definition 1.3. Let f and g be two nonconstant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid g \neq b)$ ($\overline{N}(r, a; f \mid g \neq b)$) the counting function (reduced counting function) of those a-points of f which are not the b-points of g.

Definition 1.4. Let f and g be two nonconstant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid g = b)$ ($\overline{N}(r, a; f \mid g = b)$) the counting function (reduced counting function) of those a-points of f which are the b-points of g.

DEFINITION 1.5. Let f and g be two nonconstant meromorphic functions defined in $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer k, we denote by $N(r, a; f \mid\geq k)$ ($N(r, a; f \mid\leq k)$) the counting function of those a-points of f whose multiplicities are not less (greater) than k. By $\overline{N}(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\leq k)$, we denote the corresponding reduced counting functions.

The following definition is well known.

DEFINITION 1.6. Let f be a nonconstant meromorphic function in $\mathbb{C}$. Suppose that

$$M_j[f] = a_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(p_j)})^{n_{p_j j}}$$

is a differential monomial in f, where a_j is a small function of f. We denote by $\gamma_{M_j} = \sum_{k=0}^{p_j} n_{kj}$ and by $\Gamma_{M_j} = \sum_{k=0}^{p_j}(1+k)n_{kj}$ the degree and weight of $M_j[f]$, respectively. The numbers $\gamma_P = \max_{1\leq j\leq n} \gamma_{M_j}$ and $\Gamma_P = \max_{1\leq j\leq n} \Gamma_{M_j}$ are respectively called the degree and weight of the differential polynomial $P[f] = \sum_{j=1}^{n} M_j[f]$.

2. Lemmas

In this section, we present some necessary lemmas.

LEMMA 2.1. *Let f be a nonconstant entire function and a be a nonzero finite complex number. Then $f = L = \alpha e^z$, where α is a nonzero constant, provided the following hold:*

(i) $m(r, a; f) = S(r, f)$;

(ii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$;

(iii) $N_A(r, a; f) = S(r, f)$, *where* $A = \overline{E}(a; f) \backslash (\overline{E}(a; L) \cap \overline{E}(a; L^{(1)}) \cap \overline{E}(a; f^{(1)}))$.

PROOF. Let

$$\lambda = \frac{f^{(1)} - a}{f - a}. \tag{2.1}$$

From the hypothesis, we see that λ has no simple pole and $T(r, \lambda) = S(r, f)$. From (2.1),

$$f^{(1)} = \lambda_1 f + \mu_1, \tag{2.2}$$

where $\lambda_1 = \lambda$ and $\mu_1 = a(1 - \lambda)$. Differentiating (2.2),

$$f^{(k)} = \lambda_k f + \mu_k,$$

where λ_k and μ_k are meromorphic functions satisfying $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ and $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ for $k = 1, 2, 3, \ldots$. Also, we see that $T(r, \lambda_k) + T(r, \mu_k) = S(r, f)$ for $k = 1, 2, 3, \ldots$.

Now

$$L = \Big(\sum_{k=1}^{n} a_k\lambda_k\Big) f + \sum_{k=1}^{n} a_k\mu_k = \xi f + \eta, \text{ say.} \tag{2.3}$$

Clearly, $T(r, \xi) + T(r, \eta) = S(r, f)$. Differentiating (2.3),

$$L^{(1)} = \xi f^{(1)} + \xi^{(1)} f + \eta^{(1)}. \tag{2.4}$$

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Let $z_0 \notin A$ be an a-point of f. Then, from (2.3) and (2.4), $a\xi(z_0) + \eta(z_0) = a$ and $a\xi(z_0) + a\xi^{(1)}(z_0) + \eta^{(1)}(z_0) = a$.

If $a\xi + \eta \not\equiv a$, then

$$\begin{aligned} N(r, a; f) &\le N(r, a; f \mid\le 1) + N_A(r, a; f) \\ &\le N(r, a; a\xi + \eta) + S(r, f) = S(r, f), \end{aligned}$$

which is impossible because $m(r, a; f) = S(r, f)$. Hence, $a\xi + \eta \equiv a$. Similarly, $a\xi + a\xi^{(1)} + \eta^{(1)} \equiv a$. This implies that $\xi \equiv 1$ and $\eta \equiv 0$. So, from (2.3), $L \equiv f$.

By actual calculation, we see that $\lambda_2 = \lambda^2 + \lambda^{(1)}$ and $\lambda_3 = \lambda^3 + 3\lambda\lambda^{(1)} + \lambda^{(2)}$. We now verify that, in general,

$$\lambda_k = \lambda^k + P_{k-1}[\lambda], \tag{2.5}$$

where $P_{k-1}[\lambda]$ is a differential polynomial in λ with constant coefficients such that $\gamma_{P_{k-1}} \le k - 1$ and $\Gamma_{P_{k-1}} \le k$. Also, each term of $P_{k-1}[\lambda]$ contains some derivative of λ.

Let (2.5) be true. Then

$$\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k = (\lambda^k + P_{k-1}[\lambda])^{(1)} + \lambda(\lambda^k + P_{k-1}[\lambda]) = \lambda^{k+1} + P_k[\lambda],$$

noting that differentiation does not increase the degree of a differential polynomial but increases its weight by 1. So, (2.5) is verified by mathematical induction.

Since $\xi \equiv 1$, from (2.5),

$$\sum_{k=1}^{n} a_k\lambda^k + \sum_{k=1}^{n} a_k P_{k-1}[\lambda] \equiv 1. \tag{2.6}$$

Let z_0 be a pole of λ with multiplicity $p \ge 2$. Then z_0 is a pole of $\sum_{k=1}^{n} a_k\lambda^k$ with multiplicity np and it is a pole of $\sum_{k=1}^{n} a_k P_{k-1}[\lambda]$ with multiplicity at most $(n - 1)p + 1$. Since $np > (n - 1)p + 1$, it follows that z_0 is a pole of the left-hand side of (2.6) with multiplicity np, which is impossible. So, λ is an entire function. If λ is transcendental, then by the Clunie lemma we get from (2.6) that $T(r, \lambda) = S(r, \lambda)$, which is a contradiction. If λ is a polynomial of degree $d \ge 1$, then the left-hand side of (2.6) is a polynomial of degree nd with leading coefficient $a_n \ne 0$, which is also a contradiction. Therefore, λ is a constant and, so, from (2.5), $\lambda_k = \lambda^k$ for $k = 1, 2, 3, \ldots$.

Since $\xi \equiv 1$, we see that $\sum_{k=1}^{n} a_k\lambda^k = 1$. Also, from (2.2), $f^{(2)} = \lambda f^{(1)}$ and so $f^{(1)} = \alpha\lambda e^{\lambda z}$ and $f = \alpha e^{\lambda z} + \beta$, where $\alpha \ne 0$ and β are constants.

Now

$$L = \Big(\sum_{k=1}^{n} a_k\lambda^k\Big)\alpha e^{\lambda z} = \alpha e^{\lambda z}.$$

Since $f \equiv L$, $\beta = 0$. Since $N_A(r, a; f) = S(r, f)$ and $N(r, a; f) = T(r, f) + S(r, f)$, we see that $\overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \ne \emptyset$. So, $f^{(1)} = \lambda f$ implies that $\lambda = 1$. Hence, $f = \alpha e^{z}$. This proves the lemma. □

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LEMMA 2.2. *Let f be a nonconstant entire function and a be a nonzero finite complex number. Let $A = \overline{E}(a; f)\backslash\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)})\backslash(\overline{E}(a; L) \cap \overline{E}(a; L^{(1)}))$. If $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, then $N(r, a; f^{(1)} \mid\geq 2) = S(r, f)$.*

PROOF. Let $\chi = (L - f^{(1)})/(f - a)$ and $\phi = (L^{(1)} - f^{(1)})/(f - a)$. Then

$$m(r, \chi) + m(r, \phi) = S(r, f)$$

and

$$N(r, \chi) + N(r, \phi) \leq 2(N_A(r, a; f) + N_B(r, a; f^{(1)})) = S(r, f).$$

So, $T(r, \chi) + T(r, \phi) = S(r, f)$.

First, we suppose that $L \equiv L^{(1)}$ and $L^{(1)} \equiv f^{(1)}$. Then $L^{(1)} \equiv L^{(2)} \equiv f^{(2)}$. Hence, $f^{(2)} \equiv f^{(1)}$, which shows that $f^{(1)}$ has no multiple a-points, and so $N(r, a; f^{(1)} \mid\geq 2) = S(r, f)$.

Now we suppose that $L^{(1)} \not\equiv L$. Then, by the hypothesis,

$$\begin{aligned} \overline{N}(r, a; f^{(1)}) &\leq N\left(r, 1; \frac{L^{(1)}}{L}\right) + N_B(r, a; f^{(1)}) \\ &\leq T\left(r, \frac{L^{(1)}}{L}\right) + S(r, f) \\ &= N\left(r, \frac{L^{(1)}}{L}\right) + S(r, f) \\ &\leq N(r, 0; L) + S(r, f). \end{aligned} \tag{2.7}$$

Again,

$$\begin{aligned} m(r, a; f) &\leq m\left(r, \frac{L}{f - a}\right) + m\left(r, \frac{1}{L}\right) \\ &= T(r, L) - N(r, 0; L) + S(r, f) \\ &\leq m(r, f) + m\left(r, \frac{L}{f}\right) - N(r, 0; L) + S(r, f) \\ &\leq T(r, f) - N(r, 0; L) + S(r, f) \end{aligned}$$

and so

$$N(r, 0; L) \leq N(r, a; f) + S(r, f). \tag{2.8}$$

Also,

$$\begin{aligned} N(r, a; f) &= N(r, a; f \mid f^{(1)} = a) + N_A(r, a; f) \\ &\leq \overline{N}(r, a; f^{(1)}) + S(r, f). \end{aligned} \tag{2.9}$$

From (2.7), (2.8) and (2.9),

$$\overline{N}(r, a; f^{(1)}) = N(r, a; f) + S(r, f)$$

and so

$$\overline{N}(r, a; f^{(1)} \mid f \neq a) = S(r, f). \tag{2.10}$$

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Next we suppose that $L^{(1)} \not\equiv f^{(1)}$. Then, replacing L by $f^{(1)}$ in the above argument, we can prove (2.10).

We now consider the following cases.

Case I. Let $\chi \equiv 1$. Then $L - f^{(1)} \equiv f - a$ and so $L^{(1)} - f^{(2)} \equiv f^{(1)}$. This implies that

$$f^{(2)} \equiv \phi(f - a). \tag{2.11}$$

Now we suppose that ϕ is nonconstant. Differentiating (2.11) and using it repeatedly, we get, for $n \geq 2$,

$$f^{(n)} = P_n f^{(1)} + Q_n(f - a),$$

where

$$P_n = \begin{cases} P_n^* & \text{if } n \text{ is even and } \gamma_{P_n^*} \leq \dfrac{n}{2} - 1, \\ \phi^{(n-1)/2} + P_n^* & \text{if } n \text{ is odd and } \gamma_{P_n^*} \leq \dfrac{n-1}{2} - 1 \end{cases}$$

and

$$Q_n = \begin{cases} \phi^{n/2} + Q_n^* & \text{if } n \text{ is even and } \gamma_{Q_n^*} \leq \dfrac{n}{2} - 1, \\ Q_n^* & \text{if } n \text{ is odd and } \gamma_{Q_n^*} \leq \dfrac{n-1}{2} - 1, \end{cases}$$

and P_n^* and Q_n^* are differential polynomials in ϕ with constant coefficients.

Now

$$L^{(1)} = \Big(\sum_{k=1}^{n} a_k P_{k+1}\Big) f^{(1)} + \Big(\sum_{k=1}^{n} a_k Q_{k+1}\Big)(f - a). \tag{2.12}$$

Let $n \geq 2$ be even. Then, from (2.12), $L^{(1)} = \xi f^{(1)} + \eta(f - a)$, where $\xi = a_n \phi^{n/2} + \tilde{P}_{n+1}$, $\eta = a_{n-1}\phi^{n/2} + \tilde{Q}_n$ and $\gamma_{\tilde{P}_{n+1}} \leq (n/2) - 1$, $\gamma_{\tilde{Q}_n} \leq (n/2) - 1$. Since $L^{(1)} = f^{(1)} + \phi(f - a)$, we have $(1 - \xi) f^{(1)} = (\eta - \phi)(f - a)$. If $1 - \xi \equiv 0$, then $n = 0$, which is impossible.

Let n be odd. Then, from (2.12), $L^{(1)} = \xi f^{(1)} + \eta(f - a)$, where $\xi = a_{n-1}\phi^{(n-1)/2} + \tilde{P}_n$, $\eta = a_n \phi^{(n+1)/2} + \tilde{Q}_{n+1}$ and $\gamma_{\tilde{P}_n} \leq (n - 1)/2 - 1$, $\gamma_{\tilde{Q}_{n+1}} \leq (n + 1)/2 - 1$. Since $L^{(1)} = f^{(1)} + \phi(f - a)$, we have $(1 - \xi) f^{(1)} = (\eta - \phi)(f - a)$. If $1 - \xi \equiv 0$, then $\eta \equiv \phi$ and so $n = 1$ and $a_1 = 1$. Hence, $L = f^{(1)}$, which is impossible as $\chi \equiv 1$.

Therefore, in general, $1 - \xi \not\equiv 0$ and so

$$\frac{f^{(1)}}{f - a} = \frac{\eta - \phi}{1 - \xi}.$$

Hence, $\overline{N}(r, a; f) = N(r, f^{(1)}/(f - a)) = N(r, (\eta - \phi)/(1 - \xi)) = S(r, f)$. This shows that

$$\begin{aligned} N(r, a; f) &= N_A(r, a; f) + N(r, a; f \mid f^{(1)} = a) \\ &\leq \overline{N}(r, a; f) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{2.13}$$

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Let

$$C = (\overline{E}(a; f^{(1)}) \cap \overline{E}(a; L) \cap \overline{E}(a; L^{(1)}))\backslash \overline{E}(a; f). \tag{2.14}$$

By (2.10),

$$N_C(r, a; f^{(1)}) \le n\overline{N}_C(r, a; f^{(1)}) = S(r, f). \tag{2.15}$$

Hence, by (2.13) and (2.15),

$$N(r, a; f^{(1)}) \le nN(r, a; f) + N_B(r, a; f^{(1)}) + N_C(r, a; f^{(1)}) = S(r, f). \tag{2.16}$$

Now we suppose that ϕ is a constant. Then, from (2.11),

$$f = a + c^2 e^{\lambda^2 z} - d^2 e^{-\lambda^2 z}, \tag{2.17}$$

where c, d are constants and $\lambda^4 = \phi$. Since f is nonconstant, we see that $\lambda \ne 0$.

If $c = 0$ or $d = 0$, then $N(r, a; f) = S(r, f)$ and we can deduce (2.16). Let $cd \ne 0$. Then, from (2.17),

$$f - a = e^{-\lambda^2 z}(ce^{\lambda^2 z} - d)(ce^{\lambda^2 z} + d) \tag{2.18}$$

and

$$f^{(1)} - a = e^{-\lambda^2 z}\bigg(\Big(c\lambda e^{\lambda^2 z} - \frac{a}{2c\lambda}\Big)^2 + \Big(d^2\lambda^2 - \frac{a^2}{4c^2\lambda^2}\Big)\bigg).$$

So, $f^{(1)}$ has multiple a-points only if $d\lambda = \pm a/(2c\lambda)$. Let $d\lambda = a/(2c\lambda)$. Then, from (2.18),

$$\begin{aligned} N_A(r, a; f) &= N\Big(r, -\frac{a}{2c^2\lambda^2}; e^{\lambda^2 z}\Big) \\ &= T(r, e^{\lambda^2 z}) + S(r, e^{\lambda^2 z}) \\ &= \frac{1}{2}T(r, f) + S(r, f), \end{aligned}$$

which is impossible as $N_A(r, a; f) = S(r, f)$. So, $d\lambda \ne a/(2c\lambda)$. Similarly, we can show that $d\lambda \ne -a/(2c\lambda)$. Therefore, $f^{(1)}$ has no multiple a-points and $N(r, a; f^{(1)} \mid\ge 2) = S(r, f)$.

Case II. Let $\chi \not\equiv 1$. We put

$$D = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L) \cap \overline{E}(a; L^{(1)}). \tag{2.19}$$

Let $z_0 \in D$ be a multiple a-point of $f^{(1)}$. Then clearly $\chi(z_0) = 1$ and so

$$\overline{N}_D(r, a; f^{(1)} \mid\ge 2) \le N(r, 1; \chi) = S(r, f), \tag{2.20}$$

where we denote by $N_D(r, a; f^{(1)} \mid\ge 2)$ ($\overline{N}_D(r, a; f^{(1)} \mid\ge 2)$) the counting function (reduced counting function) of those multiple a-points of $f^{(1)}$ which belong to D.

Now, using (2.15) and (2.20),

$$\begin{aligned} N(r, a; f^{(1)} \mid\ge 2) &\le N_B(r, a; f^{(1)}) + N_C(r, a; f^{(1)}) + N_D(r, a; f^{(1)} \mid\ge 2) \\ &\le n\overline{N}_D(r, a; f^{(1)} \mid\ge 2) = S(r, f). \end{aligned}$$

This proves the lemma. □

Lemma 2.3. *Let f be a nonconstant entire function and a be a nonzero finite number. Suppose that $A = \overline{E}(a; f)\backslash\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)})\backslash(\overline{E}(a; L) \cap \overline{E}(a; L^{(1)}))$. Then $\phi = (L^{(1)} - f^{(1)})/(f - a)$ is an entire function provided the following hold:*

(i) $N_A(r, a; f) + B_B(r, a; f^{(1)}) = S(r, f)$;
(ii) $\overline{E}_{1)}(r, a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})$; and
(iii) $\overline{E}_{(2}(a; f) \cap \overline{E}(0; L^{(1)}) = \emptyset$.

Proof. We note that

$$f^{(1)} = L^{(1)} - \phi(f - a). \tag{2.21}$$

Differentiating (2.21) and using it repeatedly,

$$f^{(k)} = P_k + p_k L^{(1)} + q_k(f - a),$$

where P_k is a differential polynomial in $L^{(2)}$ whose coefficients are differential polynomials in ϕ with constant coefficients, $p_k = (-1)^{k-1}\phi^{k-1} + \tilde{p}_k$ and $q_k = (-1)^k\phi^k + \tilde{q}_k$.

We note that $\tilde{p}_k$ and $\tilde{q}_k$ are differential polynomials in ϕ with constant coefficients whose terms contain some derivatives of ϕ. Further, $\gamma_{\tilde{p}_k} \le k - 2$, $\Gamma_{\tilde{p}_k} \le k - 1$, $\gamma_{\tilde{q}_k} \le k - 1$ and $\Gamma_{\tilde{q}_k} \le k$.

Now

$$L^{(1)} = \sum_{k=1}^{n} a_k f^{(k+1)} = A + \xi L^{(1)} + \eta(f - a), \tag{2.22}$$

where

$$A = \sum_{k=1}^{n} a_k P_{k+1},$$

$$\xi = (-1)^n a_n \phi^n + \sum_{k=1}^{n}(-1)^k a_k \phi^k + \sum_{k=1}^{n} a_k \tilde{p}_{k+1}$$

and

$$\eta = (-1)^{n+1} a_n \phi^{n+1} + \sum_{k=1}^{n-1}(-1)^{k+1} a_k \phi^{k+1} + \sum_{k=1}^{n} a_k \tilde{q}_{k+1}.$$

Differentiating (2.22) and using (2.21),

$$L^{(2)} = A^{(1)} + \xi L^{(2)} + (\eta + \xi^{(1)})L^{(1)} + (\eta^{(1)} - \eta\phi)(f - a). \tag{2.23}$$

Eliminating $L^{(1)}$ from (2.22) and (2.23),

$$X = Y(f - a), \tag{2.24}$$

where

$$X = (1 - \xi)L^{(2)} - (1 - \xi)(A^{(1)} + \xi L^{(2)}) - (\xi^{(1)} + \eta)A$$

and

$$Y = (\xi^{(1)} + \eta)\eta + (1 - \xi)(\eta^{(1)} - \eta\phi).$$

Since $T(r, \phi) = S(r, f)$, we see that $T(r, \xi) + T(r, \eta) + T(r, Y) = S(r, f)$.

Let $X \not\equiv 0$. Then, from (2.24),

$$T\Big(r, \frac{X}{f-a}\Big) = T(r, Y) = S(r, f).$$

Now $m(r, X/(f^{(1)} - a)) = S(r, f)$ and, by Lemma 2.2 and (2.15), we get from (2.24)

$$\begin{aligned} N\Big(r, \frac{X}{f^{(1)}-a}\Big) &\leq N(r, Y) + N\Big(r, \frac{f-a}{f^{(1)}-a}\Big) \\ &\leq N(r, a; f^{(1)} \mid\geq 2) + N_B(r, a; f^{(1)}) + N_C(r, a; f^{(1)}) + S(r, f) \\ &= S(r, f), \end{aligned}$$

where C is given by (2.14).

Therefore, $T(r, X/(f^{(1)} - a)) = S(r, f)$ and so $m(r, a; f) = S(r, f)$. Hence, by Lemma 2.1, $f = L = \alpha e^z$, which implies that $\phi \equiv 0$.

Let $X \equiv 0$. Under the hypotheses, ϕ has no simple pole. Let z_0 be a pole of ϕ with multiplicity $t \geq 2$. Then z_0 is a pole of $\tilde{p}_{n+1}$ with multiplicity at most $(t-1)\gamma_{\tilde{p}_{n+1}} + \Gamma_{\tilde{p}_{n+1}} \leq (t-1)(n-1) + n = nt - (t-1) < nt$. Hence, z_0 is a pole of ξ with multiplicity nt. Also, z_0 is a pole of $\tilde{q}_{n+1}$ with multiplicity at most $(t-1)\gamma_{\tilde{q}_{n+1}} + \Gamma_{\tilde{q}_{n+1}} \leq (t-1)n + n + 1 = nt + 1 < (n+1)t$. Hence, z_0 is a pole of η with multiplicity $(n+1)t$.

Since f is an entire function, from (2.22) we see that z_0 is a pole of A with multiplicity $(n+1)t$. A simple calculation reveals that z_0 is a pole of $\xi A^{(1)} - \xi^{(1)} A$ with multiplicity $(n+1)t + nt + 1$. Since

$$X = (1-\xi)L^{(2)} - (A^{(1)} + \xi L^{(2)}) + (\xi A^{(1)} - \xi^{(1)} A) + \xi^2 L^{(2)} + \eta A$$

and $2(n+1)t > \max\{nt, (n+1)t + 1, (n+1)t + nt + 1, 2nt\}$, we see that z_0 is a pole of X. This is impossible as $X \equiv 0$. Hence, ϕ is an entire function. This proves the lemma. □

3. Proof of Theorem 1.1

Let $\phi = (L^{(1)} - f^{(1)})/(f - a)$. Then, by Lemma 2.3, ϕ is an entire function. Also, $T(r, \phi) = m(r, \phi) = S(r, f)$. First, we suppose that $\phi \not\equiv 0$. Then

$$m(r, f) = m\Big(r, a + \frac{L^{(1)} - f^{(1)}}{\phi}\Big) \leq m(r, f^{(1)}) + S(r, f) \leq m(r, f) + S(r, f)$$

and so $T(r, f) = T(r, f^{(1)}) + S(r, f)$.

Differentiating $f = a + (L^{(1)} - f^{(1)})/\phi$,

$$\Big(1 + \Big(\frac{1}{\phi}\Big)^{(1)}\Big) f^{(1)} = \Big(\frac{1}{\phi}\Big)^{(1)} L^{(1)} + \frac{1}{\phi}(L^{(2)} - f^{(2)}).$$

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Since ϕ is entire, we have $1 + (1/\phi)^{(1)} \not\equiv 0$ and so

$$\frac{f^{(1)}}{f^{(1)} - a} = \frac{1}{1 + \left(\frac{1}{\phi}\right)^{(1)}}\left(\left(\frac{1}{\phi}\right)^{(1)} \frac{L^{(1)}}{f^{(1)} - a} + \left(\frac{1}{\phi}\right)\frac{L^{(2)} - f^{(2)}}{f^{(1)} - a}\right).$$

This implies that $m(r, f^{(1)}/(f^{(1)} - a)) = S(r, f)$ and so $m(r, a; f^{(1)}) = S(r, f)$.

Again, by Lemma 2.2 and (2.10),

$$N(r, a; f^{(1)}) = \overline{N}(r, a; f^{(1)}) + S(r, f) = \overline{N}(r, a; f^{(1)} \mid f = a) + S(r, f).$$

Therefore,

$$\begin{aligned} m(r, a; f) &= T(r, f) - N(r, a; f) + S(r, f) \\ &= T(r, f^{(1)}) - N(r, a; f) + S(r, f) \\ &= N(r, a; f^{(1)}) - N(r, a; f) + S(r, f) \\ &= \overline{N}(r, a; f^{(1)} \mid f = a) - N(r, a; f) + S(r, f) \\ &\le S(r, f). \end{aligned}$$

So, by Lemma 2.1, $f = L = \alpha e^z$.

Let $\phi \equiv 0$. Then $L^{(1)} \equiv f^{(1)}$ and so $L = f + d$, where d is a constant. Let $\psi = (L - L^{(1)})/(f - a)$. Then $m(r, \psi) = S(r, f)$ and

$$N(r, \psi) \le N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f).$$

If $z_0 \in D$, then clearly $L^{(2)}(z_0) - (1 - \psi(z_0))L^{(1)}(z_0) = 0$, where D is given by (2.19). We put $g_1 = (L^{(2)} - (1 - \psi)L^{(1)})/(f^{(1)} - a)$ and $g_2 = (L^{(2)} - (1 - \psi)L^{(1)})/(f - a)$.

Then $m(r, g_1) + m(r, g_2) = S(r, f)$. Also, by Lemma 2.2 and (2.15),

$$N(r, g_1) \le N_B(r, a; f^{(1)}) + N_C(r, a; f^{(1)}) + N(r, a; f^{(1)} \mid\ge 2) = S(r, f)$$

and

$$N(r, g_2) \le N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f),$$

where C is given by (2.14). Therefore, $T(r, g_1) + T(r, g_2) = S(r, f)$.

Let $L^{(2)} - (1 - \psi)L^{(1)} \not\equiv 0$. Then $m(r, (f^{(1)} - a)/(f - a)) = m(r, g_2/g_1) = S(r, f)$ and so $m(r, a; f) = S(r, f)$. Hence, by Lemma 2.1, $L = \alpha e^z$, where $\alpha \neq 0$ is a constant. So, $L \equiv L^{(1)} \equiv L^{(2)}$ and $\psi \equiv 0$, which contradicts the supposition that $L^{(2)} - (1 - \psi)L^{(1)} \not\equiv 0$. Therefore, $L^{(2)} - (1 - \psi)L^{(1)}$ is indeed identically zero, that is,

$$L^{(2)} - (1 - \psi)L^{(1)} \equiv 0. \tag{3.1}$$

Let $\psi \not\equiv 0$. Differentiating $L - L^{(1)} \equiv \psi(f - a)$,

$$L^{(1)} - L^{(2)} \equiv \psi^{(1)}(f - a) + \psi f^{(1)}. \tag{3.2}$$

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Eliminating $L^{(2)}$ from (3.1) and (3.2),

$$\psi L^{(1)} \equiv \psi^{(1)}(f-a)+\psi f^{(1)}. \tag{3.3}$$

Since f is nonconstant and $L^{(1)} \equiv f^{(1)}$, from (3.2) we have $\psi^{(1)} \equiv 0$ and so ψ is a constant.

First, we suppose that $a+d=0$. Then

$$\psi = \frac{L-L^{(1)}}{f-a} = 1-\frac{L^{(1)}}{f-a} = 1-\frac{f^{(1)}}{f-a}$$

and so $f^{(1)}/(f-a) = 1-\psi = c$, say, a constant. Integrating, $f = a+Ke^{cz}$, where $K \neq 0$ is a constant. Since f is nonconstant, we see that $c \neq 0$. Now

$$L^{(1)} = \sum_{k=1}^{n} a_k f^{(k+1)} = \Big(\sum_{k=1}^{n} a_k c^k\Big) f^{(1)} = \Big(\sum_{k=1}^{n} a_k c^k\Big) L^{(1)}.$$

Since $L^{(1)} \equiv f^{(1)} \not\equiv 0$, we get $\sum_{k=1}^{n} a_k c^k = 1$. So,

$$L = \sum_{k=1}^{n} a_k f^{(k)} = \Big(\sum_{k=1}^{n} a_k c^k\Big) Ke^{cz} = Ke^{cz} \quad \text{and} \quad L^{(1)} \equiv f^{(1)} \equiv Kce^{cz}.$$

Since $\overline{E}(a; f) = \emptyset$, we have, in view of (2.15),

$$N(r,a;f^{(1)}) \le N_B(r,a;f^{(1)}) + N_C(r,a;f^{(1)}) + N_D(r,a;f^{(1)}) = S(r,f),$$

where C and D are respectively given by (2.14) and (2.19); this is a contradiction. Therefore, $a+d \neq 0$.

Now

$$\frac{1}{f-a} = \frac{1}{a+d}\Big(\frac{f+d}{f-a}-1\Big) = \frac{1}{a+d}\Big(\frac{L}{f-a}-1\Big)$$

implies that $m(r,a;f) = S(r,f)$. So, by Lemma 2.1, $L = \alpha e^z$, where $\alpha \neq 0$ is a constant. This contradicts our assumption that $\psi \not\equiv 0$. Therefore, indeed, $\psi \equiv 0$ and so $L \equiv L^{(1)}$. Hence, $L = \alpha e^z$, where $\alpha \neq 0$ is a constant.

If $N(r,a;f) \neq S(r,f)$, by the hypotheses, we get $d=0$ and so $f \equiv L$. Hence, $f = L = \alpha e^z$.

Let $N(r,a;f) = S(r,f)$. Since $f = L-d = \alpha e^z - d$, we get $d = -a$. Therefore, $f = a+\alpha e^z$. This proves the theorem.

4. An open question

Is it possible to replace the hypothesis (i) of Theorem 1.1 by $\overline{N}_A(r,a;f) + \overline{N}_B(r,a;f^{(1)}) = S(r,f)$?

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INDRAJIT LAHIRI, Department of Mathematics, University of Kalyani,
West Bengal 741235, India
e-mail: ilahiri@hotmail.com

RAJIB MUKHERJEE, Department of Mathematics, Krishnanath College,
Baharampur, West Bengal 742101, India
e-mail: rajib_raju786@yahoo.com

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