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HIGH-ORDER UPWIND FINITE VOLUME ELEMENT METHOD FOR FIRST-ORDER HYPERBOLIC OPTIMAL CONTROL PROBLEMS

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Abstract

We present a high-order upwind finite volume element method to solve optimal control problems governed by first-order hyperbolic equations. The method is efficient and easy for implementation. Both the semi-discrete error estimates and the fully discrete error estimates are derived. Optimal order error estimates in the sense of L^2-norm are obtained. Numerical examples are provided to confirm the effectiveness of the method and the theoretical results.

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1. Introduction

We consider the following linear–quadratic optimal control problem [9] for the state y and the control u:

$$\begin{aligned}
&\text{minimize } J(y,u) = \frac{1}{2}\int_0^T \int_I (y - y_d)^2\,dx\,dt + \frac{\alpha}{2}\int_0^T \int_I u^2\,dx\,dt \\
&\text{subject to } \frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = f + u, \quad (x,t) \in I \times [0,T], \\
&\qquad\qquad y(x,0) = \phi(x), \quad x \in I,
\end{aligned}$$

together with some boundary conditions. Here $I = [0, L]$, α is a fixed positive number, y_d is a given function and a, f and ϕ are smooth functions. Control u is within the constraints $u_a \le u \le u_b$, where $u_a < u_b$ are given constants. The optimal

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control problems governed by hyperbolic partial differential equations (PDEs) usually describe the phenomenon where diffusive transport is negligible and convective transport is dominant (see [4]). This class of problems has many applications in different areas, such as aerodynamics, conservation laws, open-channel flow, manufacturing flow and many others.

The numerical treatment of PDE-constrained optimal control problems is playing an increasingly important role in science and engineering. There has been much work on the elliptic control problems. Many numerical methods have been applied to approximate the solutions to these problems; see, for example, [2, 3] and the references therein. We can also find some work on the parabolic optimal control problems [12, 19]. In addition, optimal control of a hyperbolic PDE is an important problem and has attracted a lot of attention in the control literature (see [11, 15]).

The goal of this paper is to propose a high-order upwind finite volume element method (HUFVEM) for the first-order hyperbolic optimal control problem and give optimal order error estimates in the sense of an L^2-norm. Although for elliptic and parabolic equations the theory on error estimates for numerical approximation schemes is quite far developed, very little has been done on numerical methods for optimal control problems for hyperbolic equations. We first apply the Lagrange multiplier method [6, 7] to obtain the optimality conditions at the continuous level consisting of the state equation, the adjoint equation and a variational inequality. Then a variational discretization concept proposed by Hinze [5] is used. With the variational discretization concept, the control space is discretized implicitly by the state equation. Since the classical upwind schemes [8] and the Lax–Friedrichs schemes [13] for first-order hyperbolic PDEs usually give the smeared solution, we propose the HUFVEM to discretize the state equation. The finite volume element method [8] has been proposed as an important numerical tool for solving PDEs. Its accuracy is higher than the finite-difference method, and nearly the same as the finite-element method, while the computational effort is less than that of the finite-element method. So, it has been widely used in computational fluid dynamics [17, 18].

For the discretization error between the solution of the continuous optimization problem $(u^*, y^*) \in L^2(0, T; U_{ad}) \times L^2(0, T; H^1(I)) \cap C(0, T; L^2(I))$ and the solution of the discretized problem $(u_h^*, y_h^*) \in L^2(0, T; U_{ad}) \times L^2(0, T; U_h)$, we will prove both the semi-discrete error estimate of the type

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(I))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(I))} \leq Ch^{3/2}$$

and the fully discrete error estimate of the type

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(I))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(I))} \leq C(\Delta t + h^{3/2}),$$

where u^* and y^* denote the control and the state with their approximations u_h^* and y_h^*, respectively; h and Δt denote the spatial and temporal grid sizes, respectively; U_h is a piecewise linear polynomial function space; and U_{ad} is the set of admissible controls. We need a regularity assumption to prove these estimates (see Theorems 4.4 and 4.7).

The remainder of this paper is organized as follows. In Section 2, we introduce the model problem and give optimality conditions of the problem. In Section 3, we discretize the optimal control problem based on the variational discretization concept and the HUFVEM. Some a priori error estimates are derived in Section 4. In Section 5, numerical examples are provided to confirm the effectiveness of our numerical method and our theoretical results. Some conclusions are given in Section 6.

2. Model problem and optimality conditions

Denote by ∂I the boundary of I. Corresponding to the function a, we divide ∂I into two parts

$$(\partial I)_- = \{x \in \partial I : an \leq 0\}, \tag{2.1}$$

$$(\partial I)_+ = \{x \in \partial I : an \geq 0\}, \tag{2.2}$$

where n is the unit outer normal vector of ∂I.

Let us consider the following first-order hyperbolic PDE system:

$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = f + u, \quad (x,t) \in I \times (0,T], \tag{2.3}$$

$$y(x,t) = \hat{y}(t), \quad (x,t) \in (\partial I)_- \times (0,T], \tag{2.4}$$

$$y(x,0) = \phi(x), \quad x \in I, \tag{2.5}$$

where $\hat{y}(t)$ is a smooth function.

For $y(\cdot,t), v(\cdot,t) \in H^1(I)$, we use Green's formula and write (2.3) in an integral form:

$$(y_t, v) - (y, (av)_x) + (anyv)|_{\partial I} = (f + u, v), \tag{2.6}$$

where $y_t = \partial y/\partial t$ and $(y, v) = \int_I yv\,dx$.

By the transposition techniques (see Lions et al. [10]), the problem (2.3)–(2.5) admits a unique solution $y \in L^2(0,T; H^1(I)) \cap C(0,T; L^2(I))$ in the sense that

$$-\int_0^T \Big(y, \frac{\partial v}{\partial t}\Big)\,dt - \int_0^T \Big(y, \frac{\partial(av)}{\partial x}\Big)\,dt + \int_0^T (anyv)|_{\partial I}\,dt = \int_0^T (f + u, v)\,dt \tag{2.7}$$

for all $v \in V$, where $u \in L^2(0,T; L^2(I))$ and

$$V = \Big\{v \in L^2(0,T; H^1(I)) \cap H^1(0,T; L^2(I)) : \frac{\partial v}{\partial t} \in L^2(0,T; L^2(I)),\ v(T) = 0\Big\}.$$

Problem 2.1. Consider the optimal control problem of minimizing

$$J(y,u) = \frac{1}{2}\int_0^T \int_I (y - y_d)^2\,dx\,dt + \frac{\alpha}{2}\int_0^T \int_I u^2\,dx\,dt$$

over all $(y,u) \in L^2(0,T; H^1(I)) \cap C(0,T; L^2(I)) \times L^2(0,T; L^2(I))$, subject to the hyperbolic system (2.3)–(2.5) and the control constraints

$$u_a \leq u \leq u_b.$$

Here $y_d \in L^2(0, T; L^2(I))$. The set of admissible controls for Problem 2.1 can be written as

$$U_{ad} = \{u \in L^2(0, T; L^2(I)) \mid u_a \le u \le u_b \text{ almost everywhere in } (0, T]\}.$$

Since the problem is linear–quadratic and convex, by applying standard techniques [6], we have the results for the existence of solutions and optimality conditions.

THEOREM 2.2. *Problem 2.1 admits a unique optimal control u^* with an associated state y^* and an adjoint state p^* that satisfy the state equation*

$$\Big(\frac{\partial y^*}{\partial t}, v\Big) - \Big(y^*, \frac{\partial (av)}{\partial x}\Big) + (any^* v)|_{\partial I} = (f + u^*, v) \quad \textit{for all } v \in H^1(I), \tag{2.8}$$

the adjoint equation

$$-\Big(\frac{\partial p^*}{\partial t}, v\Big) + \Big(p^*, \frac{\partial (av)}{\partial x}\Big) - (anp^* v)|_{\partial I} = (y^* - y_d, v) \quad \textit{for all } v \in H^1(I) \tag{2.9}$$

and the variational inequality

$$\int_0^T (\alpha u^* + p^*, w - u^*)\, dt \ge 0 \quad \textit{for all } w \in U_{ad}. \tag{2.10}$$

Moreover, the variational inequality is equivalent to

$$u^* = \mathcal{P}_{[u_a, u_b]}\Big(-\frac{1}{\alpha} p^*\Big),$$

where $\mathcal{P}_{[u_a,u_b]}(v)$ denotes the projection of $v \in \mathbb{R}$ onto the interval $[u_a, u_b]$.

The adjoint equation (2.9) is the weak form of the following hyperbolic PDE that runs backwards in time:

$$\begin{aligned} -\tfrac{\partial p^*}{\partial t} - a\tfrac{\partial p^*}{\partial x} &= y^* - y_d, \quad (x, t) \in I \times [0, T], \\ p^*(x, t) &= 0, \quad (x, t) \in (\partial I)_- \times [0, T], \\ p^*(x, T) &= 0, \quad x \in I. \end{aligned}$$

3. Discretization and error estimates

Since Problem 2.1 is an optimal control problem in infinite-dimensional space for which it is not possible to compute the solution on a computer, we have to discretize Problem 2.1 to get a finite-dimensional problem and to apply some numerical methods to solve it. We use the variational discretization concept [5] where the state is a finite volume element approximation of the state equation (2.3)–(2.5). In order to get more accuracy without grid refinement, we use the HUFVEM, which is described in the next subsection.

3.1. High-order upwind finite volume element method In this section, we will design the HUFVEM for the state equation (2.3)–(2.5). Take a mesh size h and nodes $x_j = jh$, $j = 1, 2, \ldots, N$; then we have a uniform grid T_h

$$0 = x_0 < x_1 < \cdots < x_N = L.$$

We place a dual grid T_h^* corresponding to T_h with nodes

$$0 = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_N = L,$$

where $x_{j+1/2} = (j + 1/2)h$, $j = 1, 2, \ldots, N - 1$. Write $I_0^* = [x_0, x_{1/2}]$, $I_j^* = [x_{j-1/2}, x_{j+1/2}]$, $j = 1, 2, \ldots, N - 1$ and $I_N^* = [x_{N-1/2}, x_N]$.

Denote by P_r a class of all polynomials with degree $\leq r$, and the trial function space

$$U_h = \{v_h : v_h|_{I^*} \in P_r \text{ for all } I^* \in T_h^*\}.$$

In the following, we will take U_h as a piecewise polynomial function space defined on the dual elements. Therefore, we seek $y_h \in U_h$ such that

$$\int_{I_j^*} \left[\frac{\partial y_h}{\partial t} + a\frac{\partial y_h}{\partial x}\right] v_h \, dx = \int_{I_j^*} (f + u) v_h \, dx \quad \text{for all } v_h \in U_h.$$

Employing Green's formula yields

$$\int_{I_j^*} a\frac{\partial y_h}{\partial x} v_h \, dx = -\int_{I_j^*} y_h \frac{\partial (a v_h)}{\partial x} \, dx + (a\nu y_h v_h)|_{\partial I_j^*}; \tag{3.1}$$

then we write equation (2.3) as

$$\int_{I_j^*} \frac{\partial y_h}{\partial t} v_h \, dx - \int_{I_j^*} y_h \frac{\partial (a v_h)}{\partial x} \, dx + (a\nu y_h v_h)|_{\partial I_j^*} = \int_{I_j^*} (f + u) v_h \, dx,$$

where ν is the unit outer normal vector of ∂I_j^*. Similar to (2.1) and (2.2), $(\partial I_j^*)_-$ and $(\partial I_j^*)_+$ can be defined. For $x \in \partial I_j^*$, the upwind and the downwind values of y_h at $x \in \partial I_j^*$ are defined as follows:

$$y_h^+(x) = \begin{cases} \lim\limits_{x' \to x} y_h(x') & \text{when } x \in (\partial I_j^*)_-, x' \notin I_j^*, \\ \lim\limits_{x' \to x} y_h(x') & \text{when } x \in (\partial I_j^*)_+, x' \in I_j^*, \end{cases}$$

$$y_h^-(x) = \begin{cases} \lim\limits_{x' \to x} y_h(x') & \text{when } x \in (\partial I_j^*)_-, x' \in I_j^*, \\ \lim\limits_{x' \to x} y_h(x') & \text{when } x \in (\partial I_j^*)_+, x' \notin I_j^*. \end{cases}$$

In analogy with the classical upwind scheme, y_h on the boundary of dual elements can be replaced by y_h^+ to obtain

$$\int_{I_j^*} \frac{\partial y_h}{\partial t} v_h \, dx - \int_{I_j^*} y_h \frac{\partial (a v_h)}{\partial x} \, dx + (a\nu y_h^+ v_h)|_{\partial I_j^*} = \int_{I_j^*} (f + u) v_h \, dx. \tag{3.2}$$

It follows from equation (3.1) that

$$-\int_{I_j^*} y_h \frac{\partial(a v_h)}{\partial x}\,dx = \int_{I_j^*} a\frac{\partial y_h}{\partial t}\,dx - (a v y_h^+ v_h)|_{(\partial I_j^*)_+} - (a v y_h^- v_h)|_{(\partial I_j^*)_-}.$$

Substituting it in equation (3.2) yields a semi-discrete upwind scheme

$$\int_{I_j^*} \frac{\partial y_h}{\partial t} v_h\,dx + \int_{I_j^*} a\frac{\partial y_h}{\partial x} v_h\,dx + av[y_h]v_h|_{(\partial I_j^*)_-} = \int_{I_j^*} (f+u)v_h\,dx, \tag{3.3}$$

where $[y_h^+] = y_h^+ - y_h^-$ is the jump of y_h across $(\partial I_j^*)_-$. If U_h consists of step functions, a classical upwind scheme can be derived from the above equation. To obtain highly accurate upwind schemes, piecewise high degree polynomials are taken. In the following, we use piecewise linear and constant functions to derive the upwind finite volume element schemes.

Let the trial function space U_h be composed of two groups of functions of which the first group is $\{\psi_j^0\}$ given by

$$\psi_j^0(x) = \begin{cases} 1, & x \in [x_{j-1/2}, x_{j+1/2}], \\ 0, & \text{elsewhere} \end{cases}$$

and the other one is $\{\psi_j^1\}$ defined by

$$\psi_j^1(x) = \begin{cases} x - x_j, & x \in [x_{j-1/2}, x_{j+1/2}], \\ 0, & \text{elsewhere} \end{cases}$$

for $j = 1, 2, \ldots, N-1$. Obviously, any $y_h \in U_h$ has the expression

$$y_h = y_h(x,t) = \sum_{j=1}^{N} [y_{0j}(t)\psi_j^0(x) + y_{1j}(t)\psi_j^1(x)],$$

where $y_{0j} = y_h(x_j, t),\ y_{1j} = y_h'(x_j, t)$.

Substitute y_h into (3.3) and take $v_h = \psi_j^0$ and ψ_j^1; then we have the following semi-discrete high-order upwind finite volume element schemes:

$$\begin{cases} h\dfrac{\partial y_{0j}}{\partial t} + a_j(y_{0j} - y_{0j-1}) + \dfrac{a_j h}{2}(y_{1j} - y_{1j-1}) = \displaystyle\int_{I_j^*} (f+u)\psi_j^0\,dx & \text{for } a_j \geq 0, \\ h\dfrac{\partial y_{0j}}{\partial t} + a_j(y_{0j+1} - y_{0j}) + \dfrac{a_j h}{2}(y_{1j} - y_{1j+1}) = \displaystyle\int_{I_j^*} (f+u)\psi_j^0\,dx & \text{for } a_j \leq 0 \end{cases}$$

and

$$\begin{cases} \dfrac{h^3}{12}\dfrac{\partial y_{1j}}{\partial t} - \dfrac{a_j h}{2}(y_{0j} - y_{0j-1}) + \dfrac{a_j h^2}{4}(y_{1j} + y_{1j-1}) = \displaystyle\int_{I_j^*} (f+u)\psi_j^1\,dx & \text{for } a_j \geq 0, \\ \dfrac{h^3}{12}\dfrac{\partial y_{1j}}{\partial t} + \dfrac{a_j h}{2}(y_{0j+1} - y_{0j}) - \dfrac{a_j h^2}{4}(y_{1j+1} + y_{1j}) = \displaystyle\int_{I_j^*} (f+u)\psi_j^1\,dx & \text{for } a_j \leq 0, \end{cases}$$

where $a_j = a(x_j, \cdot)$. The numerical integrations needed in the computation are approximated using the three-point Gaussian quadrature formula.

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3.2. Discretization of the optimal control problem We are now ready to define the discrete problem 3.1 using the variational discretization concept introduced by Hinze [5].

PROBLEM 3.1. Consider the minimizing problem

$$J_h(y_h, u) = \frac{1}{2}\int_0^T \int_I (y_h - y_d)^2\, dx\, dt + \frac{\alpha}{2}\int_0^T \int_I u^2\, dx\, dt$$

over all $(y_h, u) \in L^2(0, T; U_h) \times L^2(0, T; L^2(I))$ subject to

$$\Big(\frac{\partial y_h}{\partial t}, v_h\Big) - \Big(y_h, \frac{\partial (a v_h)}{\partial x}\Big) + (any_h v_h)|_{\partial I} = (f + u, v_h) \quad \text{for all } v_h \in U_h$$

and the control constraints

$$u_a \le u \le u_b.$$

Similar to Theorem 2.2, we have the following theorem for Problem 3.1.

THEOREM 3.2. *Problem 3.1 has a unique solution $u_h^*(\cdot, t) \in U_{ad}$ with associated state $y_h^*(\cdot, t) \in U_h$ and adjoint state $p_h^*(\cdot, t) \in U_h$ that satisfies the state equation*

$$\Big(\frac{\partial y_h^*}{\partial t}, v_h\Big) - \Big(y_h^*, \frac{\partial (a v_h)}{\partial x}\Big) + (any_h^* v_h)|_{\partial I} = (f + u_h^*, v_h) \quad \textit{for all } v_h \in U_h,$$

the adjoint equation

$$-\Big(\frac{\partial p_h^*}{\partial t}, v_h\Big) + \Big(p_h^*, \frac{\partial (a v_h)}{\partial x}\Big) - (anp_h^* v_h)|_{\partial I} = (y_h^* - y_d, v_h) \quad \textit{for all } v_h \in U_h$$

and the projection equation

$$u_h^* = \mathcal{P}_{[u_a, u_b]}\Big(-\frac{1}{\alpha} p_h^*\Big).$$

Moreover, the projection equation is equivalent to the variational inequality

$$\int_0^T (\alpha u_h^* + p_h^*, w - u_h^*)\, dt \ge 0 \quad \textit{for all } w \in U_{ad}. \tag{3.4}$$

From Theorem 3.2, we see that the control is implicitly discretized by projecting a discrete adjoint state onto U_{ad}.

4. Error analysis

To get error estimates between Problems 2.1 and 3.1, we introduce the auxiliary functions $\widetilde{y}(\cdot, t) \in H^1(I)$ and $\widetilde{p}(\cdot, t) \in H^1(I)$, which are solutions of the following problems:

$$\Big(\frac{\partial \widetilde{y}}{\partial t}, v\Big) - \Big(\widetilde{y}, \frac{\partial (a v)}{\partial x}\Big) + (an\widetilde{y} v)|_{\partial I} = (f + u_h^*, v) \quad \text{for all } v \in H^1(I), \tag{4.1}$$

$$-\Big(\frac{\partial \widetilde{p}}{\partial t}, v\Big) + \Big(\widetilde{p}, \frac{\partial (a v)}{\partial x}\Big) - (an\widetilde{p} v)|_{\partial I} = (y_h^* - y_d, v) \quad \text{for all } v \in H^1(I), \tag{4.2}$$

where u_h^* and y_h^* are the discrete upwind finite volume element approximations of u^* and y^*, respectively. Note that y^* and $\widetilde{y}$ are solutions of (2.6) with u^* and u_h^* replacing u, respectively, and p^* in (2.9) differs from $\widetilde{p}$ in (4.2) on the input of the right-hand side.

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4.1. Error estimates for the semi-discrete schemes First, we present the L^2 error estimates for the HUFVEM schemes. To this end, we define a bilinear form

$$a(u,v) = \sum_{j=1}^{N}\Big[\int_{I_j^*}\Big(a\frac{\partial u}{\partial x}\Big)v\,dx + (av[u]v)|_{(\partial I_j^*)_-}\Big].$$

For the error estimations later on, we need some preliminary results, which are presented below.

LEMMA 4.1. *For $a(\cdot,\cdot)$ defined above,*

$$a(v_h,v_h) \geq \gamma_0(\|v_h\|_0^2 + \|v_h\|_{\partial I}^2) \quad \textit{for all } v_h \in U_h,$$

where $\gamma_0 = \min(\sigma_0, 1/2)$, $-(\partial a/\partial x)/2 \geq \sigma_0 > 0$ and $\|v_h\|_0^2 = (v_h, v_h)_I$.

PROOF. By means of Green's formula, and since $\partial(av_h)/\partial x = v_h(\partial a/\partial x) + a(\partial v_h/\partial x)$,

$$(avv_h^2)|_{\partial I_j^*} = 2\int_{I_j^*} v_h\Big(a\frac{\partial v_h}{\partial x}\Big)dx + \int_{I_j^*} v_h^2\frac{\partial a}{\partial x}\,dx.$$

Therefore,

$$\begin{aligned}
a(v_h,v_h) &= \sum_{j}^{N}\Big[\frac{1}{2}av_xv_h^2|_{\partial I_j^*} - \frac{1}{2}\int_{I_i^*}\frac{\partial a}{\partial x}v_h^2\,dx + av[v_h]v_h|_{(\partial I_j^*)_-}\Big]\\
&= \sum_{j}^{N}\Big[\frac{1}{2}av_xv_h^2|_{\partial I_j^*} + \frac{1}{2}av_xv_h^2|_{(\partial I_j^*)_-} + av(v_h^+ - v_h^-)v_h|_{(\partial I_j^*)_-}\\
&\quad + \frac{1}{2}av\{(v_h^+)^2 - 2v_h^+v_h^- + (v_h^-)^2\}|_{(\partial I_j^*)_-} - \frac{1}{2}av[v_h]^2|_{(\partial I_j^*)_-} - \frac{1}{2}\int_{I_j^*}\frac{\partial a}{\partial x}v_h^2\,dx\Big].
\end{aligned}$$

Note that

$$v_h|_{(\partial I_j^*)_+} = v_h^+|_{(\partial I_j^*)}, \quad v_h|_{(\partial I_j^*)_-} = v_h^-|_{(\partial I_j^*)}.$$

Thus,

$$a(v_h,v_h) = \frac{1}{2}\sum_{j}^{N}\Big[av(v_h^+)^2|_{(\partial I_j^*)_-} + av(v_h^+)^2|_{(\partial I_j^*)_+} - av[v_h]^2|_{(\partial I_j^*)_-} - \int_{I_j^*}\frac{\partial a}{\partial x}v_h^2\,dx\Big].$$

But $v_h^+|_{(\partial I)_-} = 0$, so $a(v_h,v_h)$ is positive definite and

$$a(v_h,v_h) \geq \gamma_0(\|v_h\|_0^2 + \|v_h\|_{\partial I}^2) \quad \text{for all } v_h \in U_h,$$

where $\gamma_0 = \min(\sigma_0, 1/2)$, $-(\partial a/\partial x)/2 \geq \sigma_0 > 0$, $\|v_h\|_0^2 = (v_h, v_h)_I$ and

$$\|v_h\|_{\partial I}^2 = \sum_{j}^{N}|av|[v_h^2]|_{(\partial I_j^*)_-} + |av|(v_h^+)^2|_{(\partial I)_+}. \qquad \square$$

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Let us define,for $y \in H^2(I)$, the Ritz projection [8] $R_h y \in U_h$ determined by the equation

$$a(R_h y, v_h) = a(y, v_h) \quad \text{for all } v_h \in U_h. \tag{4.3}$$

Since $a(y, v_h)$ is positive definite, the Ritz projection exists and is unique.

LEMMA 4.2. *For $y \in H^2(I)$, the following estimate holds:* [8]

$$|||y - R_h y||| \le C h^{3/2} \|y\|_2, \tag{4.4}$$

where $|||\cdot|||$ is defined by

$$|||v||| = \|v\|_0^2 + \|v\|_{\partial I} + h \sum_{j=1}^{N} \int_{I_j^*} \Big(a \frac{\partial v}{\partial x}\Big)^2 \, dx.$$

LEMMA 4.3. *Let y and y_h be the solutions to equations* (2.6) *and* (3.3)*, respectively. Assume that $y(\cdot, t) \in H^2(I)$, $y_t(\cdot, t) \in H^2(I)$, $y(\cdot, 0) = \phi(x) \in H^2(I)$ and $y_h(\cdot, 0) = \phi_h(x) \in U_h$. Then the following error estimate holds:*

$$\|y - y_h\|_0 \le C\Big\{\|\phi - \phi_h\|_0 + h^{3/2}\Big[\|\phi\|_2 + \int_0^T \|y_t\|_2 dt\Big]\Big\}. \tag{4.5}$$

PROOF. Write $\rho = R_h y - y$, $e = y_h - R_h y$, where R_h is the Ritz projection defined above. Then

$$y_h - y = \rho + e.$$

It follows from (4.4) that

$$\begin{aligned} \|\rho\|_0 &\le C h^{3/2} \|y\|_2 = C h^{3/2} \Big\| y(\cdot, 0) + \int_0^t y_t \, dt \Big\|_2 \\ &\le C h^{3/2} \Big[\|\phi\|_2 + \int_0^t \|y_t\|_2 \, dt\Big]. \end{aligned} \tag{4.6}$$

Next we deal with e. Since y and y_h satisfy (2.6) and (3.3), respectively,

$$\Big(\frac{\partial y}{\partial t} - \frac{\partial y_h}{\partial t}, v_h\Big) + a(y - y_h, v_h) = 0 \quad \text{for all } v_h \in U_h.$$

This together with (4.3) yields

$$(e_t, v_h) + a(y - y_h, v_h) = -(\rho_t, v_h),$$

which means $a(e, v_h) = (y_t - R_h y_t, v_h)$. Hence,

$$(e_t, e) + a(e, e) = (y_t - R_h y_t, e). \tag{4.7}$$

Notice that $e^+|_{(\partial I)_-} = 0$ and $a(e, e) \ge 0$, where $e^+ = e|_{(\partial I)_+}$. Then, by equation (4.7),

$$\frac{d}{dt}\|e\|_0 \le C\|\rho_t\|_0,$$

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which, by integrating with respect to t, yields

$$\|e\|_0 \le C\Big[\|e(\cdot,0)\|_0 + \int_0^t \|\rho_t\|_0\,dt\Big]. \tag{4.8}$$

By virtue of Lemma 4.2,

$$\begin{aligned}\|e(\cdot,0)\|_0 &\le \|R_h y(\cdot,0) - y(\cdot,0)\|_0 + \|y(\cdot,0) - y_h(\cdot,0)\|_0\\ &\le Ch^{3/2}\|y(\cdot,0)\|_2 + \|y(\cdot,0) - y_h(\cdot,0)\|_0\\ &= Ch^{3/2}\|\phi\|_2 + \|\phi - \phi_h\|_2\end{aligned}$$

and

$$\|\rho_t\|_0 = \|y_t - R_h y_t\|_0 \le Ch^{3/2}\|y_t\|_2. \tag{4.9}$$

A combination of (4.6) and (4.8)–(4.9) leads to (4.5). This completes the proof of the lemma. □

Now we are ready to calculate the error estimates for the solutions of the continuous and discretized optimal control problems.

THEOREM 4.4. *Let* $(u^*, y^*, p^*) \in L^2(0,T;U_{ad}) \times L^2(0,T;H^1(I)) \cap C(0,T;L^2(I)) \times L^2(0,T;H^1(I)) \cap H^1(0,T;L^2(I))$ *and* $(u_h^*, y_h^*, p_h^*) \in L^2(0,T;U_{ad}) \times L^2(0,T;U_h) \times L^2(0,T;U_h)$ *be the solutions of Problems 2.1 and 3.1, respectively, and* $\widetilde{y}$ *and* $\widetilde{p}$ *be the solutions of problems (4.1) and (4.2), respectively. Assume that* $\widetilde{y}(\cdot,t) \in H^2(I)$, $\widetilde{y}_t(\cdot,t) \in H^2(I)$, $\widetilde{y}(\cdot,0) \in H^2(I)$, $y_h^*(\cdot,0) \in U_h$ *and* $\widetilde{p}(\cdot,t) \in H^2(I)$, $\widetilde{p}_t(\cdot,t) \in H^2(I)$, $\widetilde{p}(\cdot,0) \in H^2(I)$, $p_h^*(\cdot,0) \in U_h$. *Then there exists a constant* $C > 0$, *independent of* h, *such that*

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(I))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(I))} \le Ch^{3/2}.$$

PROOF. We follow the idea of the proof presented by Hinze [5]. Testing (2.10) with u_h^* and (3.4) with u^*, and summing up these inequalities,

$$\int_0^T (\alpha(u^* - u_h^*) + (p^* - p_h^*), u_h^* - u^*)\,dt \ge 0.$$

This leads to

$$\begin{aligned}\alpha\|u^* - u_h^*\|^2_{L^2(0,T;L^2(I))} &\le \int_0^T (p^* - p_h^*, u_h^* - u^*)\,dt\\ &\le \int_0^T (p^* - \widetilde{p}, u_h^* - u^*)\,dt + \int_0^T (\widetilde{p} - p_h^*, u_h^* - u^*)\,dt.\end{aligned}$$

Following (2.7), (2.8) and (4.1),

$$-\int_0^T (y^* - \widetilde{y}, v_t)\,dt - \int_0^T \Big(y^* - \widetilde{y}, \frac{\partial(av)}{\partial x}\Big)dt + \int_0^T [av(y^* - \widetilde{y})]|_{\partial I}\,dt = \int_0^T (u^* - u_h^*, v)\,dt. \tag{4.10}$$

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Setting $v = \widetilde{p} - p^*$ in (4.10) yields

$$\begin{aligned}\int_0^T (\widetilde{p} - p^*, u^* - u_h^*)\,dt &= -\int_0^T (\widetilde{y} - y^*, \widetilde{p}_t - p_t^*)\,dt - \int_0^T \Big(\widetilde{y} - y^*, \frac{\partial(a\widetilde{p})}{\partial x} - \frac{\partial(ap^*)}{\partial x}\Big)\,dt \\ &\quad + \int_0^T [av(y^* - \widetilde{y})]|_{\partial I}\,dt \\ &= \int_0^T (y_h^* - y_d, \widetilde{y} - y^*)\,dt - \int_0^T (y^* - y_d, \widetilde{y} - y^*)\,dt \\ &= \int_0^T (y_h^* - y^*, \widetilde{y} - y^*)\,dt \\ &= -\int_0^T \|y_h^* - y^*\|_0^2\,dt + \int_0^T (y_h^* - y^*, y_h^* - \widetilde{y})\,dt \\ &\le -\frac{1}{2}\|y_h^* - y^*\|^2_{L^2(0,T;L^2(I))} + \frac{1}{2}\|y_h^* - \widetilde{y}\|^2_{L^2(0,T;L^2(I))}.\end{aligned}$$

Using Young's inequality [1] gives

$$\int_0^T (\widetilde{p} - p_h^*, u_h^* - u^*)\,dt \le \frac{1}{2}\alpha\|u_h^* - u^*\|^2_{L^2(0,T;L^2(I))} + C(\alpha)\|\widetilde{p} - p_h^*\|^2_{L^2(0,T;L^2(I))}.$$

Therefore,

$$\begin{aligned}&\alpha\|u^* - u_h^*\|^2_{L^2(0,T;L^2(I))} + \|y_h^* - y^*\|^2_{L^2(0,T;L^2(I))} \\ &\qquad \le C(\|y_h^* - \widetilde{y}\|^2_{L^2(0,T;L^2(I))} + \|\widetilde{p} - p_h^*\|^2_{L^2(0,T;L^2(I))}).\end{aligned} \tag{4.11}$$

From Lemma 4.3, there exists a constant $C > 0$, independent of h, such that

$$\|\widetilde{y} - y_h^*\|_0 \le Ch^{3/2}, \quad \|\widetilde{p} - p_h^*\|_0 \le Ch^{3/2}. \tag{4.12}$$

Combining (4.11) and (4.12),

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(I))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(I))} \le Ch^{3/2}. \qquad \square$$

Remark 4.5. We need a regularity assumption to prove the estimate in Theorem 4.4. From (4.1), we obtain $-a(\partial\widetilde{y}/\partial x) = f + u_h^* - \partial\widetilde{y}/\partial t$. We assume that f on the right-hand side and the initial and boundary conditions of (4.1) are smooth enough. In the one-dimensional case, if $u_h^*(\cdot, t)$ is Lipschitz continuous, and $u_h^*(\cdot, t)$ is piecewise linear, then $u_h^*(\cdot, t) \in H^1(I)$ [16, Theorem 4.1.1]. Also, $\widetilde{y}(\cdot, t) \in H^1(I)$ and $\widetilde{y}$ is continuously differentiable with respect to time, so we assume that $\widetilde{y}(\cdot, t) \in H^2(I)$ [8, Ch. 6.2].

4.2. Error estimates for the fully discrete schemes

Using backward differencing on the time direction, (3.3) can be written as

$$\int_I \frac{y_h^n - y_h^{n-1}}{\Delta t} v_h\,dx + a(y_h^n, v_h) = \int_I (f^n + u^n) v_h\,dx \quad \text{for all } v_h \in U_h, \tag{4.13}$$

where $y_h^n = y_h(x, t_n)$, $f^n = f(x, t_n)$ and $t_n = n\Delta t$.

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LEMMA 4.6. *Let y and y_h^n be the solutions to equations (2.6) and (4.13), respectively. Assume that $y(\cdot,t) \in H^2(I)$, $y_t(\cdot,t) \in H^2(I)$, $y_{tt}(\cdot,t) \in L^2(I)$, $y(\cdot,0) = \phi(x) \in H^2(I)$ and $y_h^0 = \phi_h(x) \in U_h$. Then the following error estimate holds:*

$$\|y(t_n) - y_h^n\|_0 \le \|\phi - \phi_h\|_0 + Ch^{3/2}\|\phi\|_2 + \Delta t \int_0^{t_n} \|y_{tt}\|_0 \, dt + Ch^{3/2} \int_0^{t_n} \|y_t(t)\|_2 \, dt.$$

PROOF. Write $y_h^n - y(t_n) = \rho^n + e^n$, where $\rho^n = R_h y(t_n) - y(t_n)$, $e^n = y_h^n - R_h y(t_n)$ and $y(\cdot, t_n)$ is written as $y(t_n)$. According to equation (4.4),

$$\|\rho^n\|_0 \le Ch^{3/2}\|y(t_n)\|_2.$$

Also, observe that

$$y(t_n) = y(0) + \int_0^{t_n} y_t(t)\, dt \quad \text{and} \quad \|y(t_n)\|_2 \le \|y(0)\|_2 + \int_0^{t_n} \|y_t(t)\|_2 \, dt.$$

Thus,

$$\|\rho^n\|_0 \le Ch^{3/2}\Big(\|\phi\|_2 + \int_0^{t_n} \|y_t(t)\|_2 \, dt\Big). \tag{4.14}$$

Now we begin to deal with e^n. Denote $\overline{\partial}_t y_h^n = (y_h^n - y_h^{n-1})/\Delta t$. It follows from equation (4.13) and the definition of R_h that

$$\begin{aligned} a(e^n, v_h) &= a(y_h^n - R_h y(t_n), v_h) - (\overline{\partial}_t y_h^n, v_h) + (f^n + u^n, v_h) - a(y(t_n), v_h) \\ &= (y_t(t_n) - \overline{\partial}_t y_h^n, v_h). \end{aligned}$$

Therefore,

$$(\overline{\partial}_t e^n, e^n) + a(e^n, e^n) = (y_t(t_n) - R_h \overline{\partial}_t y(t_n), e^n) = (\omega_1^n + \omega_2^n, e^n), \tag{4.15}$$

where $\omega_1^n = y_t(t_n) - \overline{\partial}_t y(t_n)$, $\omega_1^n = \overline{\partial}_t y(t_n) - R_h \overline{\partial}_t y(t_n)$. Notice that

$$a(e^n, e^n) \ge 0, \quad (e^n)^+|_{(\partial I)_-} = 0.$$

Therefore, by equation (4.15),

$$\|e^n\|_0 \le \|e^{n-1}\|_0 + \Delta t \|\omega_1^n + \omega_2^n\|_0 \le \|e^0\|_0 + \Delta t \sum_j^N \|\omega_1^n + \omega_2^n\|_0.$$

It follows that

$$\begin{aligned} \|e^n\|_0 &= \|y_h^0 - R_h y(0)\|_0 \\ &\le \|y_h^0 - y(x,0)\|_0 + \|y(x,0) - R_h y(0)\|_0 \\ &\le \|\phi_h^0 - \phi\|_0 + Ch^{3/2}\|\phi\|_2. \end{aligned}$$

Also, note that

$$\begin{aligned} \omega_1^j &= y_t(t_j) - (y(t_j) - y(t_{j-1}))/\Delta t = \tfrac{1}{\Delta t} \textstyle\int_{t_{j-1}}^{t_j} (t - t_j) y_{tt}(t)\, dt, \\ \omega_2^j &= (I - R_h)\overline{\partial}_t y(t_j) = \tfrac{1}{\Delta t} \textstyle\int_{t_{j-1}}^{t_j} (I - R_h) y_t(t)\, dt. \end{aligned}$$

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Furthermore,

$$\Delta t \sum_{j}^{N} \|\omega_1^n + \omega_2^n\|_0 \leq \Delta t \int_0^{t_n} \|y_{tt}\|_0 \, dt + Ch^{3/2} \int_0^{t_n} \|y_t(t)\|_2 \, dt.$$

Therefore,

$$\|e^n\|_0 \leq \|\phi - \phi_h\|_0 + Ch^{3/2}\|\phi\|_2 + \Delta t \int_0^{t_n} \|y_{tt}\|_0 \, dt + Ch^{3/2} \int_0^{t_n} \|y_t(t)\|_2 \, dt. \tag{4.16}$$

Finally, a combination of equations (4.14) and (4.16) leads to the desired estimate

$$\|y(t_n) - y_h^n\|_0 \leq \|\phi - \phi_h\|_0 + Ch^{3/2}\|\phi\|_2 + \Delta t \int_0^{t_n} \|y_{tt}\|_0 \, dt + Ch^{3/2} \int_0^{t_n} \|y_t(t)\|_2 \, dt.$$

This completes the proof. □

Theorem 4.7. *Let $(u^*, y^*, p^*) \in L^2(0, T; U_{ad}) \times L^2(0, T; H^1(I)) \cap C(0, T; L^2(I)) \times L^2(0, T; H^1(I)) \cap H^1(0, T; L^2(I))$ and $(u_h^*, y_h^*, p_h^*) \in L^2(0, T; U_{ad}) \times L^2(0, T; U_h) \times L^2(0, T; U_h)$ be the solutions of Problems 2.1 and 3.1, respectively, and $\widetilde{y}$ and $\widetilde{p}$ solutions of problems (4.1) and (4.2), respectively. Assume that $\widetilde{y}(\cdot, t) \in H^2(I)$, $\widetilde{y}_t(\cdot, t) \in H^2(I)$, $\widetilde{y}_{tt}(\cdot, t) \in L^2(I)$, $\widetilde{y}(\cdot, 0) \in H^2(I)$, $y_h^*(\cdot, 0) \in U_h$ and $\widetilde{p}(\cdot, t) \in H^2(I)$, $\widetilde{p}_t(\cdot, t) \in H^2(I)$, $\widetilde{p}_{tt}(\cdot, t) \in L^2(I)$, $\widetilde{p}(\cdot, 0) \in H^2(I)$, $p_h^*(\cdot, 0) \in U_h$. Then there exists a constant $C > 0$, independent of h and sufficiently small Δt, such that*

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(I))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(I))} \leq C(\Delta t + h^{3/2}).$$

Proof. From (4.11) and Lemma 4.6, we can obtain the error estimate easily. □

We consider the case where only the state space is discretized, that is, the discretization of the control space is given implicitly by the necessary optimality conditions. In general, this is an intermediate step that gives us preliminary insights into the convergence behaviour of the discretization. We obtain the error estimates directly from classical results for the error of the finite volume element projection without using adjoint information. If the control space is discretized directly, one should first derive additional regularity results for the optimal control and the adjoint state to control the approximation error, based on which analogous convergence rates can be obtained.

5. Numerical examples

In order to confirm the theories of the previous sections, we present some numerical examples for illustration. The HUFVEM schemes and a fixed-point iteration algorithm [6] are used to solve the optimality system. To guarantee that the fixed-point iteration is convergent, we set $\alpha = 1$. The convergence order of the HUFVEM schemes is defined as

$$\text{Order} = \log_2\Big(\frac{\|y_{2h} - y_{\text{exa}}\|}{\|y_h - y_{\text{exa}}\|}\Big),$$

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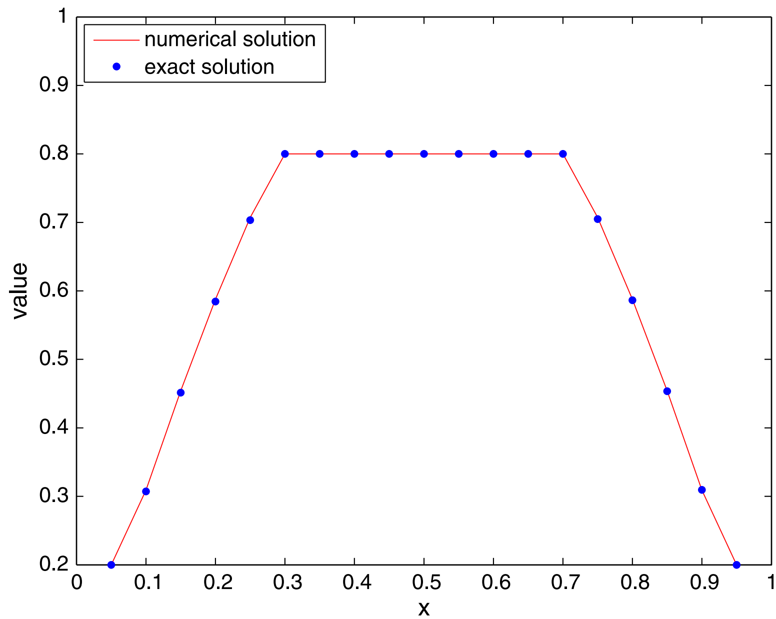

FIGURE 1. The exact solution u^* versus the discrete solution u_h^* obtained by the HUFVEM at $t = 1$ with $N = 20$.

where y_{exa} is the exact solution. We compute the L^2 and L^∞ errors using the HUFVEM schemes, where the error norms are defined as follows:

$$L^2 = \sqrt{\sum_{j=1}^{N} |(y_h)_j - (y_{\text{exa}})_j|^2 h}, \quad L^\infty = \max |y_h - y_{\text{exa}}|.$$

5.1. Example 1 In this example, we investigate the distributed hyperbolic optimal control problem with Dirichlet boundary value conditions. We set $I = [0, 1]$. The method in [14] is used to construct the analytic control, state and adjoint. Let $u^* = \min\{u_b, \max\{u_a, (t - T)^2 \sin(\pi x)\}\}$ and we have $p^* = -\alpha(t - T)^2 \sin(\pi x)$. We also take the optimal state as $y^* = e^t \sin(\pi x)$. Then we can determine the functions f and y_d accordingly. We choose $a(x, t) = (1 + x^2)/(2 + 2xt + 2x^2 + x^4)$, $u_a = 0.2$ and $u_b = 0.8$.

The exact solution u^* versus the discrete solution u_h^* obtained by the HUFVEM at $t = 1$ are plotted in Figure 1. Errors and convergence order for the state, control and adjoint using the HUFVEM are listed in Table 1. From Table 1, we can see that it has $O(h^{3/2})$ in the sense of L^2-norm. This coincides with our results in Theorem 4.4. We also show the errors and convergence order in the physical space in the sense of L^∞-norm in Table 2 for the problem considered.

5.2. Example 2 In this example, we consider the case $u_a = 0.1$ and $u_b = \infty$. We choose $a(x, t) = -(1 + x^2)/(2 + 2xt + 2x^2 + x^4)$. Let $u^* = \max\{u_a, (t - T)^2 \sin(\pi x)\}$ and

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TABLE 1. L^2 error for Example 1 by the HUFVEM at $t = 1$.

N	$\|\|u^* - u_h^*\|\|_{L^2}$	Order	$\|\|y^* - y_h^*\|\|_{L^2}$	Order	$\|\|p^* - p_h^*\|\|_{L^2}$	Order
10	3.5599e−003		4.2925e−003		3.6578e−003	
20	7.7210e−004	2.04	1.1548e−003	1.89	9.8108e−004	1.89
40	2.0840e−004	1.89	2.5225e−004	2.19	2.5903e−004	1.91
80	5.8375e−005	1.83	5.1889e−005	2.28	7.7600e−005	1.73
160	1.8879e−005	1.62	1.5117e−005	1.77	2.4944e−005	1.63

TABLE 2. L^∞ error for Example 1 by the HUFVEM at $t = 1$.

N	$\|\|u^* - u_h^*\|\|_{L^\infty}$	Order	$\|\|y^* - y_h^*\|\|_{L^\infty}$	Order	$\|\|p^* - p_h^*\|\|_{L^\infty}$	Order
10	7.9151e−003		7.5282e−003		7.9151e−003	
20	2.0198e−003	1.67	1.9418e−003	1.95	2.2043e−003	1.80
40	5.2890e−004	1.92	4.2530e−004	2.16	5.7120e−004	1.94
80	1.3708e−004	1.94	1.2093e−004	1.81	1.4205ev004	2.00
160	5.4273e−005	1.35	4.2932e−005	1.49	5.4273e−005	1.33

TABLE 3. L^2 error for Example 2 by the HUFVEM at $t = 1$.

N	$\|\|u^* - u_h^*\|\|_{L^2}$	Order	$\|\|y^* - y_h^*\|\|_{L^2}$	Order	$\|\|p^* - p_h^*\|\|_{L^2}$	Order
10	1.3949e−002		2.2657e−002		1.4388e−002	
20	4.4396e−003	1.65	6.6146e−003	1.77	4.5390e−003	1.66
40	1.4362e−003	1.63	1.9910e−003	1.73	1.4629e−003	1.63
80	4.6530e−004	1.62	6.0288e−004	1.72	4.7022e−004	1.63
160	1.4910e−004	1.64	1.8635e−004	1.69	1.4962e−004	1.65

we have $p^* = -\alpha(t-T)^2 \sin(\pi x)$. We also take the optimal state as $y^* = e^t \sin(\pi x)$. Then the functions f and y_d can be determined accordingly.

The exact solution u^* versus the discrete solution u_h^* obtained by the HUFVEM at $t = 1$ are plotted in Figure 2. Numerical results measured in the sense of L^2-norm by the HUFVEM are listed in Table 3. From it we can see that it has $O(h^{3/2})$ in the sense of L^2-norm. These results are in agreement with the theoretical predictions in Theorem 4.4. Errors and convergence order for the state, control and adjoint in the L^∞-norm are also listed in Table 4 for the problem considered.

6. Conclusion

In this paper, we have designed a HUFVEM for optimal control problems governed by first-order hyperbolic equations. Error analysis shows that under some regularity assumptions the errors obtained by the HUFVEM schemes are $O(h^{3/2})$ in the sense of L^2-norm. A numerical experiment is performed to validate the effectiveness of the schemes and the theoretical results.

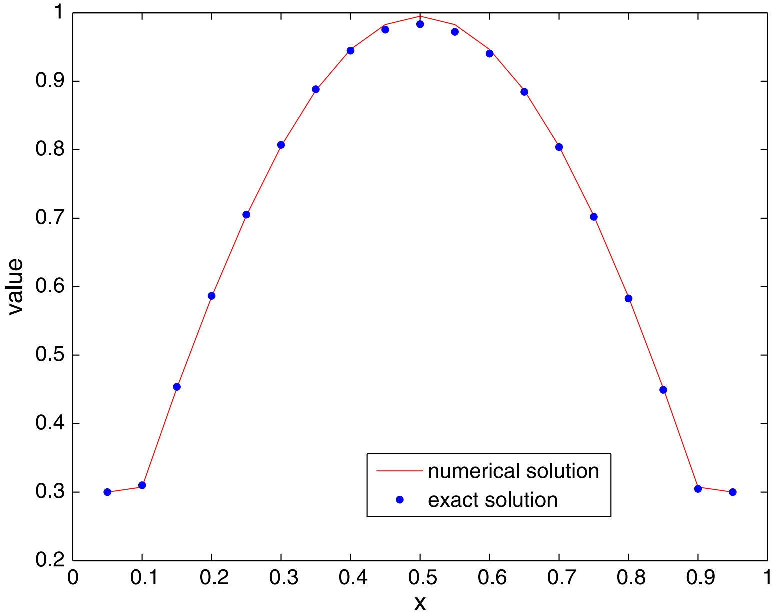

FIGURE 2. The exact solution u^* versus the discrete solution u_h^* obtained by the HUFVEM at $t = 1$ with $N = 20$.

TABLE 4. L^∞ error for Example 2 by the HUFVEM at $t = 1$.

N	$\|u^* - u_h^*\|_{L^\infty}$	Order	$\|y^* - y_h^*\|_{L^\infty}$	Order	$\|p^* - p_h^*\|_{L^\infty}$	Order
10	2.6705e−002		4.5652e−002		7.9151e−002	
20	1.1671e−002	1.19	1.7608e−002	1.37	2.2043e−002	1.20
40	5.0315e−003	1.22	6.9878e−003	1.33	5.7120e−003	1.22
80	2.2662e−003	1.15	2.9408e−003	1.24	1.4205e−003	1.20
160	9.4503e−004	1.49	1.1540e−003	1.35	5.4273e−004	1.26

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