QUASI-HOMEOMORPHISMS AND LATTICE-EQUIVALENCES OF TOPOLOGICAL SPACES

YIP KAI-WING

(Received 3 August 1970)

Communicated by P. D. Finch

1

In his paper [1], Thron introduced a concept of lattice-equivalence of topological spaces. Let $C(X)$ denote the lattice of all closed sets of a topological space X. Two topological spaces X and Y are said to be lattice-equivalent if there exists a lattice-isomorphism between $C(X)$ and $C(Y)$. It is clear that for any continuous function $f: X \to Y$, the induced map $\psi_f: C(Y) \to C(X)$, defined by $\psi_f(F) = f^{-1}(F)$, is a lattice-homomorphism. Furthermore, if $h: X \to Y$ is a homeomorphism then $\psi_h: C(Y) \to C(X)$ is a lattice-isomorphism. Thron proved among others that for T_D-spaces X and Y, any lattice-isomorphism $\psi: C(Y) \to C(X)$ can be induced by a homeomorphism $f: X \to Y$ in the above way.

Finch [2] proved that for a lattice isomorphism $\psi: C(Y) \to C(X)$ there is a homeomorphism $h: X \to Y$ such that $\psi = \psi_h$, if the following conditions are satisfied:

(i) *to each x in X there is at least one y in Y such that*

$$\bar{y} = \psi^{-1}(\bar{x})$$

(ii) *to each y in Y there is at least one x in X such that*

$$\bar{x} = \psi(\bar{y})$$

(iii) *if for each x in X and each y in Y*

$$X_x = \{\xi : \bar{\xi} = \bar{x}, \xi \in X\}$$

$$Y_y = \{\eta : \bar{\eta} = \bar{y}, n \in Y\}$$

then

$$\bar{y} = \psi^{-1}(\bar{x}) \Rightarrow |X_x| = |Y_y|.$$

(Here $|X_x|$ denotes the cardinal number of the set X_x).

In this paper, we shall study a necessary and sufficient condition for a continuous function $f: X \to Y$ so that the induced lattice-homomorphism $\psi_f: C(Y) \to C(X)$ is a lattice-isomorphism.

https://doi.org/10.1017/S1446788700009617 Published online by Cambridge University Press

2

A continuous function $f: X \to Y$ from a topological space X into a topological space Y is said to be quasi-homeomorphism if the following conditions are satisfied:

(i) For any closed set C in $X, f^{-1}[f(C)^{-}] = C$.
(ii) For any closed set F in $Y, f[f^{-1}(F)]^{-} = F$.

It is clear that any homeomorphism is a quasi-homeomorphism.

The main theorem of this paper is the following:

THEOREM 1. *A continuous function $f: X \to Y$ induces a lattice-isomorphism $\psi_f: C(Y) \to C(X)$ if and only if f is a quasi-homeomorphism.*

PROOF. *Necessity*: Let $\psi_f: C(Y) \to C(X)$ be a lattice-isomorphism and ψ_f^{-1} its inverse, we have then

(i) For any closed set C in X, $\psi_f[\psi_f^{-1}(C)] = C$, and
(ii) For any closed set F in Y, $\psi_f^{-1}[\psi_f(F)] = F$.

Therefore, it is sufficient to show that for any closed set C in X

$$\psi_f^{-1}(C) = f(C)^{-}.$$

Let $B = \psi_f^{-1}(C)$, we have $C = \psi_f(B) = f^{-1}(B)$. So we get

$$f(C) = f[f^{-1}(B)] = f(X) \cap B \subset B.$$

Since B is closed in Y.

$$f(C)^{-} \subset B^{-} = B = \psi_f^{-1}(C).$$

To prove the inverse inclusion, let $D = f(C)^{-}$. It is clear that

$$\psi_f(D) = f^{-1}(D) = f^{-1}[f(C)^{-}] \supset f^{-1}[f(C)] \supset C.$$

Therefore $f(C)^{-} = D = \psi_f^{-1}[\psi_f(D)] \supset \psi_f^{-1}(C)$.

Sufficiency: Let f be a quasi-homeomorphism. We now prove that ψ_f is a lattice-isomorphism. We define $\psi: C(X) \to C(Y)$ as follows:

$$\psi(C) = f(C)^{-} \text{ for } C \in C(X).$$

It is clear that ψ is order preserving and hence a lattice-homomorphism of $C(X)$ into $C(Y)$. Follows from the definition of a quasi-homeomorphism immediately,

(i) $\psi \circ \psi_f(F) = F$ for all $F \in C(Y)$ and
(ii) $\psi_f \circ \psi(C) = C$ for all $C \in C(X)$.

This proves that ψ_f is a lattice-isomorphism.

3

We shall study in this section some of the properties of a quasi-homeomorphism.

THEOREM 2. *Let $f: X \to Y$ be a quasi-homeomorphism. Then we have the following*:

(i) *For two closed sets F_1 and F_2 in Y, if $f^{-1}(F_1) = f^{-1}(F_2)$ then $F_1 = F_2$.*
(ii) *For two closed sets C_1 and C_2 in X, if $f(C_1)^- = f(C_2)^-$ then $C_1 = C_2$.*
(iii) *$f(X)$ is dense in Y.*

PROOF. (i) and (ii) follow from Theorem 1 immediately. We now prove (iii). It is clear that

$$f^{-1}[f(X)^-] = X = f^{-1}(Y).$$

By (i), $f(X)^- = Y$ and hence $f(X)$ is dense in Y.

The following example illustrates that the quasi-homeomorphism is not a symmetric concept, i.e., there may be two topological spaces X and Y, so that the existence of a quasi-homeomorphism $f: X \to Y$ does not imply the existence of a quasi-homeomorphism from Y to X.

EXAMPLE. Let X be an infinite set with the co-finite topology. Put $Y = X \cup \{\infty\}$, where ∞ is an object not in X, with topology $\tau = \{\phi\} \cup \{A \subset Y: Y \backslash A \text{ is a finite subset of } X\}$. It is easy to verify that $f: X \to Y$, defined by $f(x) = x$ is a quasi-homeomorphism.

We shall show that there is no quasi-homeomorphism from Y to X. Suppose $g: Y \to X$ is a quasi-homeomorphism from Y to X. By definition, for all closed sets C in X, $g[g^{-1}(C)]^- = C$. Let $x_0 = g(\infty)$. $\{x_0\}$ is a closed set in X, therefore

$$g[g^{-1}(x_0)]^- = \{x_0\}.$$

But $g^{-1}(x_0)$ is a closed set in Y and $\infty \in g^{-1}(x_0)$, then $g^{-1}(x_0) = Y$, by theorem 2(iii) we have

$$g[g^{-1}(x_0)]^- = g(Y)^- = X.$$

The contradiction shows the non-existence of quasi-homeomorphism from Y to X.

To conclude this note, we prove the following:

THEOREM 3. *For T_0-space X and Y, a quasi-homeomorphism $f: X \to Y$ is a homeomorphism if and only if it is closed.*

PROOF. The necessity is obvious. We shall show the sufficiency. First of all, if f is a closed quasi-homeomorphism then we obviously have

(i) For any closed set C in X, $f^{-1}[f(C)] = C$ and
(ii) For any closed set F in Y, $f[f^{-1}(F)] = F$.

It follows from (ii) that $f(X) = Y$.

https://doi.org/10.1017/S1446788700009617 Published online by Cambridge University Press

It remains to show that f is injective. Let x_1, x_2 to be two distinct points in X. We shall show that $f(x_1) \neq f(x_2)$. Since X is T_0, we may assume that $x_1 \notin \{\bar{x}_2\}$. By (i) we have

$$f^{-1}[f(\bar{x}_2)] = \bar{x}_2,$$

where $\bar{x}_2 = \{\bar{x}_2\}$. Suppose $f(x_1) = f(x_2)$. We would have $f(x_1) \in f(\bar{x}_2)$ and hence

$$x_1 \in f^{-1}[f(\bar{x}_2)] = \bar{x}_2.$$

The contradiction proves the injectivity of f. The proof of the theorem is complete.

References

[1] W. J. Thron, 'Lattice-equivalence of topological spaces', *Duke Math. Journ.* 29 (1962), 671–679.

[2] P. D. Finch, 'On the lattice-equivalence of topological spaces', *Journ. Austral. Math. Soc.* 6 (1966), 495–511.

United College
The Chinese University of Hong Kong
Hong Kong

https://doi.org/10.1017/S1446788700009617 Published online by Cambridge University Press