A NOTE ON THE EXTREMAL INDEX
FOR SPACE–TIME PROCESSES

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Abstract

Let \( \{ X(s, t), s = (s_1, s_2) \in \mathbb{R}^2, t \in \mathbb{R} \} \) be a stationary random field defined over a discrete lattice. In this paper, we consider a set of domain of attraction criteria giving the notion of extremal index for random fields. Together with the extremal-types theorem given by Leadbetter and Rootzen (1997), this will give a characterization of the limiting distribution of the maximum of such random fields.

Keywords: Extremal index; random field

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1. Introduction

We consider the asymptotic distribution of the maximum of a stationary random field defined over a discrete lattice in \( \mathbb{R}^3 \). We will take the first two coordinates as space and the third coordinate as time; hence, we call this random field a spatiotemporal process. We are motivated by the extremal properties of linear spatiotemporal autoregressive moving average processes (Cliff and Ord (1975); see also Cressie (1993, pp. 449–450)) given by

\[
X(t) - \sum_{k=0}^{p} B_k X(t - k) = \varepsilon(t) - \sum_{l=0}^{q} E_l \varepsilon(t - l),
\]

where \( X(t) = (X(s_i, t), i = 1, 2, \ldots, n)^\top \), is a vector process defined at spatial locations \( s_i, i = 1, 2, \ldots, n \), and at time points \( t = 1, 2, \ldots, T \); \( B_k \) and \( E_l \) are matrices of constants satisfying certain restrictions; and \( \varepsilon(t) = (\varepsilon(s_i, t), i = 1, 2, \ldots, n)^\top \), \( t = 1, 2, \ldots, T \), are independent and identically distributed random variables at space–time locations \((s_i, t)\).

The traditional way of obtaining limiting results for the maximum of a stationary sequence is as follows.

1. Prove an extremal types-theorem, which shows that under a long-range dependence condition the maximum of the field is the maximum of an approximately independent sequence of submaxima.

2. Obtain domain of attraction criteria, which characterize the limiting distribution function of the maximum in terms of the tail of the common marginal distribution and local dependence behavior of the sequence, given in terms of the extremal index.

Leadbetter and Rootzen (1997) proved an extremal-types theorem for random fields in \( \mathbb{R}^2 \), under a weak coordinatewise-mixing (CW-mixing) condition. Although we will generalize...
We are interested in the asymptotic distribution of \( \lim_{n \to \infty} x^n \). Without loss of generality, we assume that, for every \( i \), the extremal index for \( n \) can be characterized in terms of the tail of the distribution function of the process, as well as three coordinatewise conditions that describe the propensity of consecutive large values of the process to cluster in each coordinate direction.

Hence, the outline of the paper is as follows. In Section 2, we give the CW-mixing condition of Leadbetter and Rootzen (1997), adapted for spatiotemporal processes. We also prove the extremal-types theorem under this condition. Although the proof is a straightforward extension of Leadbetter and Rootzen (1997), we give the full proof for completeness and the reader’s convenience. In Section 3, we define the notion of coordinatewise extremal indices, which resembles the definition of the extremal index given by O’Brien (1987) for stationary sequences, and prove the validity of a set of domain of attraction criteria based on these three coordinatewise conditions.

2. Extremal-types theorem

Let \( X(s, t) \) be a stationary spatiotemporal process defined over a discrete lattice

\[
E_n = \{(i, j, k), \quad i = 0, 1, \ldots, n_1, \quad j = 0, 1, \ldots, n_2, \quad k = 0, 1, \ldots, n_3\},
\]

with \( n = (n_1, n_2, n_3) \). Here, we take \( (i, j) = s \) and \( k = t \) respectively to be the position in the \((x, y)\)-plane and the time at which the random field \( X(s, t) \) is evaluated. Let \( F(x) = \Pr(X(0, 0, 0) \leq x) \) be the marginal distribution function of the process.

For any subset \( B \in E_n \), define

\[
M(B) = \max_{(i, j, k) \in B} X(i, j, k).
\]

We are interested in the asymptotic distribution of \( M(E_n) \), when suitably normalized, as \( n_1 \to \infty, n_2 \to \infty, \) and \( n_3 \to \infty \) (we write \( \lim_{n \to \infty} \) for \( \lim_{n_1 \to \infty, n_2 \to \infty, n_3 \to \infty} \)); that is,

\[
\lim_{n \to \infty} P(M(E_n) \leq u_n(x)),
\]

for some suitably chosen normalizing constants \( b_n \) and \( a_n \) such that \( u_n = a_n x + b_n \).

The following extension of the CW-mixing condition of Leadbetter and Rootzen (1997) yields the extremal-types theorem needed to characterize the limiting distribution of \( M(E_n) \).

Let \( r_1, r_2, \) and \( r_3 \) be integers defining the lengths of blocks of cubes

\[
B_{ijk} = [(i - 1)r_1, ir_1] \times [(j - 1)r_2, jr_2] \times [(k - 1)r_3, kr_3],
\]

which will be used for subdivision of \( E_n \). Assume that as \( n_1 \to \infty, n_2 \to \infty, \) and \( n_3 \to \infty, \) \( r_1, r_2, \) and \( r_3 \) all tend to \( \infty \) in such a way that \( r_i = o(n_i), \) \( i = 1, 2, 3. \) Let \( m_i = [n_i / r_i], \) \( i = 1, 2, 3, \) be integers such that \( E_n \) contains \( m_n = m_1 m_2 m_3 \) complete blocks and no more than \( m_1 + m_2 + m_3 + 1 \) incomplete blocks. (Note that \([\cdot]\) denotes the integer-part function.)

Without loss of generality, we assume that, for every \( i, \) \( m_i r_i = n_i, \) in order that there be no incomplete blocks. This assumption eases notational difficulties and should not effect the asymptotic results. Let \( l_i \equiv l_i(n_i), \) \( i = 1, 2, 3, \) be integers tending to \( \infty \) in such a way that \( l_i = o(r_i). \) With this notation, the random field is said to satisfy the CW-mixing condition (Leadbetter and Rootzen (1997)) for a given family of levels \( u_n \) and separation constants \( l_i \) if the following conditions are satisfied.
Condition 1. (t-direction condition.) For each $i$ and $j$ with $0 < i \leq n_1$ and $0 < j \leq n_2$, and for cubes 
\[ B_1 = [0, i] \times [0, j] \times [0, a] \quad \text{and} \quad B_2 = [0, i] \times [0, j] \times [b, c] \]
with $0 < a < n_3$, $a + l_1 \leq b < c \leq n_3$, and $c - b \leq r_3$, we have 
\[ |P(M(B_1) \leq u_n, M(B_2) \leq u_n) - P(M(B_1) \leq u_n)P(M(B_2) \leq u_n)| \leq \alpha_3(r_3, l_3), \quad (1) \]
where the $t$-coordinate mixing function $\alpha_3$ satisfies $m_3\alpha_3(r_3, l_3) \to 0$ as $n_3 \to \infty$. Note that, with this condition, maxima defined over the cubes of size $n_1 \times n_2 \times r_3$ (or smaller in the $(x, y)$-plane) separated along the time direction by cubes of size $n_1 \times n_2 \times l_3$ are asymptotically independent as $n_3 \to \infty$.

Condition 2. (x-direction condition.) For each $j$, $0 < j \leq n_2$, and cubes 
\[ B_1 = [0, a] \times [0, j] \times [0, r_3] \quad \text{and} \quad B_2 = [b, c] \times [0, j] \times [0, r_3] \]
with $0 < a < n_1$, $a + l_1 \leq b < c \leq n_1$, and $c - b \leq r_1$, we have 
\[ |P(M(B_1) \leq u_n, M(B_2) \leq u_n) - P(M(B_1) \leq u_n)P(M(B_2) \leq u_n)| \leq \alpha_3(r_1, r_3, l_1), \quad (2) \]
where the $x$-coordinate mixing function $\alpha_3$ satisfies $m_3\alpha_3(r_3, l_3) \to 0$ as $n_3 \to \infty$ and $n_1 \to \infty$. The $x$-coordinate mixing condition says that maxima defined over strips of cubes of size $r_1 \times n_2 \times r_3$ (or smaller) are asymptotically independent, provided that they are separated along the $x$-direction by strips of cubes of size $l_1 \times n_2 \times r_3$.

Condition 3. (y-direction condition.) For cubes 
\[ B_1 = [0, r_1] \times [0, a] \times [0, r_3] \quad \text{and} \quad B_2 = [0, r_1] \times [b, c] \times [0, r_3] \]
with $0 < a < n_2$, $a + l_2 \leq b < c \leq n_2$, and $c - b \leq r_2$, we have 
\[ |P(M(B_1) \leq u_n, M(B_2) \leq u_n) - P(M(B_1) \leq u_n)P(M(B_2) \leq u_n)| \leq \alpha_2(r_1, r_2, r_3, l_2), \]
where the $y$-direction mixing function $\alpha_2$ satisfies 
\[ m_1m_2m_3\alpha_2(r_1, r_2, r_3, l_2) \to 0 \quad (3) \]
as $n_1 \to \infty$, $n_2 \to \infty$, and $n_3 \to \infty$. Again, under the $y$-direction mixing condition, the maxima of the process over cubes of size $r_1 \times r_2 \times r_3$ are asymptotically independent, provided that these cubes are separated along the $y$-direction by cubes of size $r_1 \times l_2 \times r_3$. Note that, as we cut out smaller cubes, we need stricter conditions on the corresponding mixing functions and, hence, stricter conditions on the asymptotic independence of the maxima over smaller cubes.

The following result, due to Leadbetter and Rootzen (1997), essentially says that, under the above separate mixing conditions on each of the three directions, for $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, and $k = 1, \ldots, m_3$ the $m_1m_2m_3$ processes $X(i, j, k)$ defined over the blocks 
\[ B_{ijk} = [(i - 1)r_1, ir_1] \times [(j - 1)r_2, jr_2] \times [(k - 1)r_3, kr_3], \]
each of size $r_1r_2r_3$, are asymptotically independent.
Lemma 1. Assume that the stationary random field \( X(i, j, k) \) satisfies the CW-mixing condition given above, for an appropriately chosen level \( u_n \). Then, for

\[
E_n = [0, n_1] \times [0, n_2] \times [0, n_3] = [0, m_1 r_1] \times [0, m_2 r_2] \times [0, m_3 r_3] = \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \bigcup_{k=1}^{m_3} B_{ijk},
\]

we find that

\[
P(M(E_n) \leq u_n) = P^{m_1 m_2 m_3}(M(J) \leq u_n) + o_n(1),
\]

where \( J = B_{111} = [0, r_1] \times [0, r_2] \times [0, r_3] \), as \( n_1, n_2, \) and \( n_3 \) all tend to \( \infty \).

Proof. With the above notation, write

\[
J_{1,i} = [0, m_1 r_1] \times [0, m_2 r_2] \times [(i-1) r_3, i r_3],
J_{2,i} = [0, m_1 r_1] \times [0, m_2 r_2] \times [(i-1) r_3 + l_3, i r_3 - l_3],
J_{3} = J_{1,i} - J_{2,i}.
\]

Then \( E_n = \bigcup_{i=1}^{m_3} J_{1,i} \) and, using stationarity, for \( 2 \leq m \leq m_3 \),

\[
0 \leq P\left(M\left(\bigcup_{i=1}^{m-1} J_{1,i}\right) \leq u_n, M(J_{2,m}) \leq u_n\right) - P\left(M\left(\bigcup_{i=1}^{m} J_{1,i}\right) \leq u_n\right) \\
\leq P(M(J_{m}^n) > u_n) \\
= P(M(J_{1}^n) > u_n).
\]

Furthermore,

\[
\left|P\left(M\left(\bigcup_{i=1}^{m-1} J_{1,i}\right) \leq u_n, M(J_{2,m}) \leq u_n\right) - P\left(M\left(\bigcup_{i=1}^{m-1} J_{1,i}\right) \leq u_n\right) P(M(J_{2,m}) \leq u_n)\right|
\leq o_4(r_3, l_3)
\]

and

\[
0 \leq P(M(J_{2,m}) \leq u_n) - P(M(J_{1,m}) \leq u_n) \leq P(M(J_{m}^n) > u_n);
\]

hence, it follows, using stationarity, that

\[
\left|P\left(M\left(\bigcup_{i=1}^{m} J_{1,i}\right) \leq u_n\right) - P\left(M\left(\bigcup_{i=1}^{m-1} J_{1,i}\right) \leq u_n\right) P(M(J_{1,1}) \leq u_n)\right|
\leq o_4(r_3, l_3) + 2P(M(J_{1}^n) > u_n).
\]

By applying (6) repeatedly, from CW-mixing we obtain

\[
|P(M(E_n) \leq u_n) - P^{m_3}(M(J_{1,1}) \leq u_n)| \leq 2m_3 P(M(J_{1}^n) > u_n) + o_n(1).
\]

We will now show that

\[
P(M(E_n) \leq u_n) - P^{m_3}(M(J_{1,1}) \leq u_n) \to 0.
\]
It is sufficient to show that this holds as $n \rightarrow \infty$ in a manner such that $P_{m3}(M(J^*)_n \leq u_n)$ converges to some $\rho$, $0 \leq \rho \leq 1$. If $\rho = 1$ then $P(M(J^*_1) > u_n) \rightarrow 0$, and since $m_3 \log P(M(J^*_1) \leq u_n) \rightarrow 0$ it follows that $m_1 \cdot P(M(J^*_1) > u_n) \rightarrow 0$ and that (9) is a consequence of (8).

On the other hand, if $\rho < 1$ then, since $m_3\alpha_3(r_3, l_3) \rightarrow 0$ and $l_3 = o(r_3)$, there exists a $\beta_n \rightarrow \infty$ such that $m_3\beta_n\alpha_3(r_3, l_3) \rightarrow 0$ and $\beta_n l_3 = o(r_3)$. Hence, for sufficiently large $n$, $\beta_n$ cubes congruent to $J^*_1$ and mutually separated by at least a distance $l_3$ in the $t$-direction may be chosen in $J_{2,1}$. Arguments parallel to those yielding (4)–(8) then imply that

$$P(M(J_{1,1}) \leq u_n) \leq P_{m3}(M(J^*_1) \leq u_n) + \beta_n\alpha_3(l_3, r_3),$$

whence

$$P_{m3}(M(J_{1,1}) \leq u_n) \leq P_{m3}(M(J^*_1) \leq u_n) + \beta_n m_3\alpha_3(l_3, r_3)$$

$$= (\rho + o(1))\beta_n \rightarrow 0.$$

Hence, the second term of the difference in (9) tends to 0. Finally, it follows similarly that

$$P(M(E_n \leq u_n) \leq P_{m3}(M(J_{2,1}) \leq u_n) + m_3\alpha_3(l_3, r_3),$$

which tends to 0 since (10) and, hence, (11) apply with $M(J_{2,1})$ in place of $M(J_{1,1})$. Both terms on the left-hand side of (9) therefore tend to 0 if $\rho < 1$, and the convergence again holds.

To complete the proof of the lemma it is sufficient to establish that

$$P_{m3}(M(J_{1,1}) \leq u_n) - P_{m1m3}(M([0, r_1] \times [0, n_2] \times [0, r_3]) \leq u_n) \rightarrow 0$$

and

$$P_{m1m3}(M([0, r_1] \times [0, n_2] \times [0, r_3]) \leq u_n) - P_{m1m2m3}(M(J) \leq u_n) \rightarrow 0.$$
For \( k = 0, 1, \ldots, r_3 \) let
\[
\zeta_k = \max_{0 \leq i \leq r_1, 0 \leq j \leq r_2} X(i, j, k),
\]
and for \( 0 \leq i \leq r_1 \) let
\[
\nu_i = \max_{0 \leq j \leq r_2} X(i, j, 0).
\]
Note that the \( \zeta_k \) are the maxima over the (spatial) coordinates \( \{(i, j), 0 \leq i \leq r_1, 0 \leq j \leq r_2\} \) at each time point \( k = 1, \ldots, r_3 \). We will call them the maxima of time plates. For each fixed \( i = 0, 1, \ldots, r_1 \), the \( \nu_i \) are the maxima over the coordinates \( \{(i, j), 0 \leq j \leq r_2\} \) at fixed \( k = 0 \). We will call them the maxima of \( y \)-arrays.

The local dependence structure of the stationary spatiotemporal process will be characterized by three conditions, which represent the propensity of the large values of the process to cluster along the \( t \)-, \( x \)-, and \( y \)-coordinates.

Assume that the following limits exist.

1. \( t \)-coordinate clustering propensity:
\[
\lim_{n \to \infty} P(\zeta_1 \leq u_n, \zeta_2 \leq u_n, \ldots, \zeta_{r_3} \leq u_n \mid \zeta_0 > u_n) = \theta_1 \tag{12}
\]
for some \( \theta_1, 0 < \theta_1 \leq 1 \). This condition indicates how the largest values over the \((x, y)\)-plane cluster at consecutive time points.

2. \( x \)-coordinate clustering propensity:
\[
\lim_{n \to \infty} P(\nu_1 \leq u_n, \nu_2 \leq u_n, \ldots, \nu_{r_1} \leq u_n \mid \nu_0 > u_n) = \theta_2 \tag{13}
\]
for some \( \theta_2, 0 < \theta_2 \leq 1 \). This condition indicates how largest values in the \( y \)-columns cluster along the \( x \)-direction at a fixed point of time.

3. \( y \)-coordinate clustering propensity:
\[
\lim_{n \to \infty} P(X(0, 1, 0) \leq u_n, X(0, 2, 0) \leq u_n, \ldots, X(0, r_2, 0) \leq u_n \mid X(0, 0, 0) > u_n) = \theta_3 \tag{14}
\]
for some \( \theta_3, 0 < \theta_3 \leq 1 \). This condition indicates how the large values of the process cluster along the \( y \)-direction at a fixed \( x \)-coordinate and a fixed time.

We will show, by extending the results of O’Brien (1987), that these coordinate-wise clustering conditions uniquely characterize how the process clusters in time and space and give us domain of attraction criteria.

**Theorem 1.** Assume that \( u_n = u_n x + b_n \) is chosen such that
\[
\lim_{n \to \infty} n_1^a n_2^b (1 - F(u_n)) \to \tau(x).
\]
Assume that the process \( \{X(i, j, k), (i, j, k) \in E_n\} \) satisfies the CW-mixing condition given in (1), (2), and (3) for \( i, j, k \in E_n \) and \( u_n \) chosen as above. Assume further that the limits in (12), (13), and (14) exist. Then
\[
\lim_{n \to \infty} |P(M(E_n) \leq u_n) - \exp(-\theta_1 \theta_2 \theta_3 \tau(x))| = 0.
\]
Proof. We extend O’Brien’s (1987) technique to random fields. Let

\[ M_{\zeta}^{r_3} = \max_{0 \leq k \leq r_3} \zeta_k \quad \text{and} \quad M_{\zeta_{k_1,k_2}}^{r_3} = \max_{k_1+1 \leq k \leq k_2} \zeta_k; \]

then

\[ M_{0,r_3}^{\zeta} = \max_{1 \leq k \leq r_3} \zeta_k. \]

Similarly, define

\[ M_{\nu}^{r_1} = \max_{0 \leq i \leq r_1} \nu_i \quad \text{and} \quad M_{\nu_{i_1,i_2}}^{r_1} = \max_{i_1+1 \leq i \leq i_2} \nu_i; \]

then

\[ M_{0,r_1}^{\nu} = \max_{1 \leq i \leq r_1} \nu_i. \]

Finally, let

\[ M_X^{r_2} = \max_{0 \leq j \leq r_2} X(0, j, 0) \quad \text{and} \quad M_X^{r_1,r_2} = \max_{j_1+1 \leq j \leq j_2} X(0, j, 0); \]

then

\[ M_{0,r_2}^{X} = \max_{1 \leq j \leq r_2} X(0, j, 0). \]

Thus,

\[
P(M(J) > u_n) = P\left( \max_{0 \leq i \leq r_3} \zeta_i > u_n \right) \\
= P\left( \bigcup_{i=0}^{r_3} (\zeta_i > u_n) \right) \\
= \sum_{i=0}^{r_3} P(\zeta_i > u_n, M_{\zeta_i}^{r_3} \leq u_n),
\]

which follows from the fact that, for any events \( A_1, \ldots, A_n \),

\[
\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \left( A_i \cap \bigcap_{k=i+1}^{n} A_j^{c} \right)
\]

and the events \( A_i \cap \bigcap_{k=i+1}^{n} A_j^{c}, i = 1, \ldots, n \), are mutually exclusive.

Now,

\[
P(\zeta_i > u_n, M_{\zeta_{i,r_3}}^{c} \leq u_n) = P(\zeta_i > u_n, \zeta_{i+1} \leq u_n, \ldots, \zeta_{r_3} \leq u_n) \\
= P(\zeta_0 > u_n, \zeta_1 \leq u_n, \ldots, \zeta_{r_3-i} \leq u_n) \\
\geq P(\zeta_0 > u_n, \zeta_1 \leq u_n, \ldots, \zeta_{r_3} \leq u_n) \\
= P(\zeta_0 > u_n, M_{0,r_3}^{\zeta} \leq u_n),
\]

and we have

\[
P(M(J) > u_n) \geq r_3 P(\zeta_0 > u_n, M_{0,r_3}^{\zeta} \leq u_n).
\]
The extremal index for space–time processes

Hence,

\[ P(M(J) \leq u_n) = 1 - P(M(J) > u_n) \]
\[ \leq 1 - r_3 P(\xi_0 > u_n, M_{0,r_1}^\xi \leq u_n) \]
\[ = 1 - r_3 P(M_{0,r_3}^\xi \leq u_n | \xi_0 > u_n) P(\xi_0 > u_n). \]  

(16)

Using the arguments leading to (15), we also have

\[ P(\xi_0 > u_n) = P\left( \max_{0 \leq i \leq r_3} v_i > u_n \right) \]
\[ = P(M_{r_1}^\eta > u_n) \]
\[ \geq r_1 P(v_0 > u_n, M_{0,r_1}^\eta \leq u_n) \]
\[ = r_1 P(M_{0,r_1}^\eta \leq u_n | v_0 > u_n) P(v_0 > u_n), \]  

(17)

and repeating the argument for \( P(v_0 > u_n) \) gives

\[ P(v_0 > u_n) = P\left( \max_{0 \leq j \leq r_2} X(0, j, 0) > u_n \right) \]
\[ = P(M_{r_2}^X > u_n) \]
\[ \geq r_2 P(X(0, 0, 0) > u_n, M_{0,r_2}^X \leq u_n) \]
\[ = r_2 P(M_{0,r_2}^X \leq u_n | X(0, 0, 0) > u_n) P(X(0, 0, 0) > u_n). \]  

(18)

By combining (16), (17), and (18) we obtain

\[ P(M(J) \leq u_n) \leq 1 - r_1 r_2 r_3 P(M_{0,r_3}^\xi \leq u_n | \xi_0 > u_n) P(M_{0,r_1}^\eta \leq u_n | v_0 > u_n) \]
\[ \times P(M_{0,r_2}^X \leq u_n | X(0, 0, 0) > u_n) P(X(0, 0, 0) > u_n). \]

Hence, from the CW-mixing condition, as \( n \to \infty, \)

\[ P(M(E_n) \leq u_n) \leq u_n \]
\[ \leq P^{\max(m_1 m_2 m_3)}(M(J) \leq u_n) + o_n(1), \]
\[ \leq (1 - r_1 r_2 r_3) P(M_{0,r_3}^\xi \leq u_n | \xi_0 > u_n) P(M_{0,r_1}^\eta \leq u_n | v_0 > u_n) \]
\[ \times P(M_{0,r_2}^X \leq u_n | X(0, 0, 0) > u_n) P(X(0, 0, 0) > u_n) \]
\[ \to \exp(-\theta_1 \theta_2 \theta_3 \tau(x)). \]

We will now prove the opposite inequality, namely

\[ P(M(E_n) \leq u_n) \geq \exp(-\theta_1 \theta_2 \theta_3 \tau(x)) \quad \text{as} \quad n \to \infty. \]

Let \( \eta_1, \eta_2, \) and \( \eta_3 \) be sequences of integers converging to \( \infty, \) as \( n \to \infty, \) in such a manner that \( r_1 = o(\eta_1), \eta_1 = o(n_1), r_2 = o(\eta_2), \eta_2 = o(n_2), r_3 = o(\eta_3), \) and \( \eta_3 = o(n_3). \) Let \( u_1, u_2, \) and \( u_3 \) be integers such that \( u_1 = [n_1/\eta_1], u_2 = [n_2/\eta_2], \) and \( u_3 = [n_3/\eta_3]. \) Furthermore, let

\[ \tilde{j} = [0 \leq i \leq \eta_1, 0 \leq j \leq \eta_2, 0 \leq k \leq \eta_3]. \]

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For such choices of integers, it follows from the CW-mixing condition that
\[ P(M(E_n) \leq u_n) = P^{i_{11}1_{i3}}(M(\hat{J}) \leq u_n) + o_n(1). \]

Now assume that
\[ \phi_k = \max_{0 \leq i \leq n_1, 0 \leq j \leq n_2} X(i, j, k), \]
\[ \psi_i = \max_{0 \leq j \leq n_2} X(i, j, 0). \]

Then
\[ P(M(\hat{J}) > u_n) = P(\max_{0 \leq k \leq n_3} \phi_k > u_n) \]
\[ = P(M_{n_3}^\phi > u_n) \]
\[ = P(M_{n_3}^\phi > u_n) \cup \{M_{n_3}^\phi < u_n \cap \{M_{n_3+1, n_3}^\phi > u_n\}\} \]
\[ = P(M_{n_3}^\phi > u_n) + P(\{M_{n_3}^\phi < u_n \cap \{M_{n_3+1, n_3}^\phi > u_n\}\}) \]
\[ = P(M_{n_3}^\phi > u_n) + P(M_{n_3-r_3}^\phi > u_n, M_{n_3-r_3+1, n_3}^\phi < u_n) \quad (\text{by stationarity}). \]

Note that, since \( r_3 = o(n_3) \),
\[ P(M(J) > u_n) = P(M_{n_3}^\phi > u_n) = o(P(M(\hat{J}) > u_n)) \quad \text{as } n_3 \to \infty. \]

Hence,
\[ P(M(\hat{J}) > u_n)(1 + o_n(1)) = P(M_{n_3-r_3}^\phi > u_n, M_{n_3-r_3+1, n_3}^\phi < u_n). \]

By writing the event
\[ \{M_{n_3-r_3}^\phi > u_n, M_{n_3-r_3+1, n_3}^\phi < u_n\} = \bigcup_{i=0}^{n_3-r_3} \{\phi_i > u_n\} \cap B, \]
where \( B = \{M_{n_3-r_3+1, n_3}^\phi \leq u_n\} \), in the ‘mutually exclusive’ form
\[ \bigcup_{i=0}^{n_3-r_3} ((\{\phi_i > u_n\} \cap B) \cap \bigcap_{j=i+1}^{n_3-r_3} (\{\phi_j \leq u_n\} \cup B^c)), \]
we obtain
\[ P(M_{n_3-r_3}^\phi > u_n, M_{n_3-r_3+1, n_3}^\phi \leq u_n) = \sum_{i=0}^{n_3-r_3} P(\{\phi_i > u_n\} \cap \bigcap_{j=i+1}^{n_3-r_3} \{\phi_j \leq u_n\} \cap B) \]
\[ = \sum_{i=0}^{n_3-r_3} P(\phi_i > u_n, M_{i+1, n_3}^\phi \leq u_n). \]

Since, for every \( i = 0, 1, \ldots, n_3 - r_3 \), we have \( n_3 \geq i + r_3 \) and
\[ \{M_{i+1, n_3}^\phi < u_n\} \subseteq \{M_{r_3+i, n_3}^\phi < u_n\}, \]

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The extremal index for space–time processes

we then obtain

\[
P(M_{\eta_1-r_3}^\phi > u_n, M_{\eta_1-r_3+1,r_1}^\phi \leq u_n) \leq \sum_{i=0}^{\eta_1-r_3} P(\phi_i > u_n, M_{i+1,r_3+i}^\phi \leq u_n)
\]

\[
= \sum_{i=0}^{\eta_1-r_3} P(\phi_0 > u_n, M_{1,r_3}^\phi \leq u_n)
\]

\[
\leq \eta_3 P(\phi_0 > u_n, M_{1,r_3}^\phi \leq u_n)
\]

\[
= \eta_3 P(M_{1,r_3}^\phi \leq u_n | \phi_0 > u_n) P(\phi_0 > u_n).
\]

The final inequality in (19) follows from the fact that, for every \( k = 1, \ldots, r_3, \phi_k \geq \zeta_k \) almost surely.

If we now start with

\[
P(\phi_0 > u_n) = P\left( \max_{0 \leq i \leq \eta_1} \psi_i > u_n \right)
\]

and apply the steps leading to (19), we find that

\[
P(\phi_0 > u_n) \leq \eta_1 P(M_{0,r_1}^\nu \leq u_n | \nu_0 > u_n) P(\nu_0 > u_n).
\]

(20)

By applying the steps yet again, to \( P(\nu_0 > u_n) = P(M_{\eta_2}^X > u_n) \), we obtain

\[
P(M_{\eta_2}^X > u_n) \leq \eta_2 P(M_{0,r_2}^X \leq u_n | X(0, 0, 0) > u_n) P(X(0, 0, 0) > u_n).
\]

(21)

From Lemma 1 and the CW-mixing condition, with \( \eta_1, \eta_2, \eta_3, \nu_1, \nu_2, \) and \( \nu_3 \) as defined above, as \( n \to \infty \) we find that

\[
P(M(E_n) \leq u_n) = P^{\nu_1 \nu_2 \nu_3}(M(\hat{J}) \leq u_n) + o_n(1),
\]

and from (19), (20), and (21) we have

\[
P(M(E_n) \leq u_n) \geq [1 - \eta_1 \eta_2 \eta_3 P(X(0, 0, 0) > u_n) P(M_{1,r_1}^\xi \leq u_n | \xi_0 > u_n)
\]

\[
\times P(M_{1,r_1}^\xi \leq u_n | \nu_0 > u_n) P(M_{0,r_2}^X \leq u_n | X(0, 0, 0) > u_n)
\]

\[
\times (1 + o_n(1))]^{\nu_1 \nu_2 \nu_3} + o_n(1)
\]

\[
\to \exp(-\theta_1 \theta_2 \theta_3 \tau(x)).
\]

The choice of the indices \( \theta_1, \theta_2, \) and \( \theta_3 \) is not unique. For example, we could first have sliced the cube \( J \) along the \( x \)-direction, into plates parameterized by \( j \) and \( k \), and looked at how the largest values over these plates cluster along the \( x \)-direction; then looked at how large values of the process cluster along the \( y \)-direction; and finally considered how the large values cluster in time.

Let \( \gamma_i = \max_{0 \leq j \leq r_2, 0 \leq k \leq r_3} X(i, j, k) \) and \( \tau_j = \max_{0 \leq k \leq r_3} X(0, j, k) \), and assume that the following limits exist, where \( 0 < \theta_1^*, \theta_2^*, \theta_3^* \leq 1 \).

1. \( x \)-coordinate clustering propensity:

\[
\lim_{n \to \infty} P(\gamma_1 \leq u_n, \gamma_2 \leq u_n, \ldots, \gamma_{r_1} \leq u_n | \gamma_0 > u_n) = \theta_1^*.
\]
2. $y$-coordinate clustering propensity:

$$\lim_{n \to \infty} P(\tau_1 \leq u_n, \tau_2 \leq u_n, \ldots, \tau_r \leq u_n \mid \tau_0 > u_n) = \theta_2^*.$$ 

3. $t$-coordinate clustering propensity:

$$\lim_{n \to \infty} P(X(0, 0, 1) \leq u_n, X(0, 0, 2) \leq u_n, \ldots, X(0, 0, r_3) \leq u_n \mid X(0, 0, 0) > u_n) \to \theta_3^*.$$ 

If the process is stationary but not isotropic (in the sense that the process $X(i, j, k)$ is a subsampled version of an isotropic, continuous-parameter random field), then, in general, $\theta_i^* \neq \theta_i$, $i = 1, 2, 3$. However, provided that the limits exist, the extremal index of the stationary process $\theta$ will be unique, i.e. $\theta_1^* \theta_2^* \theta_3^* = \theta_1 \theta_2 \theta_3$.

**Example 1.** Consider the spatial process

$$X(i, j) = \phi X(i - 1, j) + \varepsilon(i, j), \quad i, j = 1, 2, \ldots, n,$$

(22)

where $|\phi| < 1$ and $\{\varepsilon(i, j)\}$ are independent, identically distributed random variables with regularly varying tails such that, as $x \to \infty$,

$$P(\varepsilon(0, 0) > x) \sim x^{-\alpha} L(x)$$

for some $\alpha \in (0, 2)$ and a slowly varying function $L(x)$. Furthermore, we assume that

$$\lim_{x \to \infty} \frac{P(\varepsilon(0, 0) > x)}{P(|\varepsilon(0, 0)| > x)} = 1.$$ 

The stationary solution to (22) is given by

$$X(i, j) = \sum_{k=0}^{\infty} \phi^k \varepsilon(i - k, j).$$

From Resnick (1987), $X(0, 0)$ and $\varepsilon(0, 0)$ are tail equivalent; that is, as $x \to \infty$,

$$P(X(0, 0) > x) \sim c x^{-\alpha} L(x)$$

for some constant $c$. Let $a_n = n^{2/\alpha}$ and $u_{n^2} \equiv u_{n^2}(x) = a_n x$; then, for $x > 0$,

$$n^2 P(X(0, 0) > u_{n^2}) \sim \tau(x) \quad \text{as } n \to \infty.$$ 

Note that $X(i, j)$ is an independent sequence along the $y$-direction. Let

$$v_i = \max_{0 \leq j \leq r_n} X(i, j), \quad i = 0, 1, \ldots, r_1,$$
and let \( r_i = r_n \) and \( m_i = m_n, \ i = 1, 2. \) Then \( n = r_n m_n \) and, as \( n \to \infty, \)

\[
\begin{align*}
& n \sum_{i=1}^{r_n} P(\nu_1 > u_{n^2}, \nu_i > u_{n^2}) \\
& \quad = n \sum_{i=1}^{r_n} P \left( \bigcup_{j=0}^{r_n} \{X(1, j) > u_{n^2}\}, \bigcup_{k=0}^{r_n} \{X(i, k) > u_{n^2}\} \right) \\
& \quad \leq n \sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \sum_{k=0}^{r_n} P(X(1, j) > u_{n^2}, X(i, k) > u_{n^2}) \\
& \quad = \begin{cases} \\
& \left( n \sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \sum_{k=0}^{r_n} P(X(1, j) > u_{n^2}) P(X(i, k) > u_{n^2}) \right) \text{ if } j \neq k, \\
& \left( n \sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \sum_{k=0}^{r_n} P(X(1, j) > u_{n^2}, X(i, j) > u_{n^2}) \right) \text{ if } j = k. 
\end{cases}
\end{align*}
\]

The first term in (23) (for \( j \neq k \)) is of order \( O(1/m_n^3) \), whereas the second term (for \( j = k \)) can be written as

\[
\begin{align*}
& n \sum_{i=1}^{r_n} P(X(1, 1) > u_{n^2}, X(i, 1) > u_{n^2}) = n r_n \sum_{i=1}^{r_n} P(X(1, 1) > u_{n^2}, X(i, 1) > u_{n^2}) g_n(i), \\
& \text{where} \\
& g_n(i) = \frac{P(X(1, 1) > u_{n^2}, X(i, 1) > u_{n^2})}{P(X(1, 1) > u_{n^2}, X(i, 1) > u_{n})} \\
& = r_n \frac{P(\sum_{k=0}^{i-2} \phi^k \epsilon(i-k, 1) > u_{n^2})}{P(\sum_{k=0}^{i-2} \phi^k \epsilon(i-k, 1) > u_n)} \\
& = O \left( \frac{1}{m_n} \right),
\end{align*}
\]

uniformly in \( i = 1, 2, \ldots, r_n \). Since \( X(i, 1) \) is an (autoregressive) AR(1) process, we have

\[
\begin{align*}
& n \sum_{i=1}^{r_n} P(X(1, 1) > u_{n}, X(i, 1) > u_{n}) = O \left( \frac{1}{m_n} \right). 
\end{align*}
\]

Hence, it follows from (23) that \( \nu_1 \) satisfies the \( D'(u_{n^2}) \) condition and \( \theta_1 = 1. \)

Note that \( (X(0, j), j = 0, 1, \ldots, r_n) \) is an independent sequence. It is easy to see that \( \theta_2 = 1 \) and \( \theta := \theta_1 \theta_2 = 1. \)

On the other hand, with \( \xi_j = \max_{0 \leq i \leq r_n} X(i, j), \) we see that \( \{\xi_j\} \) is an independent sequence, and that

\[
\begin{align*}
& \theta_1^* = \lim_{n \to \infty} P(\xi_1 < u_{n^2}, \ldots, \xi_{r_n} < u_{n^2} \mid \xi > u_{n^2}) = 1.
\end{align*}
\]
Similarly, 
\[ \theta_2^* = \lim_{n \to \infty} P(X(1, 0) < u_n^2, \ldots, X(r_n, 0) < u_n^2 \mid X(0, 0) > u_n^2) = 1, \]

since the maximum of the array \( X(1, 0), X(2, 0), \ldots, X(n, 0) \) is \( O_P(n^{1/\alpha}) \) and cannot cluster above the level \( n^{2/\alpha} \). Hence, again \( \theta := \theta_1^* \theta_2^* = 1. \)

In general, the calculation of the extremal index for STARMA processes using this characterization would not be easy. However, it permits the adaptation of the runs method (see, for example, Embrechts et al. (1997, pp. 422–423)) to estimate the extremal index for these processes.

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References


