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A BOUND FOR SIMILARITY CONDITION NUMBERS OF UNBOUNDED OPERATORS WITH HILBERT–SCHMIDT HERMITIAN COMPONENTS

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Abstract

Let H be a linear unbounded operator in a Hilbert space. It is assumed that the resolvent of H is a compact operator and $H - H^*$ is a Hilbert–Schmidt operator. Various integro-differential operators satisfy these conditions. It is shown that H is similar to a normal operator and a sharp bound for the condition number is suggested. We also discuss applications of that bound to spectrum perturbations and operator functions.

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1. Introduction and statement of the main result

Two operators A and $\tilde{A}$ are said to be similar if there exists a boundedly invertible bounded operator T such that $\tilde{A} = T^{-1}AT$. The constant $\kappa_T = \|T^{-1}\|\,\|T\|$ is called the condition number. The condition number is important in applications. We refer the reader to [5], where condition number estimates are suggested for combined potential boundary integral operators in acoustic scattering, and [23], where condition numbers are estimated for second-order elliptic operators. Conditions that provide the similarity of various operators to normal and self-adjoint ones were considered by many mathematicians, cf. [1, 4, 8, 14–18, 21] and references given therein. In many cases, the condition number must be numerically calculated; see for example [2, 20]. The interesting generalizations of condition numbers of bounded linear operators in Banach spaces were explored in the paper [6].

In the present paper we consider a class of unbounded operators in a Hilbert space with Hilbert–Schmidt Hermitian components. Various integro-differential operators belong to that class. We suggest a sharp bound for the condition numbers of the considered operators. It generalizes the bounds for the condition numbers of matrices from [10, 11]. We also discuss applications of the obtained bound to spectrum perturbations and norm estimates for operator functions.

Let $\mathfrak{H}$ be a separable Hilbert space with scalar product $(.,.)$, norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. For a linear operator A in $\mathfrak{H}$, $\mathrm{Dom}(A)$ is the domain, A^* is the adjoint of A, $\sigma(A)$ denotes the spectrum of A, A^{-1} is the inverse to A, $R_\lambda(A) = (A - I\lambda)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent, $A_I := (A - A^*)/2i$ and $\lambda_k(A)$ ($k = 1, 2, \ldots$) are the eigenvalues of A taken with their multiplicities and enumerated as $|\lambda_j(A)| \le |\lambda_{j+1}(A)|$. By SN_p ($1 \le p < \infty$), we denote the Schatten–von Neumann ideal of compact operators K with the finite norm $N_p(K) := [\mathrm{tr}(KK^*)^{p/2}]^{1/p}$. The set SN_2 is the Hilbert–Schmidt ideal.

Everywhere below, H is an invertible operator in $\mathfrak{H}$ with the following properties: $\mathrm{Dom}(H) = \mathrm{Dom}(H^*)$, there exists some fixed value $p \in [1, \infty)$ such that

$$H^{-1} \in SN_p \text{ and, in addition, } H_I \in SN_2. \tag{1.1}$$

Note that instead of the condition $H^{-1} \in SN_p$, in our reasonings below, one can require the condition $(H - aI)^{-1} \in SN_p$ for some point $a \notin \sigma(H)$. Since H^{-1} is compact, $\sigma(H)$ is purely discrete. It is assumed that *all the eigenvalues $\lambda_j(H)$ of H are different*. For a fixed integer m, put

$$\delta_m(H) = \inf_{j=1,2,\ldots;j\ne m} |\lambda_j(H) - \lambda_m(H)|.$$

It is further supposed that

$$\zeta(H) := \left[\sum_{j=1}^{\infty} \frac{1}{\delta_j^2(H)}\right]^{1/2} < \infty. \tag{1.2}$$

Hence, it follows that

$$\hat{\delta}(H) := \inf_m \delta_m(H) = \inf_{j\ne k; j,k=1,2,\ldots} |\lambda_j(H) - \lambda_k(H)| > 0.$$

Denote

$$g(H) := \sqrt{2}\left[N_2^2(H_I) - \sum_{k=1}^{\infty} |\mathrm{Im}\, \lambda_k(H)|^2\right]^{1/2} \le \sqrt{2} N_2(H_I),$$

$$\tau(H) := \sum_{k=0}^{\infty} \frac{g^{k+1}(H)}{\sqrt{k!}\,\hat{\delta}^k(H)} \quad \text{and} \quad \gamma(H) := \exp\,[\zeta^2(H)\tau^2(H)].$$

It follows from condition (1.2) that $\delta_j(H) \sim j^{\alpha+1/2}$ for some $\alpha > 0$. That is, $\delta_j(H)$ increases more rapidly than $j^{1/2}$. So, we can interpret this condition to mean that the eigenvalues of H are in some sense widely separated. Note also that $g(H)$ is in some sense a 'measure of departure of H from normality'.

Now we are in a position to formulate our main result.

THEOREM 1.1. *Let conditions (1.1) and (1.2) be fulfilled. Then there are an invertible operator T and a normal operator D, acting in $\mathfrak{H}$, such that*

$$THx = DTx \quad (x \in \mathrm{Dom}(H)). \tag{1.3}$$

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Moreover,

$$\kappa_T := \|T^{-1}\|\,\|T\| \le \gamma(H).$$

The proof of this theorem is divided into a series of lemmas, which are presented in the next three sections. The theorem is sharp: if H is normal, then $g(H) = 0$ and we obtain $\gamma(H) = 1$.

To illustrate Theorem 1.1, consider the case $H = S + K$, where $K \in SN_2$ and S is a positive-definite self-adjoint operator with a discrete spectrum, whose eigenvalues are different and

$$\lambda_{j+1}(S) - \lambda_j(S) \ge b_0 j^{\alpha} \quad (b_0 = \text{const} > 0; \alpha > 1/2; j = 1, 2, \ldots).$$

It can be directly checked that the condition $\|R_\lambda(S)\|\,\|K\| < 1$ implies $\lambda \notin \sigma(H)$. Since $\|R_\lambda(S)\| \le \rho^{-1}(S, \lambda)$, we have $\lambda \notin \sigma(H)$ provided $\|K\| < \rho(S, \lambda)$. Hence, $\|K\| \ge \rho(S, \mu)$ for any $\mu \in \sigma(H)$. This implies the relation

$$\sup_k \inf_j |\lambda_k(H) - \lambda_j(S)| \le \|K\|.$$

Thus, if

$$2\|K\| < \inf_j (\lambda_{j+1}(S) - \lambda_j(S)),$$

then $\hat{\delta}(H) \ge \inf_j (\lambda_{j+1}(S) - \lambda_j(S) - 2\|K\|)$ and (1.2) holds with

$$\zeta(H) \le \zeta_1(S, K), \quad \text{where } \zeta_1(S, K) := \sum_{j=1}^{\infty} (\lambda_{j+1}(S) - \lambda_j(S) - 2\|K\|)^{-2} < \infty. \tag{1.4}$$

EXAMPLE 1.2. Consider in $L^2(0, 1)$ the problem

$$-u''(x) + (Ku)(x) = \lambda u(x) \quad (0 < x < 1); \quad u(0) = u(1) = 0,$$

where K is a Hilbert–Schmidt operator. So, H is defined by $H = -d^2/dx^2 + K$ with

$$\text{Dom}\,(H) = \{v \in L^2(0, 1) : v'' \in L^2(0, 1), v(0) = v(1) = 0\}.$$

Take $S = -d^2/dx^2$ with $\text{Dom}\,(S) = \text{Dom}\,(H)$. Then $\lambda_j(S) = \pi^2 j^2$ $(j = 1, 2, \ldots)$ and $\lambda_{j+1}(S) - \lambda_j(S) = \pi^2(2j + 1)$. So, if $2\|K\| < 3\pi^2$, then $\hat{\delta}(H) = 3\pi^2 - 2\|K\|$ and, due to (1.4),

$$\zeta^2(H) \le \sum_{j=1}^{\infty} (\pi^2(2j + 1) - 2\|K\|)^{-2} < \infty.$$

In addition, $g(H) \le \sqrt{2} N_2(K)$. Now one can directly apply Theorem 1.1.

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2. Auxiliary results

Let B_0 be a bounded linear operator in $\mathfrak{H}$ having a finite chain of invariant projections P_k $(k = 1, \dots, n; n < \infty)$:

$$0 \subset P_1\mathfrak{H} \subset P_2\mathfrak{H} \subset \cdots \subset P_n\mathfrak{H} = \mathfrak{H} \tag{2.1}$$

and

$$P_k B_0 P_k = B_0 P_k \quad (k = 1, \dots, n). \tag{2.2}$$

That is, $P_k B_0$ maps $P_k\mathfrak{H}$ into $P_k\mathfrak{H}$ for each k. Put

$$\Delta P_k = P_k - P_{k-1} \quad (P_0 = 0) \quad \text{and} \quad A_k = \Delta P_k B_0 \Delta P_k.$$

It is assumed that the spectra $\sigma(A_k)$ of A_k in $\Delta P_k\mathfrak{H}$ satisfy the condition

$$\sigma(A_k) \cap \sigma(A_j) = \emptyset \quad (j \neq k;\ j, k = 1, \dots, n). \tag{2.3}$$

Lemma 2.1. *One has*

$$\sigma(B_0) = \bigcup_{k=1}^{n} \sigma(A_k).$$

Proof. Put

$$\hat{D} = \sum_{k=1}^{n} A_k \quad \text{and} \quad W = B_0 - \hat{D}.$$

Due to (2.2), we have $WP_k = P_{k-1}WP_k$. Hence,

$$\begin{aligned} W^n &= W^n P_n = W^{n-1} P_{n-1} W P_n = W^{n-2} P_{n-2} W P_{n-1} W P_n \\ &= W^{n-2} P_{n-2} W^2 = W^{n-3} P_{n-3} W^3 = \cdots = P_0 W^n = 0. \end{aligned}$$

So, W is nilpotent. Similarly, taking into account that

$$(\hat{D} - \lambda I)^{-1} W P_k = (\hat{D} - \lambda I)^{-1} P_{k-1} W P_k = P_{k-1} (\hat{D} - \lambda I)^{-1} W P_k,$$

we prove that $((\hat{D} - \lambda I)^{-1} W)^n = 0$ $(\lambda \notin \sigma(D))$. Thus,

$$\begin{aligned} (B_0 - \lambda I)^{-1} &= (\hat{D} + W - \lambda I)^{-1} = (I + (\hat{D} - \lambda I)^{-1} W)^{-1} (\hat{D} - \lambda I)^{-1} \\ &= \sum_{k=0}^{n-1} (-1)^k ((\hat{D} - \lambda I)^{-1} W)^k (\hat{D} - \lambda I)^{-1}. \end{aligned}$$

Hence, it easily follows that $\sigma(\hat{D}) = \sigma(B_0)$. Since A_k are mutually orthogonal, this proves the lemma. □

Under conditions (2.1) and (2.2), put

$$Q_k = I - P_k, B_k = Q_k B_0 Q_k \quad \text{and} \quad C_k = \Delta P_k B_0 Q_k.$$

Since B_j is a block triangular operator matrix, according to the previous lemma

$$\sigma(B_j) = \bigcup_{k=j+1}^{n} \sigma(A_k) \quad (j = 0, \dots, n).$$

We need the following result.

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THEOREM 2.2 (Rosenblum [22]). *Let $\mathfrak{H}$ be a Hilbert space and let A, B, Q be bounded linear operators on $\mathfrak{H}$. Suppose that $\sigma(A) \cap \sigma(B) = \emptyset$. Under these conditions, the operator equation $AX - XB + Q = 0$ has a unique solution X given by*

$$X = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} Q (zI - B)^{-1}\, dz,$$

where Γ is a piecewise-smooth closed curve with $\sigma(A) \subset ext(\Gamma)$ and $\sigma(B) \subset int(\Gamma)$.

Due to (2.3),

$$\sigma(B_j) \cap \sigma(A_j) = \emptyset \quad (j = 1, \ldots, n).$$

Under this condition, according to the Rosenblum theorem, the equation

$$A_j X_j - X_j B_j = -C_j \quad (j = 1, \ldots, n-1) \tag{2.4}$$

has a unique solution (see also [7, Section I.3] and [3]).

LEMMA 2.3. *Let condition (2.3) hold and X_j be a solution to (2.4). Then*

$$\begin{aligned}(I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1)\, B_0\, (I + X_1)(I + X_2) \cdots (I + X_{n-1})\\ = A_1 + A_2 + \cdots + A_n = \hat{D}.\end{aligned} \tag{2.5}$$

PROOF. Since $X_j = \Delta P_j X_j Q_j$, we have $X_j A_j = B_j X_j = X_j C_j = C_j X_j = 0$. Clearly, $Q_j B_0 P_j = 0$. Thus, $B_0 = A_1 + B_1 + C_1$ and, consequently,

$$\begin{aligned}(I - X_1) B_0 (I + X_1) &= (I - X_1)(A_1 + B_1 + C_1)(I + X_1)\\ &= A_1 + B_1 + C_1 - X_1 B_1 + A_1 X_1 = A_1 + B_1.\end{aligned}$$

Furthermore, $B_1 = A_2 + B_2 + C_2$. Hence,

$$\begin{aligned}(Q_1 - X_2) B_1 (Q_1 + X_2) &= (Q_1 - X_1)(A_2 + B_2 + C_2)(Q_1 + X_1)\\ &= A_2 + B_2 + C_2 - X_2 B_2 + A_2 X_2 = A_2 + B_2.\end{aligned}$$

Therefore,

$$\begin{aligned}(I - X_2)(A_1 + B_1)(I + X_2) &= (P_1 + Q_1 - X_2)(A_1 + B_1)(P_1 + Q_1 + X_2)\\ &= A_1 + (Q_1 - X_2)(A_1 + B_1)(Q_1 + X_2) = A_1 + A_2 + B_2.\end{aligned}$$

Consequently,

$$\begin{aligned}(I - X_2)(A_1 + B_1)(I + X_2) &= (I - X_2)(I - X_1) B_0 (I + X_1)(I + X_2)\\ &= A_1 + A_2 + B_2.\end{aligned}$$

Continuing this process and taking into account that $B_{n-1} = A_n$, we obtain the required result. □

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Take

$$\hat{T}_n = (I + X_1)(I + X_2) \cdots (I + X_{n-1}). \tag{2.6}$$

It is simple to see that the inverse to $I + X_j$ is the operator $I - X_j$. Thus,

$$\hat{T}_n^{-1} = (I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1)$$

and (2.5) can be written as

$$\hat{T}_n^{-1} B_0 \hat{T}_n = \operatorname{diag}(A_k)_{k=1}^n.$$

By the inequalities between the arithmetic and geometric means, we get

$$\|\hat{T}_n\| \le \prod_{k=1}^{n-1}(1 + \|X_k\|) \le \left(1 + \frac{1}{n-1}\sum_{k=1}^{n-1}\|X_k\|\right)^{n-1} \tag{2.7}$$

and

$$\|\hat{T}_n^{-1}\| \le \left(1 + \frac{1}{n-1}\sum_{k=1}^{n-1}\|X_k\|\right)^{n-1}. \tag{2.8}$$

3. The finite-dimensional case

In this section we apply Lemma 2.3 to an $n \times n$ matrix A whose eigenvalues are different and are enumerated in the increasing way of their absolute values. We define

$$\hat{\delta}(A) := \min_{j,k=1,\dots,n;k\ne j} |\lambda_j(A) - \lambda_k(A)| > 0. \tag{3.1}$$

Hence, there are an invertible matrix $T_n \in \mathbb{C}^{n\times n}$ and a normal matrix $D_n \in \mathbb{C}^{n\times n}$ such that

$$T_n^{-1} A T_n = D_n. \tag{3.2}$$

In this case,

$$g(A) := \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^n |\operatorname{Im}\lambda_k(A)|^2\right]^{1/2} \le \sqrt{2}N_2(A_I).$$

As is shown in [9, Theorem 2.3.1 and Lemma 2.3.2],

$$g^2(A) = N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2 \le N_2^2(A) - |\operatorname{tr}(A^2)|. \tag{3.3}$$

Furthermore, for a fixed $m \le n$, put

$$\delta_m(A) = \inf_{j=1,2,\dots,n;j\ne m} |\lambda_j(A) - \lambda_m(A)|, \quad \zeta(A) = \left(\sum_{k=1}^{n-1}\frac{1}{\delta_k^2(A)}\right)^{1/2},$$

$$\tau_n(A) := \sum_{k=0}^{n-2}\frac{g^{k+1}(A)}{\sqrt{k!}\hat{\delta}^k(A)}$$

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and

$$\gamma_n(A) := \left(1 + \frac{\zeta(A)\tau_n(A)}{n-1}\right)^{2(n-1)}.$$

We need the following result.

Lemma 3.1. *Let condition (3.1) be fulfilled. Then there is an invertible operator T_n such that (3.2) holds with $\kappa_{T_n} := \|T_n^{-1}\|\,\|T_n\| \le \gamma_n(A)$.*

Proof. Let $\{e_k\}$ be the Schur basis (the orthogonal normal basis of the triangular representation) of matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

with $a_{jj} = \lambda_j(A)$. Besides, according to (3.3),

$$\sum_{k=2}^{n-1}\sum_{j=1}^{k-1} |a_{jk}|^2 = g^2(A)$$

(see also [9, Lemma 2.3.2]). To apply Lemma 2.3, take $P_j = \sum_{k=1}^{j}(., e_k)e_k$, $B_0 = A$, $\Delta P_k = (., e_k)e_k$,

$$Q_j = \sum_{k=j+1}^{n}(., e_k)e_k, \quad A_k = \Delta P_k A \Delta P_k = \lambda_k(A)\Delta P_k,$$

$$B_j = Q_j A Q_j = \begin{pmatrix} a_{j+1,j+1} & a_{j+1,j+2} & \cdots & a_{j+1,n} \\ 0 & a_{j+2,j+2} & \cdots & a_{j+2,n} \\ . & . & . & \cdots \\ 0 & 0 & . & a_{nn} \end{pmatrix},$$

$$C_j = \Delta P_j A Q_j = \begin{pmatrix} a_{j,j+1} & a_{j,j+2} & \cdots & a_{j,n} \end{pmatrix}$$

and

$$D_n = \operatorname{diag}(\lambda_k(A)). \tag{3.4}$$

In addition,

$$A = \begin{pmatrix} \lambda_1(A) & C_1 \\ 0 & B_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \lambda_2(A) & C_2 \\ 0 & B_2 \end{pmatrix}, \ldots, B_j = \begin{pmatrix} \lambda_{j+1}(A) & C_{j+1} \\ 0 & B_{j+1} \end{pmatrix}$$

$(j < n)$. So, B_j is an upper-triangular $(n-j)\times(n-j)$ matrix. Equation (2.4) takes the form

$$\lambda_j(A)X_j - X_jB_j = -C_j.$$

Since $X_j = X_jQ_j$, we can write $X_j(\lambda_j(A)Q_j - B_j) = C_j$. Therefore,

$$X_j = C_j\,(\lambda_j(A)Q_j - B_j)^{-1}. \tag{3.5}$$

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The inverse operator is understood in the sense of subspace $Q_j\mathbb{C}^n$. Hence,

$$\|X_j\| \le \|C_j\| \, \|(\lambda_j(A)Q_j - B_j)^{-1}\|.$$

Besides,

$$\|C_j\|^2 = \sum_{k=j+1}^{n} |a_{jk}|^2$$

and, due to [9, Corollary 2.1.2],

$$\|(\lambda_j(A)Q_j - B_j)^{-1}\| \le \sum_{k=0}^{n-j-1} \frac{g^k(B_j)}{\sqrt{k!}\delta_j^{k+1}(A)}.$$

But $g(B_j) = g(Q_jB_jQ_j) \le g(A)$ $(j \ge 1)$. So,

$$\|(\lambda_j(A)Q_j - B_j)^{-1}\| \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\delta_j^{k+1}(A)} = \frac{\tau_n(A)}{g(A)\delta_j(A)}$$

and thus

$$\|X_j\| \le \frac{\|C_j\|\tau_n(A)}{g(A)\delta_j(A)}.$$

Take $T_n = \hat{T}_n$ as in (2.6) with X_k defined by (3.5). Besides, (2.7) and (2.8) imply

$$\|T_n\| \le \left(1 + \frac{1}{n-1} \sum_{j=1}^{n-1} \|X_j\|\right)^{n-1} \le \left(1 + \frac{\tau_n(A)}{g(A)(n-1)} \sum_{j=1}^{n-1} \frac{\|C_j\|}{\delta_j(A)}\right)^{n-1}$$

and

$$\|T_n^{-1}\| \le \left(1 + \frac{\tau_n(A)}{g(A)(n-1)} \sum_{j=1}^{n-1} \frac{\|C_j\|}{\delta_j(A)}\right)^{n-1}.$$

But, by the Schwarz inequality,

$$\left(\sum_{j=1}^{n-1} \frac{\|C_j\|}{\delta_j(A)}\right)^2 \le \sum_{j=1}^{n-1} \|C_j\|^2 \sum_{k=1}^{n-1} \frac{1}{\delta_k^2(A)}.$$

In addition,

$$\sum_{j=1}^{n-1} \|C_j\|^2 = \sum_{j=1}^{n-1} \sum_{k=k=j+1}^{n} |a_{jk}|^2 = g^2(A).$$

Thus, $\|T_n\|^2 \le \gamma_n(A)$ and $\|T_n^{-1}\|^2 \le \gamma_n(A)$. This proves the lemma. □

It should be noted that a result similar to Lemma 3.1 has been established in the paper [12], but Lemma 3.1 is sharper than that result.

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4. Proof of Theorem 1.1

LEMMA 4.1. *Under the hypothesis of Theorem 1.1, operator H^{-1} has a complete system of root vectors.*

PROOF. For any real c with $-ic \notin \sigma(H)$ with the notation $H_R = (H + H^*)/2$,

$$(H + icI)^{-1} = (I + i(H_R + icI)^{-1} H_I)^{-1}(H_R + icI)^{-1}.$$

Recall the Keldysh theorem, cf. [13, Theorem V. 8.1] and [19].

THEOREM 4.2 (Keldysh). *Let $A = S(I + K)$, where $S = S^* \in SN_p$ for some $p \in [0, \infty)$ and K is compact. In addition, from $Af = 0$ ($f \in \mathfrak{H}$) it follows that $f = 0$. Then A has a complete system of root vectors.*

Take into account that $(H + icI)^{-1} = H^{-1}(I + icH^{-1})^{-1} \in SN_p$. So, $(H_R + icI)^{-1} \in SN_p$ and, by the Keldysh theorem, operator $(H + icI)^{-1}$ has a complete system of root vectors. Since $(H + icI)^{-1}$ and H^{-1} commute, H^{-1} has a complete system of root vectors, as claimed. □

From the previous lemma, it follows that there is an orthonormal (Schur) basis $\{\hat{e}_k\}_{k=1}^{\infty}$ in which H^{-1} is represented by a triangular matrix (see [13, Lemma I.4.1]). Denote $\hat{P}_k = \sum_{j=1}^{k} (., \hat{e}_j)\hat{e}_j$. Then

$$H^{-1}\hat{P}_k = \hat{P}_k H^{-1}\hat{P}_k \quad (k = 1, 2, \ldots).$$

Besides,

$$\Delta\hat{P}_k H^{-1} \Delta\hat{P}_k = \lambda_k^{-1}(H)\Delta\hat{P}_k \quad (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1}, k = 1, 2, \ldots; \hat{P}_0 = 0).$$

Put

$$D = \sum_{k=1}^{\infty} \lambda_k \Delta\hat{P}_k \quad (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1}, k = 1, 2, \ldots) \quad \text{and} \quad V = H - D.$$

We have

$$H\hat{P}_k f = \hat{P}_k H\hat{P}_k f \quad (k = 1, 2, \ldots; f \in \mathrm{Dom}(H)). \tag{4.1}$$

Indeed, $H^{-1}\hat{P}_k$ is an invertible $k \times k$ matrix and, therefore, $H^{-1}\hat{P}_k\mathfrak{H}$ is dense in $\hat{P}_k\mathfrak{H}$. Since $\Delta\hat{P}_j\hat{P}_k = 0$ for $j > k$, we have $0 = \Delta\hat{P}_j H H^{-1}\hat{P}_k = \Delta\hat{P}_j H\hat{P}_k H^{-1}\hat{P}_k$. Hence, $\Delta\hat{P}_j H f = 0$ for any $f \in \hat{P}_k H$. This implies (4.1).

Furthermore, put $H_n = HP_n$. Due to (4.1),

$$\|H_n f - Hf\| \to 0 \quad (f \in \mathrm{Dom}(H)) \quad \text{as } n \to \infty.$$

From Lemma 3.1 and (3.4) with $A = H_n$, it follows that in $\hat{P}_n\mathfrak{H}$ there is a invertible operator T_n such that $T_n H_n = \hat{P}_n D T_n$ and $\|T_n\|^2 \le \gamma_n(H_n) \le \gamma(H)$. So, there is a weakly convergent subsequence T_{n_j} whose limit we denote by T. It is simple to check that $T_n = P_n T$. So, in fact, the pointed subsequence converges strongly. Thus, $T_{n_j} H_{n_j} f \to THf$ and therefore $\hat{P}_{n_j} D T_{n_j} f = T_{n_j} H_{n_j} f \to THf$. Letting $n_j \to \infty$, we arrive at the required result. □

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5. Applications of Theorem 1.1

Rewrite (1.3) as $Hx = T^{-1}DTx$. Let ΔP_k be the eigenprojections of the normal operator D and $E_k = T^{-1}\Delta P_k T$. Then

$$Hx = \sum_{k=1}^{\infty} \lambda_k(H)E_k x \quad (x \in \mathrm{Dom}(H)).$$

Let $f(z)$ be a scalar function defined and bounded on the spectrum of H. Put

$$f(H) = \sum_{k=1}^{\infty} f(\lambda_k(H))E_k.$$

Theorem 1.1 immediately implies the following corollary.

COROLLARY 5.1. *Let conditions (1.1) and (1.2) hold. Then*

$$\|f(H)\| \leq \gamma(H) \sup_k |f(\lambda_k(H))|.$$

In particular,

$$\|e^{-Ht}\| \leq \gamma(H)e^{-\beta(H)t} \quad (t \geq 0),$$

where $\beta(H) = \inf_k \mathrm{Re}\, \lambda_k(H)$ and

$$\|R_\lambda(H)\| \leq \frac{\gamma(H)}{\rho(H,\lambda)} \quad (\lambda \notin \sigma(H)), \tag{5.1}$$

where $\rho(H,\lambda) = \inf_k |\lambda - \lambda_k(H)|$.

Let A and $\tilde{A}$ be linear operators. Then the quantity

$$sv_A(\tilde{A}) := \sup_{t\in\sigma(\tilde{A})} \inf_{s\in\sigma(A)} |t-s|$$

is said to be the variation of $\tilde{A}$ with respect to A.

Now let $\tilde{H}$ be a linear operator in $\mathfrak{H}$ with $\mathrm{Dom}(H) = \mathrm{Dom}(\tilde{H})$ and

$$q := \|H - \tilde{H}\| < \infty. \tag{5.2}$$

From (5.1), it follows that $\lambda \notin \sigma(\tilde{H})$, provided $q\gamma(H) < \rho(H,\lambda)$. So, for any $\mu \in \sigma(\tilde{H})$, we have $q\gamma(H) \geq \rho(H,\mu)$. This inequality implies our next result.

COROLLARY 5.2. *Let conditions (1.1), (1.2) and (5.2) hold. Then* $sv_H(\tilde{H}) \leq q\gamma(H)$.

Now consider unbounded perturbations. To this end, put

$$H^{-\nu} = \sum_{k=1}^{\infty} \lambda_k^{-\nu}(H)E_k \quad (0 < \nu \leq 1).$$

We define H^ν similarly. We have

$$\|H^\nu R_\lambda(H)\| \leq \frac{\gamma(H)}{\psi_\nu(H,\lambda)} \quad (\lambda \notin \sigma(H)),$$

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where

$$\psi_\nu(H,\lambda) = \inf_k |(\lambda - \lambda_k(H))\lambda_k^{-\nu}(H)|.$$

Now let $\tilde{H}$ be a linear operator in $\mathfrak{H}$ with $\mathrm{Dom}(H) = \mathrm{Dom}(\tilde{H})$ and

$$q_\nu := \|(H - \tilde{H})H^{-\nu}\| < \infty. \tag{5.3}$$

Take into account that

$$R_\lambda(H) - R_\lambda(\tilde{H}) = R_\lambda(H)(\tilde{H} - H)R_\lambda(\tilde{H}) = R_\lambda(\tilde{H})(\tilde{H} - H)H^{-\nu}H^{\nu}R_\lambda(H),$$

$\lambda \notin \sigma(\tilde{H})$, provided the conditions (5.3) and $q_\nu\gamma(H) < \psi_\nu(H,\lambda)$ hold. So, for any $\mu \in \sigma(\tilde{H})$, we have

$$q_\nu\gamma(H) \geq \psi(H,\mu). \tag{5.4}$$

The quantity

$$\nu - \mathrm{rsv}_H(\tilde{H}) := \sup_{t\in\sigma(\tilde{H})} \inf_{s\in\sigma(H)} |(t-s)s^{-\nu}|$$

is said to be the ν-relative spectral variation of operator $\tilde{H}$ with respect to H. Now (5.4) implies the following corollary.

COROLLARY 5.3. *Let conditions (1.1), (1.2) and (5.3) hold. Then* $\nu - \mathrm{rsv}_H(\tilde{H}) \leq q_\nu\gamma(H)$.

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