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APPLICATION OF BILINEAR METHOD TO INTEGRABLE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. Two coupled bilinear equations are considered, and then two new coupled differential-difference systems are found. Also two special reductions of these two systems are studied. By using Hirota's method, Bäcklund transformation and superposition formulae, soliton solutions to these equations are presented.

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1. Introduction. During the past twenty-nine years or so, Hirota's bilinear method [**1–6**] has become a powerful tool to find exact solutions of nonlinear equations in both continuous and discrete cases. There are two steps in applying Hirota method, namely, firstly to transform the equations under consideration into bilinear equations and then use perturbation techniques to solve them. We say that an equation is a Hirota bilinear equation if it is bilinear with respect to dependent variables and all the derivatives and differences appearing in the equation can be expressed in terms of the Hirota bilinear operators $D_z^m D_t^k$ and bilinear difference operator $exp(\delta D_n)$ defined by [**2–5**]

$$D_z^m D_t^k\ a\bullet b \equiv \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^m \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(z,t)b(z',t')|_{z'=z,t'=t}$$

and

$$\exp(\delta D_n)a(n)\bullet b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\right]a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta)$$

respectively. Hirota's method has been successfully applied to search for integrable equations by testing for 3-soliton, 4-soliton and even N-soliton solutions and Bäcklund transformations. (See, e.g., [**6**,**7**].) Recently, it has been shown that several integrable lattices could be transformed into the following coupled bilinear form [**8–12**]:

$$F_1(D_t, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n)\bullet f(n) = 0, \tag{1}$$

$$F_2(D_t, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n)\bullet f(n) = 0, \tag{2}$$

where F_i $(i = 1, 2)$ are two even order polynomials in $D_t, D_z, \sinh(\alpha_1 D_n), \ldots$ and $\sinh(\alpha_l D_n)$, and l is a given positive integer; the $\alpha_i, i = 1, 2, \cdots, l,$ are l different constants, and

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$$F_i(0,0,\cdots,0)=0.$$

As an example, by the dependent variable transformation

$$b(n)=\left(\ln\frac{f(n+\frac{1}{2})}{f(n-\frac{1}{2})}\right)_t,\quad c(n)=\frac{f(n+\frac{5}{2})f(n-\frac{3}{2})}{f(n+\frac{3}{2})f(n-\frac{1}{2})},\tag{3}$$

the so-called Belov-Chaltikian lattice [**13**]

$$b_t(n)=b(n)(b(n+1)-b(n-1))-c(n)+c(n-1),\tag{4}$$
$$c_t(n)=c(n)(b(n+2)-b(n-1)),\tag{5}$$

is transformed into the following bilinear form [**11**]

$$(D_t^2e^{\frac{1}{2}D_n}-D_ze^{\frac{1}{2}D_n})f(n)\bullet f(n)=0,\tag{6}$$
$$(D_ze^{D_n}-D_t^2e^{D_n}+2e^{2D_n}-2e^{D_n})f(n)\bullet f(n)=0.\tag{7}$$

The purpose of this paper is to search for new integrable differential-difference systems which can be written in bilinear form of type (1) and (2), and to further study their integrable properties such as Bäcklund transformations and soliton solutions.

The paper is organized as follows. In the next section, a new coupled differential-difference system connected with bilinear equations (6) and (7) is proposed. As a further reduction, a differential-difference equation is considered. By using Mathematica, 3-soliton solutions are obtained. Section 3 is devoted to another new integrable differential-difference system. A Bäcklund transformation and the corresponding nonlinear superposition formula are presented. We also consider a reduced differential-difference equation and obtain soliton solutions. The conclusion and discussion are given in Section 4.

2. A differential-difference system connected with the Belov-Chaltikian lattice. In this section, we will consider the bilinear equations (6) and (7). Firstly, we derive a new differential-difference system (i.e., z flow) from the bilinear form (6) and (7). For this purpose, we set

$$u(n)=\ln\frac{f(n+2)f(n)}{f^2(n+1)},\quad v(n)=\frac{f_t(n+1)}{f(n+1)}-\frac{f_t(n)}{f(n)},\tag{8}$$

and then from (6) and (7) we can deduce the following system

$$\begin{aligned}&u_z(n)-v(n+1)(v(n+2)+v(n+1))+v(n)(v(n)+v(n-1))\\&\quad+e^{u(n+2)+u(n+1)+u(n)}-e^{u(n)+u(n-1)+u(n-2)}=0,\end{aligned}\tag{9}$$

$$\begin{aligned}&v_z(n)-v(n)(v(n+1)+v(n)+v(n-1))(v(n+1)-v(n-1))\\&\quad+v(n)[v(n-1)v(n-2)-v(n+2)v(n+1)+e^{u(n+2)+u(n+1)+u(n)}\\&\quad-e^{u(n-1)+u(n-2)+u(n-3)}]+(v(n+2)+v(n+1)+v(n))e^{u(n+1)+u(n)+u(n-1)}\\&\quad-(v(n)+v(n-1)+v(n-2))e^{u(n)+u(n-1)+u(n-2)}=0.\end{aligned}\tag{10}$$

Since this system (9) and (10) shares the same bilinear form with the Belov-Chaltikian lattice, we can easily obtain its soliton solutions. For example, we have the following one-soliton solution of (9) and (10):

$$u(n) = \ln\frac{f(n+2)f(n)}{f^2(n+1)}, \quad v(n) = \frac{f_t(n+1)}{f(n+1)} - \frac{f_t(n)}{f(n)},$$
$$f(n) = 1 + \exp(\eta), \quad \eta = pn + qz + rt + \eta^0, \tag{11}$$

with

$$q = \lambda^{-2}(e^{2p} - 1), \quad r = \lambda^{-1}(e^p - 1), \quad \lambda = \pm\sqrt{\frac{e^{\frac{1}{2}p} - e^{\frac{3}{2}p}}{e^{\frac{3}{2}p} - e^{-\frac{3}{2}p}}}$$

and two-soliton solution of (9) and (10):

$$u(n) = \ln\frac{f(n+2)f(n)}{f^2(n+1)}, \quad v(n) = \frac{f_t(n+1)}{f(n+1)} - \frac{f_t(n)}{f(n)},$$
$$f(n) = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_2}{\lambda_1 - \lambda_2} e^{\eta_1} + \frac{\lambda_1 - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_2} + \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_1 + \eta_2},$$
$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0,$$

with

$$q_i = \lambda_i^{-2}(e^{2p_i} - 1), \quad r_i = \lambda_i^{-1}(e^{p_i} - 1), \quad \lambda_i = \pm\sqrt{\frac{e^{\frac{1}{2}p_i} - e^{\frac{3}{2}p_i}}{e^{\frac{3}{2}p_i} - e^{-\frac{3}{2}p_i}}}.$$

Besides, by the same dependent variable transformations as (3), we can derive the following system from (6) and (7)

$$\begin{aligned} &b_z(n) - b(n)(b(n+1) + b(n) + b(n-1))(b(n+1) - b(n-1)) \\ &\quad + b(n)[b(n-1)b(n-2) - b(n+2)b(n+1) + c(n+1) - c(n-2)] \\ &\quad + c(n)(b(n+2) + b(n+1) + b(n)) \\ &\quad - c(n-1)(b(n) + b(n-1) + b(n-2)) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &c_z(n) + c(n)[c(n+2) + c(n+1) - c(n-1) - c(n-2)] \\ &\quad + c(n)[b(n-1)(b(n) + b(n-1) + b(n-2)) \\ &\quad - b(n+2)(b(n+3) + b(n+2) + b(n+1))] = 0. \end{aligned} \tag{13}$$

The z-variable appearing in (12) and (13) might be viewed as another time variable (as in the continuous case) and then the system (12)–(13) obtained would be a member of the same hierarchy of the Belov-Chaltikian lattice.

Next, we consider a special reduction of the system (9) and (10). Let $v(n) = 0$, we have from (9) and (10) the following lattice [**12**]:

$$u_z(n) + e^{u(n+2)+u(n+1)+u(n)} - e^{u(n)+u(n-1)+u(n-2)} = 0. \tag{14}$$

In the following, we want to search for soliton solutions of the lattice (14). To this end, set $u(n) = \ln\frac{f(n+2)f(n)}{f^2(n+1)}$ and then (14) can be tranformed into the following bilinear form

$$D_z f(n+1)\bullet f(n) = h(n+1)f(n) + h(n)f(n+1), \tag{15}$$

$$\begin{aligned} &D_z f(n+1)\bullet f(n-1) + 2f(n+2)f(n-2) - 2f(n+1)f(n-1) \\ &= h(n+1)f(n-1) + h(n-1)f(n+1). \end{aligned} \tag{16}$$

It can easily be verified that (15) and (16) have the following one-soliton solution

$$f(n) = 1 + A_1 e^{\eta} + A_2 e^{2\eta}, \quad h(n) = e^{\eta},$$

where

$$\begin{aligned} A_1 &= \frac{1}{2\cosh(p) - 2\cosh(2p)}, \\ A_2 &= \frac{1}{2\cosh(p) - 2\cosh(2p)} \times \frac{1}{2\sinh(p)} \times \frac{\sinh(p) - \sinh(2p)}{\cosh(4p) - \cosh(2p)}, \\ \eta &= pn + \frac{\cosh(4p) - \cosh(2p)}{\sinh(p) - \sinh(2p)} z + \eta_0, \end{aligned}$$

with p and η_0 being arbitrary constants. It is noted that in the continuous case the Kaup-Kupershmidt equation also has a soliton solution of such a kind. We now proceed to search for 3-soliton solution

$$\begin{aligned} f(n) = {}& 1 + A_1 e^{\eta_1} + A_2 e^{\eta_2} + A_3 e^{\eta_3} + A_4 e^{2\eta_1} + A_5 e^{2\eta_2} + A_6 e^{2\eta_3} \\ &+ A_7 e^{\eta_1+\eta_2} + A_8 e^{\eta_1+\eta_3} + A_9 e^{\eta_2+\eta_3} + A_{10} e^{2\eta_1+\eta_2} + A_{11} e^{2\eta_1+\eta_3} \\ &+ A_{12} e^{\eta_1+2\eta_2} + A_{13} e^{2\eta_2+\eta_3} + A_{14} e^{\eta_1+2\eta_3} + A_{15} e^{\eta_2+2\eta_3} + A_{16} e^{\eta_1+\eta_2+\eta_3} \\ &+ A_{17} e^{2\eta_1+2\eta_2} + A_{18} e^{2\eta_1+2\eta_3} + A_{19} e^{2\eta_2+2\eta_3} + A_{20} e^{2\eta_1+\eta_2+\eta_3} \\ &+ A_{21} e^{\eta_1+2\eta_2+\eta_3} + A_{22} e^{\eta_1+\eta_2+2\eta_3} + A_{23} e^{2\eta_1+2\eta_2+\eta_3} \\ &+ A_{24} e^{2\eta_1+\eta_2+2\eta_3} + A_{25} e^{\eta_1+2\eta_2+2\eta_3} + A_{26} e^{2\eta_1+2\eta_2+2\eta_3}, \end{aligned} \tag{17}$$

$$\begin{aligned} h(n) = {}& B_1 e^{\eta_1} + B_2 e^{\eta_2} + B_3 e^{\eta_3} + B_4 e^{\eta_1+\eta_2} + B_5 e^{\eta_1+\eta_3} \\ &+ B_6 e^{\eta_2+\eta_3} + B_7 e^{2\eta_1+\eta_2} + B_8 e^{2\eta_1+\eta_3} + B_9 e^{\eta_1+2\eta_2} \\ &+ B_{10} e^{2\eta_2+\eta_3} + B_{11} e^{\eta_1+2\eta_3} + B_{12} e^{\eta_2+2\eta_3} + B_{13} e^{\eta_1+\eta_2+\eta_3} \\ &+ B_{14} e^{2\eta_1+\eta_2+\eta_3} + B_{15} e^{\eta_1+2\eta_2+\eta_3} + B_{16} e^{\eta_1+\eta_2+2\eta_3} \\ &+ B_{17} e^{2\eta_1+2\eta_2+\eta_3} + B_{18} e^{2\eta_1+\eta_2+2\eta_3} + B_{19} e^{\eta_1+2\eta_2+2\eta_3}, \end{aligned} \tag{18}$$

where

$$\eta_i = p_i n + q_i z + \eta_i^0, \quad q_i = \frac{x_i^4 + x_i^{-4} - x_i^2 - x_i^{-2}}{x_i - x_i^{-1} - x_i^2 + x_i^{-2}}, \quad e^{p_i} \equiv x_i$$

with p_i and η_i^0 being arbitrary parameters ($i = 1, 2, 3$). Without loss of generality, we set $B_1 = B_2 = B_3 = 1$. Substituting them into (15) and (16) and by using Mathematica [**15**], we get the coefficients listed in an Appendix.

3. Another new integrable differential-difference system. In this section, we propose the coupled bilinear equations

$$(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}) f(n) \bullet f(n) = 0, \tag{19}$$

$$(D_z e^{D_n} + D_t^2 e^{D_n} - 2e^{D_n} + 2) f(n) \bullet f(n) = 0, \tag{20}$$

which are very similar to (6) and (7). It can be shown that by the dependent variable transformation $u(n) = (\ln f(n))_t$ we can deduce the following $t-$flow from (19) and (20)

$$\begin{aligned}(u(n+1) &+ u(n) + u(n-1))_{tt} + (u(n+1) - u(n))(u_t(n+1) - u_t(n)) \\ &+ (u(n) - u(n-1))(u_t(n) - u_t(n-1)) + (u(n+1) - u(n-1))(u_t(n+1) \\ &- u_t(n-1)) + 2u(n) - u(n-1) - u(n+1) + (u(n+1) + u(n-1) - 2u(n)) \\ &\times (u(n+1) + u(n) + u(n-1))_t + (u(n+1) + u(n-1) - 2u(n)) \\ &\times [(u(n+1) - u(n-1))^2 - (u(n+1) - u(n))(u(n) - u(n-1))] = 0, \end{aligned} \tag{21}$$

which is nothing but the lattice proposed in [**12**] under the rescaling transformation $t \longrightarrow 2t$, $u(n) \longrightarrow \frac{1}{2}u(n)$. We now deduce a new system, i.e. $z-$flow from (19) and (20). In this regard, we set

$$u(n) = \ln\frac{f(n+1)}{f(n)}, \quad v(n) = \frac{f_t(n+1)}{f(n+1)} - \frac{f_t(n)}{f(n)}.$$

From (19) and (20) we can deduce the following system

$$\begin{aligned} u_z(n+1) &+ u_z(n) + u_z(n-1) + v(n)(v(n+1) + v(n) + v(n-1)) \\ &+ e^{u(n)-u(n+1)} + e^{u(n-1)-u(n)} - 2 = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} (v(n+1) &+ v(n) + v(n-1))_z + v(n-1)e^{u(n-1)-u(n)} - v(n+1)e^{u(n)-u(n+1)} \\ &+ v(n)(v(n+1) + v(n) + v(n-1))^2 - (v(n+1) + v(n) + v(n-1)) \\ &\times [(2u(n+1) + u(n))_t - 2(1 - e^{u(n)-u(n+1)})] = 0. \end{aligned} \tag{23}$$

We have the following results for the bilinear equations (19) and (20):

PROPOSITION 1. *A Bäcklund transformation for (19) and (20) is*

$$(D_t e^{\frac{1}{2}D_n} + \lambda^{-1} e^{-\frac{1}{2}D_n} + \mu e^{\frac{1}{2}D_n}) f(n) \bullet g(n) = 0, \tag{24}$$

$$(D_z e^{\frac{1}{2}D_n} - \lambda^{-1} D_t e^{-\frac{1}{2}D_n} - \lambda^{-1}\mu e^{-\frac{1}{2}D_n} - \omega e^{\frac{1}{2}D_n}) f(n) \bullet g(n) = 0, \tag{25}$$

$$\begin{aligned} [-D_z e^{-\frac{D_n}{2}} &+ \omega e^{-\frac{D_n}{2}} + 2\mu D_t e^{-\frac{1}{2}D_n} \\ &+ \mu^2 e^{-\frac{1}{2}D_n} + D_t^2 e^{-\frac{1}{2}D_n} + \theta e^{\frac{1}{2}D_n} - 2e^{-\frac{1}{2}D_n}] f(n) \bullet g(n) = 0 \end{aligned} \tag{26}$$

where λ, μ, ω and θ are arbitrary constants.

PROPOSITION 2. *Let f_0 be a solution of equations (19) and (20) and suppose that $f_i (i = 1, 2)$ are solutions of (19) and (20) which are related to f_0 under the BT equations (24)–(26) with parameters $(\lambda_i, \mu_i, \omega_i, \theta_i)$, i.e., $f_0 \overset{(\lambda_i,\mu_i,\omega_i,\theta_i)}{\longrightarrow} f_i$ $(i = 1, 2)$, $\lambda_1\lambda_2 \neq 0$, $f_j \neq 0$ $(j = 0, 1, 2)$. Then f_{12} defined by*

$$\exp\left(-\frac{1}{2}D_n\right)f_0\bullet f_{12} = k\left[\lambda_1\exp\left(-\frac{1}{2}D_n\right) - \lambda_2\exp\left(\frac{1}{2}D_n\right)\right]f_1\bullet f_2 \tag{27}$$

is a new solution which is related to f_1 *and* f_2 *under the BT(24)–(26) with parameters* $(\lambda_2, \mu_2, \omega_2, \theta_2)$, $(\lambda_1, \mu_1, \omega_1, \theta_1)$ *respectively. Here k is a nonzero constant.*

These results can be proved by using Hirota's bilinear operator identities. We do not give the details of the proof. Instead by using the Bäcklund transformation (24)–(26) and superposition formula (27), we write down the one-soliton solution of (19) and (20)

$$f(n) = 1 + \exp(\eta), \quad \eta = pn + qt + rz + \eta^0,$$

with

$$q = \lambda^{-1}(e^p - 1), \quad r = \lambda^{-2}(e^{2p} - 1), \quad \lambda = \pm\sqrt{e^{2p} + e^p + 1},$$

and two-soliton solutions of (19) and (20)

$$f(n) = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_2}{\lambda_1 - \lambda_2}e^{\eta_1} + \frac{\lambda_1 - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2}e^{\eta_2} + \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2}e^{\eta_1+\eta_2},$$
$$\eta_i = p_i n + q_i t + r_i z + \eta_i^0,$$

with

$$q_i = \lambda_i^{-1}(e^{p_i} - 1), \quad r_i = \lambda_i^{-2}(e^{2p_i} - 1), \quad \lambda_i = \pm\sqrt{e^{2p_i} + e^{p_i} + 1}.$$

Next, we consider a special reduction of the system (22) and (23). Let $v(n) = 0$, we have from (22) and (23) the following lattice [**14**]:

$$u_z(n+1) + u_z(n) + u_z(n-1) + e^{u(n-1)-u(n)} + e^{u(n)-u(n+1)} - 2 = 0. \tag{28}$$

In the following we also search for soliton solutions of the lattice (28). To this end set $u(n) = \ln\frac{f(n+1)}{f(n)}$ and equation (28) can be transformed into the bilinear form

$$D_z f(n+1)\bullet f(n) = h(n+1)f(n) + h(n)f(n+1), \tag{29}$$
$$D_z f(n+1)\bullet f(n-1) - 2f(n+1)f(n-1) + 2f^2(n)$$
$$= -h(n+1)f(n-1) - h(n-1)f(n+1). \tag{30}$$

It can be easily verified that (29) and (30) have the following one-soliton solution

$$f(n) = 1 + A_1 e^{\eta} + A_2 e^{2\eta}, \quad h(n) = e^{\eta},$$

where

$$A_1 = \frac{1 + 2\cosh(p)}{2\cosh(p) - 2},$$
$$A_2 = \frac{1 + 2\cosh(p)}{2\cosh(p) - 2} \times \frac{1}{2\sinh(p)} \times \frac{\sinh(p) + \sinh(2p)}{\cosh(2p) - 1},$$
$$\eta = pn + \frac{\cosh(2p) - 1}{\sinh(p) + \sinh(2p)}z + \eta_0,$$

with p and η_0 being arbitrary constants. By using Hirota's method, we can also obtain 3-soliton solutions [**14**].

4. Conclusion and discussion. In summary, we have considered two coupled bilinear equations. As a result, two new coupled differential-difference systems are found. The first new system shares the same bilinear equations with the Belov-Chaltikian lattice while the second new system shares the same bilinear equations with the lattice proposed in [**12**]. We have also considered two special reductions of these two systems. By using the Hirota method, soliton solutions of the two reduced equations are obtained with the assistance of Mathematica. It is noted that these two reduced equations exhibit soliton solutions of the Kaup-Kupershmidt equation type. Since integrable systems share many common integrable properties, it would be interesting to study some other integrable properties of these new systems found in the paper. Besides, one could start to think about the construction of the Belov-Chaltikian hierarchy in view of the fact that (12)–(13) is a member of the same hierarchy of the Belov-Chaltikian lattice.

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5. Appendix. The coefficients of the 3-soliton solution (17) and (18).

$$A_i = \frac{x_i^2}{-1 + x_i + x_i^3 - x_i^4} \quad (i = 1, 2, 3),$$

$$A_{i+3} = \frac{x_i^5}{(x_i - 1)^4 (x_i + 1)^2 (1 + x_i + x_i^2)^2}, \ (i = 1, 2, 3),$$

$$A_7 = A_1 A_2 Q_{12}, \quad A_8 = A_1 A_3 Q_{13}, \quad A_9 = A_2 A_3 Q_{23},$$

$$Q_{ij} = \frac{(x_i + x_j)(1 - x_i x_j - x_i^2 x_j^2 + x_i^3 x_j^3) + x_i^3 + x_j^3 - 4x_i^2 x_j^2 + x_i^4 x_j + x_i x_j^4}{(x_i x_j - 1)^2 (1 + x_i + x_j + 3x_i x_j + x_i^2 x_j + x_i x_j^2 + x_i^2 x_j^2)},$$

$$(i, j = 1, 2, 3),$$

$$A_{10} = A_2 A_4 P_{12}, \quad A_{11} = A_3 A_4 P_{13}, \quad A_{12} = A_1 A_5 P_{12},$$

$$A_{13} = A_3 A_5 P_{23}, \quad A_{14} = A_1 A_6 P_{13}, \quad A_{15} = A_2 A_6 P_{23},$$

$$P_{ij} = \frac{(x_i - x_j)^2 (x_i^2 + x_j^2 + x_i + x_j + x_i^2 x_j + x_i x_j^2 + 3x_i x_j)}{(x_i x_j - 1)^2 (1 + x_i + x_j + 3x_i x_j + x_i^2 x_j + x_i x_j^2 + x_i^2 x_j^2)}, \quad (i, j = 1, 2, 3),$$

$$A_{17} = A_4 A_5 P_{12}^2, \quad A_{18} = A_4 A_6 P_{13}^2, \quad A_{19} = A_5 A_6 P_{23}^2,$$

$$A_{20} = A_2 A_3 A_4 P_{12} P_{13} Q_{23}, \quad A_{21} = A_1 A_3 A_5 P_{12} P_{23} Q_{13}, \quad A_{22} = A_1 A_2 A_6 P_{13} P_{23} Q_{12},$$

$$A_{23} = A_3 A_4 A_5 P_{12}^2 P_{13} P_{23}, \quad A_{24} = A_2 A_4 A_6 P_{13}^2 P_{12} P_{23}, \quad A_{25} = A_1 A_5 A_6 P_{23}^2 P_{12} P_{13},$$

$$A_{26} = A_4 A_5 A_6 P_{12}^2 P_{13}^2 P_{23}^2, \quad , B_4 = R_{12}, \quad B_5 = R_{13}, \quad B_6 = R_{23},$$

$$R_{ij} = -(x_i - x_j)^2(x_i + x_j + x_i^3 + x_j^3 + 3x_i^2x_j + 3x_ix_j^2 + x_ix_j + 3x_i^2x_j^2 + 3x_i^2x_j^3 + 3x_i^3x_j^2 + x_i^4x_j + x_ix_j^4 + x_ix_j^3 + x_i^3x_j + x_i^3x_j^3 + x_i^3x_j^4 + x_i^4x_j^3)/ [(x_i - 1)^2(x_j - 1)^2(1 + x_i + x_i^2)(1 + x_j + x_j^2) \times (1 + x_i + x_j + 3x_ix_j + x_i^2x_j + x_ix_j^2 + x_i^2x_j^2)], \quad (i, j = 1, 2, 3),$$

$$B_7 = A_4P_{12}, \quad B_8 = A_4P_{13}, \quad B_9 = A_5P_{12}, \quad B_{10} = A_5P_{23},$$

$$B_{11} = A_6P_{13}, \quad B_{12} = A_6P_{23}, \quad B_{14} = A_4P_{12}P_{13}B_6,$$

$$B_{15} = A_5P_{12}P_{23}B_5, \quad B_{16} = A_6P_{13}P_{23}B_4, \quad B_{17} = A_4A_5P_{12}^2P_{13}P_{23},$$

$$B_{18} = A_4A_6P_{13}^2P_{12}P_{23}, \quad B_{19} = A_5A_6P_{23}^2P_{12}P_{13},$$

$$A_{16} = \{P_4(p_1 + p_2 + p_3)[-A_1A_9P_1(-p_1 + p_2 + p_3) - A_2A_8P_1(p_1 - p_2 + p_3) - A_3A_7P_1(p_1 + p_2 - p_3) + (A_9 + B_6A_1)P_2(-p_1 + p_2 + p_3) + (A_8 + B_5A_2)P_2(p_1 - p_2 + p_3) + (A_7 + B_4A_3)P_2(p_1 + p_2 - p_3)] - P_2(p_1 + p_2 + p_3)[-A_1A_9P_3(-p_1 + p_2 + p_3) - A_2A_8P_3(p_1 - p_2 + p_3) - A_3A_7P_3(p_1 + p_2 - p_3) + (A_9 + B_6A_1)P_4(-p_1 + p_2 + p_3) + (A_8 + B_5A_2)P_4(p_1 - p_2 + p_3) + (A_7 + B_4A_3)P_4(p_1 + p_2 - p_3)]\} /[P_1(p_1 + p_2 + p_3)P_4(p_1 + p_2 + p_3) - P_2(p_1 + p_2 + p_3)P_3(p_1 + p_2 + p_3)]$$

$$B_{13} = \{P_3(p_1 + p_2 + p_3)[-A_1A_9P_1(-p_1 + p_2 + p_3) - A_2A_8P_1(p_1 - p_2 + p_3) - A_3A_7P_1(p_1 + p_2 - p_3) + (A_9 + B_6A_1)P_2(-p_1 + p_2 + p_3) + (A_8 + B_5A_2)P_2(p_1 - p_2 + p_3) + (A_7 + B_4A_3)P_2(p_1 + p_2 - p_3)] - P_1(p_1 + p_2 + p_3)[-A_1A_9P_3(-p_1 + p_2 + p_3) - A_2A_8P_3(p_1 - p_2 + p_3) - A_3A_7P_3(p_1 + p_2 - p_3) + (A_9 + B_6A_1)P_4(-p_1 + p_2 + p_3) + (A_8 + B_5A_2)P_4(p_1 - p_2 + p_3) + (A_7 + B_4A_3)P_4(p_1 + p_2 - p_3)]\} /[P_1(p_1 + p_2 + p_3)P_4(p_1 + p_2 + p_3) - P_2(p_1 + p_2 + p_3)P_3(p_1 + p_2 + p_3)]$$

where

$$P_j\left(\sum_{i=1}^{3}\sigma_ip_i\right) \equiv P_j\left(\sum_{i=1}^{3}\sigma_ip_i, \sum_{i=1}^{3}\sigma_iq_i\right), \quad (j = 1, 2, 3, 4)$$

$$P_1(x, y) \equiv y\sinh\left(\frac{x}{2}\right), \quad P_2(x, y) \equiv \cosh\left(\frac{x}{2}\right),$$

$$P_3(x, y) \equiv y\sinh(x) + 2\cosh(2x) - 2\cosh(x), \quad P_4(x, y) \equiv \cosh(x), \quad \sigma_i = \pm 1.$$

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