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PREFRATTINI SUBGROUPS AND COVER-AVOIDANCE PROPERTIES IN 𝔘-GROUPS

M. J. TOMKINSON

1. Introduction. W. Gaschütz [**5**] introduced a conjugacy class of subgroups of a finite soluble group called the prefrattini subgroups. These subgroups have the property that they avoid the complemented chief factors of G and cover the rest. Subsequently, these results were generalized by Hawkes [**12**], Makan [**14**; **15**] and Chambers [**2**]. Hawkes [**12**] and Makan [**14**] obtained conjugacy classes of subgroups which avoid certain complemented chief factors associated with a saturated formation or a Fischer class. Makan [**15**] and Chambers [**2**] showed that if W, D and V are the prefrattini subgroup, $\mathfrak{F}$-normalizer and a strongly pronormal subgroup associated with a Sylow basis **S**, then any two of W, D and V permute and the products and intersections of these subgroups have an explicit cover-avoidance property. It was also shown by Makan [**15**] that W, D and V generate a distributive sublattice of the subgroup lattice of G.

Our aim here is to present these ideas in a more unified setting and also to consider the extension of the results to the class $\mathfrak{U}$ of locally finite groups with a satisfactory Sylow structure. The class $\mathfrak{U}$ was introduced in [**3**], in which a theory of saturated formations was developed for each QS-closed subclass $\mathfrak{K}$ of $\mathfrak{U}$. Further results from the theory of finite soluble groups have also been extended to the class $\mathfrak{U}$ (see e.g. [**6**; **7**; **10**]) and this paper may be considered as a continuation of this programme.

The main results concern the prefrattini subgroups of $\mathfrak{U}$-groups. The difficulties which one expects in dealing with maximal subgroups in infinite groups are largely surmounted by using two results of B. Hartley. The first of these (Theorem E and Lemma 4.2 of [**8**]) reduces the definition of $\mathfrak{U}$ to being the class of locally finite groups in which each subgroup has conjugate Sylow Π-subgroups for each set of primes Π and also shows that:

THEOREM 1.1. *A $\mathfrak{U}$-group G has a series*

$$1 \leqq G_1 \leqq G_2 \leqq G_3 \leqq G,$$

where $G_i \lhd G$, G_1 is locally nilpotent, G_2/G_1 is divisible abelian of finite rank, G_3/G_2 is abelian with a finite Sylow p-subgroup for each prime p, and G/G_3 is finite.

We shall mainly use the corollary of this result that $G/O_{p'pp'}(G)$ is finite.

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The second result of Hartley on which our results depend heavily is the following [**11**]:

THEOREM 1.2. *Let G be a periodic abelian-by-finite group and V an $\mathfrak{M}_c$-module over $\mathbf{Z}_pG$. If $\Phi(V) = 0$, then every composition factor of V is complemented.*

We follow Hawkes' approach in defining the prefrattini subgroups. If $\mathbf{S}$ is a Sylow basis of G and $S_{p'}$ is the associated Sylow p'-subgroup of G, then we define

$$W_p(\mathbf{S}) = \bigcap \{M | M \text{ is maximal in } G \text{ and } M \geqq S_{p'}\}.$$

The prefrattini subgroup of G associated with $\mathbf{S}$ is defined to be

$$W(\mathbf{S}) = \bigcap_p W_p(\mathbf{S}).$$

Since the Sylow bases of a $\mathfrak{U}$-group are conjugate [**3**, Theorem 2.10], this definition ensures that the prefrattini subgroups form a characteristic conjugacy class of subgroups of G. Our main result is:

THEOREM A. *Let $G \in \mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$. Then a prefrattini subgroup of G avoids every complemented chief factor of G and covers the rest.*

Here $L\mathfrak{N}$ denotes the class of locally nilpotent groups, $\mathfrak{A}$ the class of abelian groups and $\mathfrak{S}^*$ the class of finite soluble groups. The above result cannot be extended to apply to the whole of $\mathfrak{U} \subseteq (L\mathfrak{N})\mathfrak{A}^2\mathfrak{S}^*$ but the class of groups considered does contain the more well-known subclasses of $\mathfrak{U}$, in particular $(L\mathfrak{N})\mathfrak{S}^*$ and $\mathfrak{C}$, the class of homomorphic images of periodic soluble linear groups.

Our application of Theorem 1.2 in the proof of Theorem A will depend on the following extension of a theorem of Gaschütz [**4**] which we prove in Section Three.

THEOREM B. *Let A be an abelian normal p-subgroup of the $\mathfrak{U}$-group G such that G/A is a finite extension of a p'-group. If $A \cap \Phi(G) = 1$, then A has a complement in G.*

In order to present the permutability results more systematically, we make the following definition:

Let $\mathbf{S}$ be a Sylow basis of the $\mathfrak{U}$-group G. A set $\mathscr{B} = \{B_p\}$ of subgroups of G, one for each prime p, is called a *CAR-system* associated with $\mathbf{S}$ (or a $\mathbf{S}$*CAR-system*) if

(i) $B_p \geqq S_{p'}$ for each prime p,

(ii) B_p either covers or avoids each chief factor of G.

(CAR = cover-avoidance, reducing.)

The intersection $B = \bigcap_p B_p$ is called a CAR-subgroup associated with $\mathbf{S}$ (or a $\mathbf{S}$*CAR-subgroup*).

If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}$-group G, then $D = \bigcap_p D_p$ where $D_p = N_G(C_p \cap S_{p'})$ if $p \in \pi(\mathfrak{F})$ and $D_p = S_{p'}$ if $p \notin \pi(\mathfrak{F})$. (See [**3**] for notation.) We see that D is an example of a CAR-subgroup. Theorem A shows that a prefrattini subgroup $W = \bigcap_p W_p$ is a CAR-subgroup if $G \in (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$.

Chambers [**2**] calls a subgroup V *strongly pronormal* if, for each prime p, V has a Sylow p-subgroup P which is a Sylow p-subgroup of its normal closure P^G. We see in Section Two that these subgroups are a very special example of CAR-subgroups and will have a natural position in any general discussion of these subgroups.

We also show that the strongly pronormal subgroups into which $\mathbf{S}$ reduces permute with every $\mathbf{S}CAR$-subgroup (2.5) and that in a $\mathfrak{U}_A$-group (i.e. a $\mathfrak{U}$-group in which every p-subgroup is abelian) every CAR-subgroup is strongly pronormal (2.8).

Our results for prefrattini subgroups are, in fact, given in a more general form in Section Four than has been stated in Theorem A. We consider $\mathscr{X}$-prefrattini subgroups, which are the intersection of the maximal subgroups in a suitable set $\mathscr{X}$ which contain $S_{p'}$. Theorem B is used to show that the set of all maximal subgroups and also certain other sets can be used for $\mathscr{X}$.

We then show that if D is an $\mathfrak{F}$-normalizer of G then we can define associated sets $\mathscr{X}_{\mathfrak{F}}$ and $\mathscr{X}_{\bar{\mathfrak{F}}}$. The $\mathscr{X}_{\mathfrak{F}}$-prefrattini subgroup is shown to be the product of D and the $\mathscr{X}$-prefrattini subgroup.

The result of Makan [**15**] concerning the distributivity of the lattice generated by V, W and D is obtained as a special case of a general result which again depends heavily on a strongly pronormal subgroup being a very special type of CAR-subgroup.

Finally in Section Seven we observe that the proof that the prefrattini subgroups of a finite soluble group are CAR-subgroups can be shortened by using a very elementary lemma. Unfortunately this method involves the consideration of a minimal normal subgroup and cannot be used in the class $\mathfrak{U}$.

2. *CAR*-subgroups.

PROPOSITION 2.1. *Let $\mathscr{B} = \{B_p\}$ be a $\mathbf{S}CAR$-system of the $\mathfrak{U}$-group G and $B = \bigcap_p B_p$ the corresponding $\mathbf{S}CAR$-subgroup. Then*

(i) $\mathbf{S}$ *reduces into B and $B \cap S_p = B_p \cap S_p$,*

(ii) *B covers those p-chief factors covered by B_p and avoids those which are avoided by B_p.*

LEMMA 2.2. *If $\mathscr{B} = \{B_p\}$ is a $\mathbf{S}CAR$-system of the $\mathfrak{U}$-group G and $N \triangleleft G$, then $BN/N = \{B_pN/N\}$ is a $(\mathbf{S}N/N)CAR$-system of G/N. If B and $\bar{B}$ are the corresponding CAR-subgroups, then $\bar{B} = BN/N$.*

Proof. The first part of the lemma is clear. By (2.1), $(B_pN \cap S_pN)/N$ is a Sylow p-subgroup of $\bar{B}$. But $(B_pN \cap S_pN)/N$ and $(B_p \cap S_p)N/N$ are both Sylow p-subgroups of B_pN/N and so $(B_pN \cap S_pN)/N = (B_p \cap S_p)N/N$ is a Sylow p-subgroup of BN/N. Hence $\bar{B} = BN/N$.

We now observe that new $\mathbf{S}CAR$-systems can be constructed from a given $\mathbf{S}CAR$-system $\mathscr{B}$ and an arbitrary collection of normal subgroups V_p.

THEOREM 2.3. *Let* $\mathscr{B} = \{B_p\}$ *be a* $\mathbf{S}CAR$*-system of the* $\mathfrak{U}$*-group* G *and, for each prime* p, *let* V_p *be a normal subgroup of* G. *Then*

(i) $\mathscr{J} = \{B_pV_p\}$ *is a* $\mathbf{S}CAR$*-system of* G*; in particular* $\mathscr{V} = \{S_{p'}V_p\}$ *is a* $\mathbf{S}CAR$*-system of* G*;*

(ii) *if* $J = \bigcap_p B_pV_p$ *and* $V = \bigcap_p S_{p'}V_p$, *then* $J = BV$;

(iii) J *avoids the* p*-chief factor* H/K *of* G *if and only if* V *avoids* H/K *and* B *avoids* HV_p/KV_p*;*

(iv) $\mathscr{I} = \{B_p \cap S_{p'}V_p\}$ *is a* $\mathbf{S}CAR$*-system of* G*;*

(v) *if* $I = \bigcap_p (B_p \cap S_{p'}V_p) = B \cap V$, *then* I *covers the* p*-chief factor* H/K *if and only if* V *covers* H/K *and* B *covers* $(H \cap V_p)/(K \cap V_p)$.

Proof. It is clear that $\mathscr{J}$ and $\mathscr{I}$ are $\mathbf{S}CAR$-systems. We must therefore prove that $J = BV$ and the cover-avoidance properties (iii) and (v).

$$S_{p'}.(B_p \cap S_p)(V_p \cap S_p) = B_p(V_p \cap S_p) = B_p(V_p \cap S_{p'})(V_p \cap S_p) = B_pV_p$$

and so $(B_p \cap S_p)(V_p \cap S_p)$ is a Sylow p-subgroup of B_pV_p. Hence J has a Sylow p-subgroup

$$B_pV_p \cap S_p = (B_p \cap S_p)(V_p \cap S_p) = (B_p \cap S_p)(V_pS_{p'} \cap S_p) = (B \cap S_p)(V \cap S_p).$$

$$\begin{aligned}(B_pV_p \cap S_p)(B_qV_q \cap S_q) &= (B_p \cap S_p)(V_pS_{p'} \cap S_p)(B_q \cap S_q)(V_qS_{q'} \cap S_q)\\ &= (B \cap S_p)(V_pS_{p'} \cap S_p(B_q \cap S_q))(V \cap S_q)\\ &= (B \cap S_p)(V_pS_{p'} \cap B_q \cap S_pS_q)(V \cap S_q)\\ &= (B \cap S_p)(B_q \cap S_q(V_pS_{p'} \cap S_p))(V \cap S_q)\\ &= (B \cap S_p)(B_q \cap S_q)(V_pS_{p'} \cap S_p)(V \cap S_q)\\ &= (B \cap S_p)(B \cap S_q)(V \cap S_p)(V \cap S_q).\end{aligned}$$

Therefore $J = \langle B_pV_p \cap S_p | p \text{ a prime}\rangle \leqq BV \leqq \langle B, V\rangle \leqq J$ and so $J = BV$.

If V or B covers H/K, then clearly J covers H/K. So let H/K be a p-chief factor avoided by both V and B. Then $H \cap KV_p = K$ and $H.KV_p = HV_p$. If B avoids HV_p/KV_p, then $KV_pB_p \cap HV_p = KV_p$ and so $K.V_pB_p \cap H = KV_p \cap H = K$, i.e., BV avoids H/K. If B covers HV_p/KV_p, then $KV_pB_p \geqq HV_p \geqq H$ and so BV covers H/K. This proves (iii).

If V or B avoid H/K, then clearly I avoids H/K. So let H/K be a p-chief factor covered by both V and B. Then $K(V_p \cap H) = H$ and $K \cap (V_p \cap H) = V_p \cap K$. If B covers $(H \cap V_p)/(K \cap V_p)$, then $B_p(K \cap V_p) \geqq H \cap V_p$ and so $(B_p \cap S_{p'}V_p)(K \cap V_p) \geqq H \cap V_p \cap S_{p'}V_p = H \cap V_p$. Therefore $(B_p \cap S_{p'}V_p)K \geqq (H \cap V_p)K = H$ and I covers H/K. If B avoids $(H \cap V_p)/(K \cap V_p)$, then $B_p \cap H \cap V_p \leqq K \cap V_p \leqq K$. Thus $B_p \cap S_{p'}V_p \cap H \leqq K$ and I avoids H/K.

The cover-avoidance properties of J and I can be given in a clearer form if we impose a fairly natural condition on B. A CAR-subgroup B is called *perspective* if whenever H/K and H_1/K_1 are chief factors of G such that $H \cap K_1 = K$, $HK_1 = H_1$ and B covers H_1/K_1, then B covers H/K.

An $\mathfrak{F}$-normalizer is an example of a perspective CAR-subgroup.

COROLLARY 2.4. *Let $\mathscr{B}$ be a perspective $\mathbf{S}CAR$-system of the $\mathfrak{U}$-group G and let $\mathscr{V} = \{S_{p'} V_p\}$ where $V_p \lhd G$. Then*

(i) *$J = BV$ avoids those chief factors which are avoided by both V and B and covers the rest;*

(ii) *$I = B \cap V$ covers those chief factors which are covered by both V and B and avoids the rest.*

We now show that these CAR-subgroups V are just the strongly pronormal subgroups considered by Chambers [**2**].

THEOREM 2.5. *A subgroup V of the $\mathfrak{U}$-group is strongly pronormal in G if and only if V is a $\mathbf{S}CAR$-subgroup of the form $\bigcap_p S_{p'} V_p$, where $\mathbf{S}$ is a Sylow basis reducing into V and each V_p is a normal subgroup of G.*

Proof. Let V be strongly pronormal in G and let $\mathbf{S}$ be a Sylow basis of G reducing into V. Let $V_p = (S_p \cap V)^G$; then $V \leqq \bigcap_p S_{p'} V_p$. But $\bigcap_p S_{p'} V_p$ has a Sylow p-subgroup $(S_p \cap S_{p'} V_p) = S_p \cap V_p = S_p \cap V$ and so $V = \bigcap_p S_{p'} V_p$.

Conversely, let $V = \bigcap_p S_{p'} V_p$ be a $\mathbf{S}CAR$-subgroup with $V_p \lhd G$. Then $S_p \cap V = S_p \cap S_{p'} V_p = S_p \cap V_p$ is a Sylow p-subgroup of the normal subgroup V_p and so V is p-normally embedded for each prime p.

COROLLARY 2.6. *Let V be a strongly pronormal subgroup of the $\mathfrak{U}$-group G and let $\mathbf{S}$ be a Sylow basis of G reducing into V. Then V permutes with every $\mathbf{S}CAR$-subgroup of G.*

Using the characterization given in (2.5) together with (2.4) for the special case of an $\mathfrak{F}$-normalizer we have:

COROLLARY 2.7. *Let $\mathfrak{K}$ be a QS-closed subclass of $\mathfrak{U}$ and let $\mathfrak{F}$ be a saturated $\mathfrak{K}$-formation. If G is a $\mathfrak{K}$-group, V is a strongly pronormal subgroup of G, $\mathbf{S}$ is a Sylow basis of G reducing into V and D is the $\mathfrak{F}$-normalizer of G associated with $\mathbf{S}$, then:*

(i) *DV avoids the $\mathfrak{F}$-eccentric chief factors avoided by V and covers the rest;*

(ii) *$D \cap V$ covers the $\mathfrak{F}$-central chief factors covered by V and avoids the rest.*

In view of the following result, the permutability of strongly pronormal subgroups is particularly useful in $\mathfrak{U}_A$-groups.

THEOREM 2.8. *Let G be a $\mathfrak{U}_A$-group. Then every CAR-subgroup of G is strongly pronormal.*

Proof. G has p-length one for each prime p [**6**, Lemma 2.2]. Let $K = O_{p'p}(G)$ and let $B = \bigcap_p B_p$ be a $\mathbf{S}CAR$-subgroup of G. Then $B_p = S_{p'}(K \cap B_p)$ and

$K \cap B_p \lhd B_p$. Also, since $B_p \geqq O_{p'}(G)$ and $K/O_{p'}(G)$ is abelian, $K \cap B_p \lhd K$. Therefore $K \cap B_p \lhd KB_p \geqq KS_{p'} = G$. Thus $B_p = S_{p'}V_p$, where $V_p = K \cap B_p \lhd G$ and the characterization in (2.5) completes the proof.

COROLLARY 2.8. *Let G be a $\mathfrak{U}_A$-group. Then any two* **S***CAR-subgroups of G are permutable.*

It should be noted that it is not sufficient in (2.8) to assume that G has p-length one for each prime p. Hawkes [**13**] constructs a finite soluble group with p-length one for each prime p but in which the basis normalizers are not 2-normally embedded.

3. The complementation theorem and its consequences.

Proof of Theorem B. Let $Q/A = O_{p'}(G/A)$ so that G/Q is finite. Let S be a Sylow p'-subgroup of Q, so that $Q = AS$ and $A \cap S = 1$. By a Frattini argument, $G = QN_G(S)$ and so there is a finite subgroup F of $N_G(S)$ such that $G = QF$. Choose F such that $G = QF$ and $|A \cap F|$ is minimal. $A \cap F$ is normalized by A (since A is abelian) and by F (since $A \lhd G$) and is centralized by S (since $[A \cap F, S] \leqq A \cap S = 1$). Thus $A \cap F \lhd AFS = G$.

If $A \cap F \neq 1$ then, since $A \cap \Phi(G) = 1$, there is a maximal subgroup M of G such that $A \cap F \cap M < A \cap F$. Since M is maximal in G, $G = (A \cap F)M$ and so $F = (A \cap F)M \cap F = (A \cap F)(M \cap F)$ and $G = QF = Q(M \cap F)$ contrary to the minimality of $A \cap F$. Thus $A \cap F = 1$ and hence $A \cap FS = 1$, since $A \cap FS$ is a normal p-subgroup of FS and so is contained in every Sylow p-subgroup of FS, and in particular, is contained in F.

It should be noted that a $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}$-group G may have a normal abelian subgroup A such that $A \cap \Phi(G) = 1$ but A has no complement. This is shown by Example 4.1 of [**10**] which is a split extension of an elementary abelian p-group A by the group

$$H = \operatorname*{Dr}_{i=1}^{\infty} C_{q_i},$$

for suitably chosen primes q_i, such that A is not completely reducible as a $\mathbf{Z}_pH$-module. Thus A contains a normal subgroup N of G which has no G-admissible complement in A and hence has no complement in G.

However, combining Theorem B with Theorem 1.2 we shall be able to prove results about complemented chief factors of $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-groups which are sufficient for our purposes. Here and even more in the next section we shall require more information about complemented chief factors. The following lemma, which states exactly what the complements are, is essentially Theorem 2.2 of [**7**], the additions all being straightforward.

LEMMA 3.1. *Let H/K be a chief factor of the $\mathfrak{U}$-group G complemented by M and let $L = \mathrm{core}_G(M)$. Then*

(i) *G/L has a unique minimal normal subgroup* $C/L = O_p(G/L) = \rho(G/L)$;

(ii) $C = C_G(H/K)$ *and* $H/K \overset{G}{\cong} C/L$,

(iii) *if Q/C is a non-trivial normal p'-subgroup of G/C and $M \geqq S_{p'}$ then* $M = N_G(Q \cap LS_{p'}) = LN_G(Q \cap S_{p'})$.

THEOREM 3.2. *Let A be a normal p-subgroup of the $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G. If $A \cap \Phi(G) = 1$ and H/K is a chief factor of G with $H \leqq A$, then H/K has a complement in G.*

Proof. If M is a maximal subgroup of G not containing A, then $M \cap A \lhd A$ [**7**, Lemma 2.3] and so $M \cap A \lhd AM = G$. Thus $A/(A \cap M)$ is a chief factor of G and so is elementary abelian. Also, $C_G(A/(A \cap M)) \geqq O_{p'p}(G)$ [**3**, Theorem 3.8] and since $A \cap \Phi(G) = 1$ we see that A is elementary abelian and $C = C_G(A) \geqq O_{p'p}(G)$. Hence G/C is a finite extension of a p'-group.

Let M_i, $i \in I$, be the maximal subgroups of G not containing A. If $C_i = C_G(A/(A \cap M_i))$ then $C_i \geqq C$ and it follows from (3.1) that

$$A/(A \cap M_i) \overset{G}{\cong} C/(C \cap M_i) \overset{G}{\cong} C_i/(C_i \cap M_i).$$

Let $L = C \cap \bigcap_{i \in I} M_i$ so that $L \lhd G$ and $A \cap L = 1$. Using Theorem B, with C/L and G/L replacing A and G, we see that there is a subgroup U of G with $CU = G$ and $C \cap U = L$.

Since $A \overset{G}{\cong} AL/L$, we have $H/K \overset{G}{\cong} HL/KL$. Applying Theorem 1.2 to the $\mathbf{Z}_p(G/C)$-module C/L, we see that there is a subgroup $N \lhd G$ with $N.HL = C$ and $N \cap HL = KL$. Thus $NH = C$ and $N \cap H = KL \cap H = K$. It is clear that $UN.C = G$ and $UN \cap C = N(U \cap C) = N$. Therefore $UN.H = UNC = G$ and $UN \cap H = N \cap H = K$. Thus UN is a complement of H/K.

Our discussion of prefrattini subgroups will depend on certain sets of maximal subgroups. We define a set $\mathscr{X}$ of maximal subgroups of the group G to be *solid* if it satisfies the following two conditions:

(S1) if $M \in \mathscr{X}$ and $g \in G$, then $M^g \in \mathscr{X}$;

(S2) if the chief factors H/K_i, $i \in I$, each have a complement in $\mathscr{X}$ then every chief factor H/K, with $K \geqq \bigcap_{i \in I} K_i$, is complemented and all the complements of H/K are in $\mathscr{X}$.

If H/K has a complement in $\mathscr{X}$ then it is called an $\mathscr{X}$-*complemented* chief factor; otherwise H/K is called an $\mathscr{X}$-*Frattini* chief factor. Taking $K_i = K$ in (S2), we have

(S3) every complement of an $\mathscr{X}$-complemented chief factor is in X.

If $\mathscr{X} = \mathscr{X}(G)$ is a solid set of maximal subgroups of G and $N \lhd G$ then we shall also use $\mathscr{X}$ to denote the solid set of maximal subgroups of G/N of the form M/N with $M \in \mathscr{X}(G)$. Then a chief factor $(H/N)/(K/N)$ of G/N is $\mathscr{X}$-complemented if and only if H/K is an $\mathscr{X}$-complemented chief factor of G.

The set of all maximal subgroups of a group G clearly satisfies (S1) and

Theorem 3.2 shows that if $G \in \mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$ then this set also satisfies (S2). Thus we have.

COROLLARY 3.3. *If $G \in \mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$, then the set of all maximal subgroups of G is solid.*

We show that there are other interesting solid sets of maximal subgroups.

THEOREM 3.4. *Let $G \in \mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$ and let $\mathscr{Y}$ denote a class of chief factors of G of rank $\leqq n$ such that $\mathscr{Y}$ is closed under G-isomorphism. If $\mathscr{X}$ is the set of those maximal subgroups which complement a chief factor in $\mathscr{Y}$, then $\mathscr{X}$ is solid.*

Proof. Again it is clear that $\mathscr{X}$ satisfies (S1). So let H/K_i, $i \in I$, be $\mathscr{X}$-complemented chief factors and let H/K be a p-chief factor with $K \geqq \bigcap_{i\in I} K_i$. We may assume that each H/K_i is a p-chief factor and that $\bigcap_{i\in I} K_i = 1$. If $C_i = C_G(H/K_i)$, then there is a finite bound on $|G : C_i|$ dependent only on n. Using the notation of Theorem 1.1, $C_i \geqq G_2$ and there is a normal subgroup of finite index in G_3 contained in C_i, for each i. Thus $G/C_G(H)$ is finite and so, by Theorem B of [**10**], H is completely reducible as a $\mathbf{Z}_pG$-module. Therefore

$$H/K \overset{G}{\cong} H/K_i \text{ for some } i$$

and so $H/K \in \mathscr{Y}$ and since H/K has a complement (3.2) it must be $\mathscr{X}$-complemented.

It seems unlikely that the condition bounding the ranks of the chief factors can be omitted from Theorem 3.4. Because of this it does not show that the sets of $\mathfrak{F}$-abnormal maximal subgroups and of $\mathfrak{F}$-normal maximal subgroups are solid. However, we can, in fact, prove rather more than this.

THEOREM 3.5. *Let $\mathfrak{K}$ be a QS-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ a saturated $\mathfrak{K}$-formation. If $\mathscr{X}$ is any solid set of maximal subgroups of the $\mathfrak{K} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G then*

$$\mathscr{X}_{\mathfrak{F}} = \{M \in \mathscr{X} \mid M \text{ is } \mathfrak{F}\text{-abnormal in } G\}$$

and

$$\mathscr{X}_{\bar{\mathfrak{F}}} = \{M \in \mathscr{X} \mid M \text{ is } \mathfrak{F}\text{-normal in } G\}$$

are both solid sets.

Proof. Again it is clear that $\mathscr{X}_{\mathfrak{F}}$ and $\mathscr{X}_{\bar{\mathfrak{F}}}$ satisfy (S1). So let H/K_i, $i \in I$, be $\mathscr{X}$-complemented chief factors of G and let H/K be a chief factor such that $K \geqq \bigcap_{i\in I} K_i = 1$. If each H/K_i is $\mathscr{X}_{\mathfrak{F}}$-complemented, then each H/K_i is $\mathfrak{F}$-eccentric. If D is an $\mathfrak{F}$-normalizer of G, then D avoids each H/K_i and so $D \cap H = 1$. Therefore D avoids H/K and H/K is $\mathfrak{F}$-eccentric. H/K is $\mathscr{X}$-complemented and since H/K is $\mathfrak{F}$-eccentric each complement must be $\mathfrak{F}$-abnormal i.e., H/K is $\mathscr{X}_{\mathfrak{F}}$-complemented.

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If each H/K_i is $\mathscr{X}_{\mathfrak{F}}$-complemented, then each H/K_i is $\mathfrak{F}$-central. Therefore $C_G(H/K) \geqq \bigcap_{i \in I} C_G(H/K_i) \geqq C_p$ and so H/K is $\mathfrak{F}$-central. H/K is $\mathscr{X}$-complemented and since it is $\mathfrak{F}$-central, each complement must be $\mathfrak{F}$-normal.

Since $\mathscr{X}_{\mathfrak{F}}$ and $\mathscr{X}_{\bar{\mathfrak{F}}}$ are solid sets it is clear that, if $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ are saturated $\mathfrak{K}$-formations, we can construct solid sets of the form $\mathscr{X}_{\mathfrak{C}_1\mathfrak{C}_2\ldots\mathfrak{C}_n}$, where $\mathfrak{C}_i$ denotes either $\mathfrak{F}_i$ or $\bar{\mathfrak{F}}_i$.

4. $\mathscr{X}$-prefrattini subgroups. If $\mathscr{X}$ is a solid set of maximal subgroups of the $\mathfrak{U}$-group G and $\mathbf{S}$ is a Sylow basis of G, then we define

$$W_p^{\mathscr{X}}(\mathbf{S}) = \bigcap \{M \in \mathscr{X} | M \geqq S_{p'}\}.$$

Clearly $W_p^{\mathscr{X}}(\mathbf{S}) \geqq S_{p'}$ and so $\mathbf{S}$ reduces into $W_p^{\mathscr{X}}(\mathbf{S})$. The *$\mathscr{X}$-prefrattini subgroup* of G associated with $\mathbf{S}$ is defined to be

$$W^{\mathscr{X}}(\mathbf{S}) = \bigcap_p W_p^{\mathscr{X}}(\mathbf{S}).$$

Since the Sylow bases of a $\mathfrak{U}$-group are conjugate [**3**, Theorem 2.10], this definition ensures that the $\mathscr{X}$-prefrattini subgroups form a characteristic conjugacy class of subgroups of G. To show that they satisfy the expected cover-avoidance property we need to show that $\mathscr{W}^{\mathscr{X}} = \{W_p^{\mathscr{X}}(\mathbf{S})\}$ is a $\mathbf{S}CAR$-system and that $W_p^{\mathscr{X}}(\mathbf{S})$ avoids a p-chief factor H/K if and only if H/K is $\mathscr{X}$-complemented.

It is sufficient to prove that if $N \lhd G$, then $W_p^{\mathscr{X}}(\mathbf{S}N/N) = W_p^{\mathscr{X}}(\mathbf{S})N/N$. For if H/N is an $\mathscr{X}$-complemented p-chief factor of G, then H/N is avoided by $W_p^{\mathscr{X}}(\mathbf{S})$. If H/N is a p-chief factor with no complement in $\mathscr{X}$, then $W_p^{\mathscr{X}}(\mathbf{S}N/N) \geqq H/N$ and so $W_p^{\mathscr{X}}(\mathbf{S})$ covers H/N.

By using induction on the length of a p-series for N, we may clearly assume that N is a p-group. We must therefore prove:

THEOREM 4.1. *Let N be a normal p-subgroup of the $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G and let $\mathscr{X}$ be a solid set of maximal subgroups of G. If $W_p = W_p^{\mathscr{X}}(\mathbf{S})$ and $\bar{W}_p/N = W_p^{\mathscr{X}}(\mathbf{S}N/N)$, then $NW_p = \bar{W}_p$.*

The following stronger form of Theorem A will then follow:

THEOREM 4.2. *Let $\mathscr{X}$ be a solid set of maximal subgroups of the $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G. Then the $\mathscr{X}$-prefrattini subgroups of G avoid the $\mathscr{X}$-complemented chief factors of G and cover the rest.*

With the notation of Lemma 3.1, we have seen that each complement M_λ of H/K containing $S_{p'}$ is of the form $L_\lambda N_G(Q \cap S_{p'})$. We have to consider the intersection of these complements and will make use of the following:

LEMMA 4.3. *If Q and L_λ, $\lambda \in \Lambda$, are normal subgroups, and S_Π is a Sylow Π-subgroup of the $\mathfrak{U}$-group G, then*

$$\bigcap_{\lambda \in \Lambda} (L_\lambda N_G(Q \cap S_\Pi)) = \left(\bigcap_{\lambda \in \Lambda} L_\lambda\right) N_G(Q \cap S_\Pi).$$

The proof of (4.3) is identical to that of Lemma 4.9 of [**16**].

If M is a maximal subgroup of G not containing N, then $N \cap M \lhd N$ [**7**, Lemma 2.3] and so $N \cap M \lhd NM = G$. Thus $N \cap W_p \lhd G$ and using (3.2) and the definition of a solid set of maximal subgroups we have:

LEMMA 4.4. *Each chief factor of G between N and $N \cap W_p$ is $\mathscr{X}$-complemented.*

LEMMA 4.5. *Let N be a normal p-subgroup of the $\mathfrak{U} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G such that $N \cap W_p = 1$. If $C_G(H/K) = C$ for each chief factor H/K of G below N, then $NW_p = \bar{W}_p$.*

Proof. Let M_λ, $\lambda \in \Lambda$, be the members of $\mathscr{X}$ containing $S_{p'}$ but not N. Then $W_p = \bar{W}_p \cap \bigcap_{\lambda \in \Lambda} M_\lambda$. Let $L_\lambda = \text{core}_G(M_\lambda) = C \cap M_\lambda$ and $Q/C = O_{p'}(G/C)$, so that $M_\lambda = L_\lambda N_G(Q \cap S_{p'})$ (3.1). If $L = \bigcap_{\lambda \in \Lambda} L_\lambda$, then

$$\bigcap_{\lambda \in \Lambda} M_\lambda = \bigcap_{\lambda \in \Lambda} L_\lambda N_G(Q \cap S_{p'}) = LN_G(Q \cap S_{p'}), \quad \text{by (4.3)},$$

and so $NW_p = \bar{W}_p \cap NLN_G(Q \cap S_{p'})$.

C/L is a normal p-subgroup of G/L and L is the intersection of C and certain members of $\mathscr{X}$. Therefore each chief factor of G between C and L is $\mathscr{X}$-complemented (4.4). Let H_i/K_i, $i \in I$, be the chief factors of G between C and L then there is a chief factor C/L_i such that $L_i \cap H_i = K_i$ and C/L_i is complemented by $L_iN_G(Q \cap S_{p'})$. Thus $L_iN_G(Q \cap S_{p'}) \in \mathscr{X}$ and so $\bar{W}_p \leqq \bigcap_{i \in I} L_iN_G(Q \cap S_{p'}) = NLN_G(Q \cap S_{p'})$. Thus $NW_p = \bar{W}_p$, as required.

Proof of Theorem 4.1. Let $A = O_{p'p}(G)$ and $B = O_{p'pp'}(G)$ so that G/B is finite (1.1). We may assume that $N \cap W_p = 1$ so that each chief factor of G below N is $\mathscr{X}$-complemented (4.4). There are only finitely many subgroups $C_1, \ldots, C_k$ containing B which occur as centralizers of chief factors of G below N. The $C_1, \ldots, C_k$ can be ordered so that if $C_i \geqq C_j$, then $i \leqq j$. For each $r = 1, \ldots, k$, we define

$$N_r = \bigcap \{K_\lambda | N/K_\lambda \text{ is a chief factor of } G \text{ and } C_G(N/K_\lambda) = C_s, \text{ for some } s = 1, \ldots, r\}.$$

This gives a series

$$N = N_0 \geqq N_1 \geqq \ldots \geqq N_k \geqq 1$$

of normal subgroups of G. Clearly no chief factor of G below N_k is centralized by B. Let H/K be a chief factor of G between N_{r-1} and N_r. By (3.2), there is a chief factor N/K_λ such that $H \cap K_\lambda = K$ and $C_G(H/K) = C_G(N/K_\lambda)$. Since $K_\lambda \ngeqq N_{r-1}$, it is clear that $C_G(H/K) \neq C_t$ for any $t < r$. But also $[C_r, N_{r-1}] \leqq N_r$ and so $C_G(H/K) \geqq C_r$. It follows from the ordering of $C_1, \ldots, C_k$ that $C_G(H/K) = C_r$, for each chief factor H/K of G between N_{r-1} and N_r.

An induction argument using (4.5) shows that we may assume that $N_k = N$ so that no chief factor of G below N is centralized by B.

Let M_λ, $\lambda \in \Lambda$, be the members of $\mathscr{X}$ containing $S_{p'}$ but not N, so that $W_p = \bar{W}_p \cap \bigcap_{\lambda \in \Lambda} M_\lambda$.

Write L_λ for $A \cap M_\lambda$ and $L = \bigcap_{\lambda \in \Lambda} L_\lambda$. If A/L_λ has distinct complements M_λ, $\bar{M}_\lambda$ containing $S_{p'}$ and if $C = C_G(A/L_\lambda)$, then $C/(C \cap M_\lambda \cap \bar{M}_\lambda)$ is the direct product of two minimal normal p-subgroups with centralizer C. Therefore if $D = A(C \cap M_\lambda \cap \bar{M}_\lambda)$ then $C_G(C/D) = C$. But also $[B, C] \leqq B \cap C \leqq D$ since B/A is a p'-group and this is contrary to $C = C_G(N/(N \cap L_\lambda)) \ngeqq B$.

Thus A/L_λ has the unique complement M_λ containing $S_{p'}$. By a Frattini argument $AN_G(B \cap S_{p'}) = G$. Therefore $A \cap L_\lambda N_G(B \cap S_{p'}) \lhd AN_G(B \cap S_{p'}) = G$ and so $A \cap L_\lambda N_G(B \cap S_{p'})$ is equal to A or L_λ. If $A \leqq L_\lambda N_G(B \cap S_{p'})$, then $[A, B \cap S_{p'}] \leqq A \cap L_\lambda (B \cap S_{p'}) = L_\lambda$ and so $C_G(A/L_\lambda) \geqq A(B \cap S_{p'}) = B$. This contradiction shows that $A \cap L_\lambda N_G(B \cap S_{p'}) = L_\lambda$ and so $L_\lambda N_G(B \cap S_{p'})$ is a complement of A/L_λ. Hence $M_\lambda = L_\lambda N_G(B \cap S_{p'})$ and $\bigcap_{\lambda \in \Lambda} M_\lambda = LN_G(B \cap S_{p'})$ (4.3). Therefore $NW_p = \bar{W}_p \cap NLN_G(B \cap S_{p'})$.

But as in the proof of (4.5) there are $\mathscr{X}$-complemented chief factors A/A_i, $i \in I$, such that $\bigcap_{i \in I} A_i = NL$ and A/A_i has the complement $A_i N_G(B \cap S_{p'})$. Thus

$$\bar{W}_p \leqq \bigcap_{i \in I} A_i N_G(B \cap S_{p'}) = NLN_G(B \cap S_{p'})$$

and we have $NW_p = \bar{W}_p$, as required.

Gaschütz [**5**] showed that the prefrattini subgroups of finite soluble groups can be characterized as those subgroups which cover each Frattini chief factor and which are contained in a conjugate of each maximal subgroup. No similar result can be obtained for $\mathfrak{U}$-groups as B. Hartley [**9**] has constructed a locally finite p-group G with a proper subgroup U which supplements each non-trivial normal subgroup N of G. The Frattini subgroup Φ is the unique prefrattini subgroup of G. $\Phi = G^p > 1$ and so $U\Phi = G$ and $U \ngeqq \Phi$. Thus $U \cap \Phi < \Phi$. Since $U \cap \Phi < \Phi$, it is contained in each maximal subgroup of G.

If H/K is a Frattini chief factor of G, then $\Phi K \geqq H$ and so $(U \cap \Phi)K = (U \cap \Phi)(K \cap \Phi)K = (U(K \cap \Phi) \cap \Phi)K$. If $K \cap \Phi = 1$ then $U(H \cap \Phi) = G$ and $U \cap (H \cap \Phi) \lhd U(H \cap \Phi) = G$ and so $U \cap (H \cap \Phi) = 1$ and U complements $H \cap \Phi$. But this would imply that $U \geqq \Phi$ and so we must have $K \cap \Phi > 1$. Therefore $U(K \cap \Phi) = G$ and $(U \cap \Phi)K = (U(K \cap \Phi) \cap \Phi)K = \Phi K \geqq H$. Thus $U \cap \Phi$ also covers each Frattini chief factor of G.

5. $\mathscr{X}_{\mathfrak{F}}$-prefrattini subgroups. We saw in (3.5) that $\mathscr{X}_{\mathfrak{F}}$ and $\mathscr{X}_{\bar{\mathfrak{F}}}$ are solid sets and so we have cover-avoidance properties for $\mathscr{X}_{\mathfrak{F}}$-prefrattini subgroups and $\mathscr{X}_{\bar{\mathfrak{F}}}$-prefrattini subgroups given by Theorem 4.2. If a chief factor has an $\mathfrak{F}$-abnormal complement it is $\mathfrak{F}$-eccentric and if it has an $\mathfrak{F}$-normal complement it is $\mathfrak{F}$-central. We therefore have the following result:

THEOREM 5.1. *Let $\mathfrak{K}$ be a QS-closed subclass of $\mathfrak{U}$, $\mathfrak{F}$ a saturated $\mathfrak{K}$-formation and $\mathscr{X}$ a solid set of maximal subgroups of the $\mathfrak{K} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G. Then*

(i) *the $\mathscr{X}_{\mathfrak{F}}$-prefrattini subgroups of G avoid the $\mathfrak{F}$-eccentric $\mathscr{X}$-complemented chief factors of G and cover the rest;*

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(ii) *the $\mathscr{X}\mathfrak{F}$-prefrattini subgroups of G avoid the $\mathfrak{F}$-central $\mathscr{X}$-complemented chief factors of G and cover the rest.*

Our main aim in this section is to obtain the alternative characterization of a $\mathscr{X}\mathfrak{F}$-prefrattini subgroup as the product of an $\mathscr{X}$-prefrattini subgroup W and $\mathfrak{F}$-normalizer D. We also consider the intersection $D \cap W$ and the $\mathscr{X}\mathfrak{F}$-prefrattini subgroups. We state our results in terms of the CAR-systems introduced in Section Two.

THEOREM 5.2. *Let $\mathfrak{K}$ be a QS-closed subclass of $\mathfrak{U}$, $\mathfrak{F}$ a saturated $\mathfrak{K}$-formation and $\mathscr{X}$ a solid set of maximal subgroups of the $\mathfrak{K} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*$-group G. Let $W_p = W_p^{\mathscr{X}}(\mathbf{S})$ and $D_p = N_G(C_p \cap S_{p'})$ so that $\mathscr{W}^{\mathscr{X}} = \{W_p\}$ and $\mathscr{D} = \{D_p\}$ are $\mathbf{S}CAR$-systems of G. Then*

(i) *$\{W_pD_p\}$ is a $\mathbf{S}CAR$-system of G and $\bigcap_p W_pD_p = WD$ is the $\mathscr{X}\mathfrak{F}$-prefrattini subgroup of G associated with $\mathbf{S}$;*

(ii) *$\{W_p \cap D_p\}$ is a $\mathbf{S}CAR$-system of G. $W \cap D = \bigcap_p (W_p \cap D_p)$ covers the $\mathfrak{F}$-central $\mathscr{X}$-Frattini chief factors and avoids the rest.*

Proof. First note that the $\mathfrak{F}$-normal maximal subgroups of G contain $G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G, and the $\mathfrak{F}$-abnormal maximal subgroups which contain $S_{p'}$ also contain D_p. (This result does not seem to have been recorded in this form but can be deduced from the proof of Theorem 4.1 of [**7**] or proved directly as in Lemma 6.3 of [**16**], using Lemma 3.1.)

(i) $W_p = \bigcap \{M \in \mathscr{X}\mathfrak{F} | M \geqq S_{p'}\} \cap \bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\}$ and so

$$D_pW_p = \bigcap \{M \in \mathscr{X}\mathfrak{F} | M \geqq S_{p'}\} \cap D_p(\bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\}).$$

Since $DG^{\mathfrak{F}} = G$ [**3**, Theorem 4.6 (iv)] and $D_p \geqq D \cap S_p$, we have $D_p(\bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\}) = G$ and so $D_pW_p = \bigcap \{M \in \mathscr{X}\mathfrak{F} | M \geqq S_{p'}\}$ and $\bigcap_p D_pW_p$ is the $\mathscr{X}\mathfrak{F}$-prefrattini subgroup of G associated with $\mathbf{S}$.

A similar argument shows that $\bigcap_p D_pW_p = DW = WD$.

(ii) We have to show that $W_p \cap D_p$ covers the $\mathfrak{F}$-central $\mathscr{X}$-Frattini p-chief factors and avoids all other p-chief factors. It is clearly sufficient to prove that if $N \lhd G$, then $N(W_p \cap D_p) = NW_p \cap ND_p$. By induction on a p-series for N we may clearly assume that N is a p-group.

$W_p \cap D_p = \bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\} \cap D_p$. If $N \leqq G^{\mathfrak{F}}$ then

$$N(W_p \cap D_p) = \bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\} \cap ND_p = NW_p \cap ND_p$$

and so by factoring out $N \cap G^{\mathfrak{F}}$, we may assume that $N \cap G^{\mathfrak{F}} = 1$. But then N is a normal p-subgroup with every chief factor of G below N being covered by D. Now $D \cap N \lhd D$ and $[D \cap N, G^{\mathfrak{F}}] \leqq N \cap G^{\mathfrak{F}} = 1$ so $D \cap N \lhd DG^{\mathfrak{F}} = G$. Therefore $D \cap N = N$ and $N \leqq D$. N is contained in each Sylow p-subgroup of D and so $N \leqq D_p$. Thus $N(W_p \cap D_p) = N(\bigcap \{M \in \mathscr{X}\bar{\mathfrak{F}} | M \geqq S_{p'}\}) \cap D_p = NW_p \cap ND_p$.

THEOREM 5.3. *With the above notation, $\{W_pG^{\mathfrak{F}}\}$ is a $\mathbf{S}CAR$-system and $\bigcap_p W_pG^{\mathfrak{F}} = WG^{\mathfrak{F}}$ is the $\mathscr{X}\bar{\mathfrak{F}}$-normalizer of G associated with $\mathbf{S}$.*

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Proof. As in the last result,

$$G^{\mathfrak{F}} W_p = G^{\mathfrak{F}}(\bigcap \{M \in \mathscr{X}_{\mathfrak{F}} | M \geqq S_{p'}) \cap \bigcap \{M \in \mathscr{X}_{\bar{\mathfrak{F}}} | M \geqq S_{p'}\}.$$

Now, if $M \in \mathscr{X}_{\mathfrak{F}}$ and $M \geqq S_{p'}$ then $M \geqq D$ and so $G^{\mathfrak{F}}(\bigcap \{M \in \mathscr{X}_{\mathfrak{F}} | M \geqq S_{p'}\}) \geqq G^{\mathfrak{F}} D = G$. Therefore $G^{\mathfrak{F}} W_p = \bigcap \{M \in \mathscr{X}_{\bar{\mathfrak{F}}} | M \geqq S_{p'}\}$, as required.

A similar argument shows that $\bigcap_p W_p G^{\mathfrak{F}} = WG^{\mathfrak{F}}$.

If $\mathfrak{F}_1$, $\mathfrak{F}_2$ are saturated $\mathfrak{K}$-formations, D_1 and D_2 are the $\mathfrak{F}_1$- and $\mathfrak{F}_2$-normalizers of G associated with **S** and $W = W^{\mathfrak{X}}(\mathbf{S})$, then we see from (5.2) and the fact that $\mathscr{X}_{\mathfrak{F}_1}$ and $\mathscr{X}_{\mathfrak{F}_2}$ are solid sets (3.5) that the product WD_1D_2 can be written in any order even though D_1 and D_2 may not permute. This situation can be seen in the following example:

Let $H = \langle x, y, z | x^7 = y^7 = z^7 = [x, z] = [y, z] = 1, [x, y] = z \rangle$ and let G be the split extension of H by a cyclic subgroup $\langle a \rangle$ of order 6 such that $a^{-1}xa = x^2$, $a^{-1}ya = y^{-1}$, $a^{-1}za = z^5$. Let **S** be the Sylow basis $\{ \langle a^3 \rangle, \langle a^2 \rangle, H\}$. The prefrattini subgroup of G associated with **S** is $W = \langle z \rangle$. If D_1 and D_2 are the $\mathfrak{N}\mathfrak{A}_2$- and $\mathfrak{N}\mathfrak{A}_3$-normalizers of G associated with **S**, then $D_1 = \langle x, a \rangle$ and $D_2 = \langle y, a \rangle$. ($\mathfrak{A}_p$ denotes the class of abelian groups of exponent p.) Then $\langle D_1, D_2 \rangle = G \neq D_1D_2$. But $WD_1 = \langle x, z, a \rangle$ so that $WD_1D_2 = D_2WD_1 = G$ and similarly $WD_2D_1 = D_1WD_2 = G$.

6. Distributivity conditions. Let $V = \bigcap_p S_{p'} V_p$ be a strongly pronormal subgroup of the $\mathfrak{U}$-group G into which the Sylow basis **S** reduces. Also let $B = \bigcap_p B_p$ and $C = \bigcap_p C_p$ be two **S**CAR-subgroups of G such that $BC = CB$. By (2.6), V permutes with both B and C and so V, B, C generate a modular sublattice of the subgroup lattice of G.

It follows therefore that this sublattice is distributive if $V(B \cap C) = VB \cap VC$ [**1**, Theorem 12 Chapter II (p. 37)]. $V(B \cap C)$ has a Sylow p-subgroup $(V_p \cap S_p)(B_p \cap C_p \cap S_p) = V_p(B_p \cap C_p) \cap S_p$. Also **S** reduces into $VB \cap VC$ which has a Sylow p-subgroup $V_p(B_p \cap S_p) \cap V_p(C_p \cap S_p) \cap S_p = V_pB_p \cap V_pC_p \cap S_p$. These two subgroups are equal if $V_p(B_p \cap C_p) = V_pB_p \cap V_pC_p$ and so we have proved:

THEOREM 6.1. *Let $V = \bigcap_p S_{p'} V_p$ be a strongly pronormal subgroup of the $\mathfrak{U}$-group G into which the Sylow basis* **S** *reduces. Let $B = \bigcap_p B_p$ and $C = \bigcap_p C_p$ be two* **S***CAR-subgroups of G satisfying the conditions:*

(i) *$BC = CB$ and*

(ii) *$N(B_p \cap C_p) = NB_p \cap NC_p$, for all $N \lhd G$.*

Then V, B, C generate a distributive sublattice of the subgroup lattice of G.

We have seen that these conditions are satisfied if B is an $\mathscr{X}$-prefrattini subgroup and C an $\mathfrak{F}$-normalizer of the $(\mathfrak{K} \cap (L\mathfrak{N})\mathfrak{A}\mathfrak{S}^*)$-group G (5.2).

Makan [**15**] obtains a rather more general result for finite soluble groups. He defines a pair of **S**CAR-subgroups to be *compatible* if they permute and if their intersection covers those chief factors covered by both of them. This condition is similar to the conditions (i) and (ii) in (6.1) but it seems unlikely that a condition on chief factors will be strong enough in $\mathfrak{U}$-groups. Makan is able to prove that if V, B, C are pairwise compatible $SCAR$-subgroups then they generate a distributive lattice but we have been unable to weaken the condition that V is strongly pronormal.

7. Finite soluble groups. Finally, we note that in the class of finite soluble groups a very much simpler proof can be given to show that $\mathscr{W}^{\mathfrak{X}}$ is a $SCAR$-system. Using the notation of Section Four, we have to show that if $N \lhd G$, then $W_pN = \bar{W}_p$. If G is finite then we may assume that N is a minimal normal subgroup of G. If $M_1, \ldots, M_n$ are the members of $\mathscr{X}$ which contain $S_{p'}$ but not N then

$$W_p = \bar{W}_p \cap \bigcap_{i=1}^{n} M_i \quad \text{and so} \quad NW_p = \bar{W}_p \cap N\left(\bigcap_{i=1}^{n} M_i\right).$$

LEMMA 7.1. (Hawkes [**12**, Lemma 2.4]). *If $L_i = \text{core}_G M_i$, then $N(M_i \cap M_j)$ is a maximal subgroup of G complementing the chief factor $NL_i/N(L_i \cap L_j)$. Hence $N(M_i \cap M_j) \in \mathscr{X}$ and so $N(M_i \cap M_j) \geqq \bar{W}_p$.*

We have to show that $N(\bigcap_{i=1}^n M_i) \geqq \bar{W}_p$ and this follows from the following elementary result:

LEMMA 7.2. *Let N be a normal subgroup of the group G and let M_i, $i \in I$, be subgroups of G such that $M_i \cap N = 1$. Then, for each $j \in I$,*

$$N\left(\bigcap_{i \in I} M_i\right) = \bigcap_{i \in I} N(M_i \cap M_j).$$

Proof. Let $n_i m_i = n_k m_k \in N(M_i \cap M_j) \cap N(M_k \cap M_j)$, with the obvious notation for elements. Then $n_k^{-1} n_i = m_k m_i^{-1} \in N \cap M_j = 1$ and so $m_i = m_k \in M_i \cap M_k \cap M_j$. Hence if $n_i m_i \in \bigcap_{i \in I} N(M_i \cap M_j)$, then $m_i \in \bigcap_{i \in I} M_i$ and the result follows.

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University of Glasgow,
Glasgow, Scotland