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ON THE ORDERING OF MULTI-POINT BOUNDARY VALUE FUNCTIONS

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We are concerned with the nth-order linear differential equation

$$y^{(n)}+\sum_{k=0}^{n-1} p_k(x)y^{(k)} = 0 \tag{1}$$

where the coefficients are continuous. Aliev [1, 2] showed, in papers unavailable to the author that for $n=4$

$$r_{31}(t)\geq r_{211}(t)$$
$$r_{211}(t)\geq \min\,[r_{31}(t), r_{22}(t)]$$

(see Definition 2). Theorems 1 and 5 give respectively nth-order generalizations of these two results. The results in this paper and the results in [3] also generalize the recent results of Mathsen [4] in the special context of the differential equation (1). In this paper we are concerned with the ordering of the boundary value functions $r_{i_1\ldots i_k}(t)$. First we need the following definition.

DEFINITION 1. A nontrivial solution $y(x)$ of (1) is said to have an $(i_1,\ldots, i_k)$, $\sum_{m=1}^{k} i_m=n$, distribution of zeros on $[t, b]$ provided there are numbers $t\leq t_1<\cdots<t_k\leq b$ such that $y(x)$ has a zero at each t_m, $m=1,\ldots, k$, with multiplicity at least i_m.

DEFINITION 2. For any real number t, the number $r_{i_1i_2\ldots i_k}(t)$ is the infimum of the set of $b>t$ such that there is a nontrivial solution of (1) having an $(i_1, i_2,\ldots, i_k)$ distribution of zeros on $[t, b]$. If there is no such b we write $r_{i_1i_2\ldots i_k}(t)=\infty$.

For linear homogeneous differential equations the study of the distribution of zeros of solutions is closely related to the study of the uniqueness of solutions of boundary value problems. In particular, if $t\leq t_1<\cdots<t_k<r_{i_1\ldots i_k}(t)$, then there is a unique solution of (1) satisfying the conditions

$$y^{(m_j)}(t_j) = A_{m_jj}$$

where A_{m_jj} is a constant, $j=1,\ldots, k$, $m_j=0, 1,\ldots, i_j-1$.

Azbelev and Caljuk [5] studied the functions $r_{12}(t)$ and $r_{21}(t)$ for third-order equations. Hanan [6] studied third-order equations when $r_{12}(t)=\infty$ and when $r_{21}(t)=\infty$. Dolan [7] made a further study of these functions for third-order. In particular Dolan studied third-order equations when $r_{12}(t)<r_{21}(t)$ and when

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$r_{21}(t) < r_{12}(t)$. The author [8] made a similar study for fourth-order equations. For more related results see [9], [10], [11].

As usual $\eta_1(t)$ will denote the first conjugate point of (1) for $x=t$. Recall that [12]

$$\eta_1(t) = \min_{i+j=n} r_{ij}(t)$$

and [13], [14]

$$\eta_1(t) = r_{1\ldots 1}(t).$$

DEFINITION 3. A fundamental set of solutions $\{u_j(x, t)\}, j=0, \ldots, n-1$, is defined by the initial conditions at $x=t$

$$u_j^{(i)}(t, t) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n-1.$$

THEOREM 1. *For any real number* t,

$$\begin{aligned} r_{(n-1)1}(t) \geq r_{(n-2)11}(t) &\geq \cdots \geq r_{21\ldots 1}(t) \\ &\geq r_{1\ldots 1}(t) = \eta_1(t). \end{aligned}$$

Proof. Assume $\alpha \equiv r_{(n-k-1)1\ldots 1}(t) > r_{(n-k)1\ldots 1}(t)$, $1 \leq k \leq n-1$. Then there is a solution $u(x)$ of (1) with a $(n-k, 1, \ldots, 1)$ distribution of zeros on $[t, \alpha)$. The solution $u(x)$ has zeros at the points $t_1, \ldots, t_{k+1}$ where $t \leq t_1 < t_2 < \cdots < t_{k+1} < \alpha$ of multiplicities at least $n-k, 1, \ldots, 1$ respectively and no other distinct zeros on $[t_1, \alpha)$. Let $x_i \in (t_i, t_{i+1})$, $i=1, \ldots, k$ and $x_{k+1} \in (t_{k+1}, \alpha)$, then since $x_{k+1} < \alpha$ there is a unique solution of (1) satisfying

$$\begin{aligned} v^{(m)}(t_1) &= 0, \quad m = 0, 1, \ldots, n-k-2 \\ v(x_i) &= v_i, \quad i = 1, \ldots, k+1 \end{aligned}$$

where

$$v_i u(x_i) > 0, \qquad |v_i| < |u(x_i)|, \quad i = 1, \ldots, k+1. \tag{2}$$

Since $v(x)$ has a zero of order at least $n-k-1$ at t_1,

$$v(x) = \sum_{d=n-k-1}^{n-1} a_d u_d(x, t_1)$$

where the a_d, $d=n-k-1, \ldots, n-1$ are constants and $u_d(x, t_1)$ are defined in Definition 3. Consider the $k+1$ equations in $k+1$ unknowns

$$\sum_{d=n-k-1}^{n-1} a_d u_d(x_i, t_1) = v_i, \quad i = 1, \ldots, k+1.$$

It is easy to see that we can pick the v_i, $i=1, \ldots, k+1$ satisfying (2) so that $a_{n-k-1} \neq 0$. Hence $v(x)$ has a zero at t_1 of order exactly $n-k-1$. Note then that either $v(x)$ or $u(x)-v(x)$ has at least one zero in (t_1, x_1). We will obtain a contradiction by showing that either $v(x)$ or $u(x)-v(x)$ has at least $k+1$ distinct zeros in (t_1, α).

Let p be the number of distinct zeros of $u(x)$ of even order in (t_1, α), then $k-p$ is the number of distinct zeros of $u(x)$ of odd order in (t_1, α).

Observe that if $u(x)$ has an even order zero at t_i, then there are three possibilities:

(I) $v(x)$ has at least two distinct zeros in (x_{i-1}, x_i),

(II) $u(x)-v(x)$ has at least two distinct zeros in (x_{i-1}, x_i),

(III) $u(x)$ and $v(x)$ both have at least one zero in (x_{i-1}, x_i).

Let l be the number of times that (III) occurs. If $u(x)$ has a zero of odd order at t_i, then both $v(x)$ and $u(x)-v(x)$ have a zero in (x_{i-1}, x_i). These last two remarks account for the fact that $v(x)$ and $u(x)-v(x)$ have at least $k-p+l$ distinct zeros in (t_1, α). Now if case (I) (II) holds more than the largest integer less than $(p-l+1)/2$ times, then $v(x)$ $\{u(x)-v(x)\}$ has at least $k+1$ distinct zeros in (t_1, α) which is a contradiction. The only possibility that remains is that $p-l$ is an even integer and both cases (I) and (II) occur $(p-l)/2$ times. But in this case both $v(x)$ and $u(x)-v(x)$ have at least k distinct zeros in (x_1, α). But recall that either $v(x)$ or $u(x)-v(x)$ has a zero in (t_1, x_1) and so we again obtain a contradiction.

A theorem similar to Theorem 1 is the following.

THEOREM 2. *For* $k=1, \ldots, n-2$

$$r_{1\ldots 1(n-k)}(t) \geq \min\,[r_{1\ldots 1(n-k-1)1}(t), r_{1\ldots 1(n-k-1)}(t)].$$

Proof. The author would like to thank the referee for his suggestions concerning the following proof. Assume

$$\alpha \equiv \min\,[r_{1\ldots 1(n-k-1)1}(t), r_{1\ldots 1(n-k-1)}(t)] > r_{1\ldots 1(n-k)}(t).$$

Let $u(x)$ be a solution of (1) with a $(1, \ldots, 1, n-k)$ distribution of zeros on $[t, \alpha)$. The solution $u(x)$ has zeros at points $t_1, t_2, \ldots, t_{k+1}$ where $t \leq t_1 < t_2 < \cdots < t_{k+1} < \alpha$ of multiplicities at least $1, 1, \ldots, 1, (n-k)$ respectively and no other distinct zeros on $[t_1, \alpha)$. Let $x_i \in (t_i, t_{i+1})$, $i=1, \ldots, k$ and $x_{k+1} \in (t_{k+1}, \alpha)$ then define $v(x)$ to be the solution of (1) satisfying

$$\begin{aligned} v^{(m)}(t_{k+1}) &= 0, & m &= 0, 1, \ldots, n-k-2 \\ v(x_i) &= v_i, & i &= 1, \ldots, k \\ v(x_{k+1}) &= u(x_{k+1}) \neq 0 \end{aligned}$$

where

$$v_i u(x_i) > 0, \qquad |v_i| < |u(x_i)|, \quad i = 1, \ldots, k. \tag{3}$$

As in Theorem 1 one can show that you can pick v_i, $i=1, \ldots, k$ satisfying (3) so that $v(x)$ has a zero at t_{k+1} of order exactly $n-k-1$. If $2 \leq i \leq k$ and u has a zero of odd order at t_i then v and $v-u$ each have at least one zero in (x_{i-1}, x_i) (Case A at t_i), while if u has a zero of even order at t_i then there are the following three cases at t_i.—B: v has at least two zeros in (x_{i-1}, x_i); C: not case B and $v-u$ has at least two zeros in (x_{i-1}, x_i); D: not case B or case C then v and $v-u$ each have one zero in (x_{i-1}, x_i) (at t_i). Suppose B occurs μ times, C occurs η times, and D occurs ν times. Then v and $v-u$ have at least $k-1+\mu-\eta$ and $k-1+\eta-\mu$ distinct zeros respectively in (x_1, x_k). One or the other gives an immediate contradiction if $|\mu-\eta|\geq 2$. If $\mu-\eta=-1$ we obtain an immediate contradiction from $v-u$. If $\mu-\eta=1$ either v has a zero in $[t_1, x_1)$ and we have a contradiction, or $v-u$ does. Finally if $\mu-\eta=0$, then both v and $v-u$ have at least $k-1$ distinct zeros on (x_1, x_k). Since $v-u$ has a zero at x_{k+1}, $v-u$ cannot have a zero in (t_1, x_1). Hence v has a zero in $[t_1, x_1)$ and therefore cannot have a zero in (x_k, t_{k+1}). It follows that $v-u$ has a zero in (x_k, t_{k+1}) which is a contradiction.

THEOREM 3. *If $p\geq 2$, then*

$$r_{p1q}(t) \geq \min\,[r_{(p+1)q}(t), r_{(p-1)11q}(t)].$$

If $q\geq 2$, then

$$r_{p1q}(t) \geq \min\,[r_{p(q+1)}(t), r_{p11(q-1)}(t)].$$

Proof. Assume

$$\alpha \equiv \min\,[r_{(p+1)q}(t), r_{(p-1)11q}(t)] > r_{p1q}(t).$$

Let $u(x)$ be a solution of (1) with a $(p, 1, q)$ distribution of zeros on $[t, \alpha)$. Then $u(x)$ has a zero at t_1 of order exactly p, a zero at t_2 of order at least one, a zero at t_3 of order at least q, and no other distinct zeros on $[t_1, t_3]$ where $t\leq t_1<t_2<t_3<\alpha$. Let $x_i\in(t_i, t_{i+1})$, $i=1, 2$, then since $t_3<\alpha$ there is a unique solution $v(x)$ of (1) satisfying

$$\begin{aligned} v^{(m)}(t_1) &= 0, & m &= 0, 1, \ldots, p-2\\ v^{(k)}(t_3) &= 0, & k &= 0, 1, \ldots, q-1\\ v(x_1) &= \tfrac{1}{2}u(x_1) & &\\ v(x_2) &= \tfrac{1}{3}u(x_2) & & \end{aligned}$$

Clearly $u(x)$ and $v(x)$ are linearly independent and $v(x)$ has a zero at t_1 of order either exactly $p-1$ or exactly p. If the order of the zero of $v(x)$ at t_1 is exactly $p-1$, then it is easy to see that either $v(x)$ or $u(x)-v(x)$ has a $(p-1, 1, 1, q)$ distribution of zeros on $[t_1, t_3]$ which is a contradiction. If $v(x)$ has a zero at t_1 of order exactly p, then there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a zero of order $p+1$ at t_1 and of order q at t_3 which is a contradiction. The second inequality is proved similarly.

It is very easy to prove a more general theorem than this by assigning $v(x)$ to have the same zeros as $u(x)$ to the left of t_1 and to the right of t_3. In this case the first inequality in Theorem 3 becomes for $i_k\geq 2$

$$r_{i_1\ldots i_k 1 i_{k+1}\ldots i_\nu}(t) \geq \min\,[r_{i_1\ldots(i_k+1)i_{k+1}\ldots i_\nu}(t), r_{i_1\ldots(i_k-1)11i_{k+1}\ldots i_\nu}(t)].$$

Some similar remarks will hold for some of the following theorems but these will be left to the reader to notice.

THEOREM 4. *Let $p \geq 2$ then*

$$r_{p2}(t) \geq \min [r_{p11}(t), r_{(p-1)3}(t)],$$

and if $q \geq 3$ then

$$r_{pq}(t) \geq \min [r_{(p-1)(q+1)}(t), r_{p1(q-1)}(t), r_{p1(q-2)1}(t)].$$

Proof. Assume

$$\alpha \equiv \min [r_{(p-1)(q+1)}(t), r_{p1(q-1)}(t), r_{p1(q-2)1}(t)] > r_{pq}(t)$$

for $q \geq 3$ (if $q=2$, $\alpha \equiv \min [r_{p11}(t), r_{(p-1)3}(t)]$). Let $u(x)$ be a solution of (1) with a zero of order p at t_1 and a zero of order q at t_2 where $t \leq t_1 < t_2 < \alpha$. Clearly $u(x) \neq 0$ on (t_1, t_2) and $u^{(q)}(t_2) \neq 0$. Let $v(x)$ be a solution of (1) satisfying for $t_1 < x_1 < t_2 < x_2 < \alpha$ the boundary conditions

$$\begin{aligned} v^{(m)}(t_1) &= 0, \quad m = 0, 1, \ldots, p-1 \\ v(x_1) &= \tfrac{1}{2}u(x_1) \\ v(x_2) &= \tfrac{1}{3}u(x_2) \\ v^{(k)}(t_2) &= 0, \quad k = 0, 1, \ldots, q-3. \end{aligned}$$

(If $q=2$ then there is no boundary condition prescribed at t_2.) We can assume $u(x) \neq 0$ for $t_2 < x \leq x_2$. Since $\alpha < r_{(p-1)(q+1)}(t)$, $v(x)$ has a zero at t_2 of order exactly $q-2$ or $q-1$. If $v(x)$ has a zero at t_2 of order exactly $q-1$ $\{q-2\}$ then either $v(x)$ or $u(x)-v(x)$ has a $(p, 1, q-1)$ $\{(p, 1, q-2, 1)\}$ distribution of zeros on $[t_1, \alpha)$ which is a contradiction.

THEOREM 5.

$$\begin{aligned} r_{p11}(t) &\geq \min [r_{(p+1)1}(t), r_{p2}(t)] \\ r_{11p}(t) &\geq \min [r_{1(p+1)}(t), r_{2p}(t)]. \end{aligned}$$

For $q \geq 2$

$$r_{p1q}(t) \geq \min [r_{(p+1)q}(t), r_{p2(q-1)}(t), r_{(p+1)1(q-1)}(t)]$$

and for $p \geq 2$

$$r_{p1q}(t) \geq \min [r_{p(q+1)}(t), r_{(p-1)2q}(t), r_{(p-1)1(q+1)}(t)].$$

Proof. Assume

$$\alpha \equiv \min [r_{(p+1)q}(t), r_{p2(q-1)}(t), r_{(p+1)1(q-1)}(t)] > r_{p1q}(t)$$

for $q \geq 2$ (if $q=1$, $\alpha \equiv \min [r_{(p+1)1}(t), r_{p2}(t)] > r_{p11}(t)$). Let $u(x)$ be a solution of (1) with a $(p, 1, q)$ distribution of zeros on $[t, \alpha)$. Then $u(x)$ has a zero of order exactly p at t_1, of order exactly 1 at t_2, and a zero of order at least q at t_3 where $t \leq t_1 < t_2 < t_3 < \alpha$. Since $\alpha \leq r_{(p+1)q}(t)$, there is a unique solution $v(x)$ of (1) satisfying

$$\begin{aligned} v^{(m)}(t_1) &= 0, \quad m = 0, 1, \ldots, p \\ v^{(k)}(t_3) &= 0, \quad k = 0, \ldots, q-2 \\ v^{(q-1)}(t_3) &= 1 \end{aligned}$$

for $q \geq 2$ (if $q=1$, then the boundary condition at t_3 becomes $v(t_3)=1$). It follows from Theorem 1 [12] that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in (t_2, t_3) which is a contradiction.

The key to the proof of Theorem 5 was to obtain a solution of (1) with a double zero which was a linear combination of two other solutions. Leighton and Nehari ([15] Lemma 1.2) were the first to use such a technique. In fact their result gave a very simple proof of the Sturm separation theorem. Sherman ([12] Theorem 1) stated a more general result. Gustafson [16] then stated a result which enabled one to find a linear combination of k solutions of (1) with certain distributions of zeros to have a zero of order k. A special case of Gustafson's result will be used in the following theorem to find a linear combination of three solutions with a triple zero. One could find many more results by using this same technique but we will be content here just to prove the following theorem.

THEOREM 6. *For any real number* t,

$$\begin{aligned} r_{p12}(t) &\geq \min\,[r_{(p+1)11}(t), r_{p21}(t), r_{p3}(t)] \\ r_{21p}(t) &\geq \min\,[r_{11(p+1)}(t), r_{p21}(t), r_{p3}(t)] \\ r_{p21}(t) &\geq \min\,[r_{(p+1)11}(t), r_{p12}(t), r_{p3}(t)] \\ r_{12p}(t) &\geq \min\,[r_{11(p+1)}(t), r_{21p}(t), r_{3p}(t)] \end{aligned}$$

Proof. Assume

$$\alpha \equiv \min\,[r_{(p+1)11}(t), r_{p21}(t), r_{p3}(t)] > r_{p12}(t).$$

Let $u_1(x)$ be a solution of (1) with a $(p, 1, 2)$ distribution of zeros on $[t, \alpha)$. The solution $u_1(x)$ has zeros at points t_1, t_2, t_3 where $t \leq t_1 < t_2 < t_3 < \alpha$ of multiplicities $p, 1, 2$ respectively. Since $\alpha > r_{p12}(t)$ these multiplicities are exact and, since $r_{p111}(t) > \alpha$, $u_1(x)$ has no other zeros on $[t_1, \alpha)$. We can assume $u_1(x) > 0$ on (t_2, t_3).

Since $r_{p21}(t) > t_3$ there is a unique solution $u_2(x)$ of (1) satisfying

$$\begin{aligned} u_2^{(m)}(t_1) &= 0, \quad m = 0, \ldots, p-1 \\ u_2^{(l)}(t_2) &= 0, \quad l = 0, 1 \\ u_2(t_3) &= 1 \end{aligned}$$

and $u_2(x) > 0$ on (t_2, t_3). Since $r_{p3}(t) > t_3$, it is easy to see that $u_2''(t_2) > 0$. Since $r_{(p+1)11}(t) > t_3$ there is a unique solution $u_3(x)$ of (1) satisfying

$$\begin{aligned} u_3^{(k)}(t_1) &= 0, \quad k = 0, \ldots, p \\ u_3(t_2) &= 1 \\ u_3(t_3) &= 0. \end{aligned}$$

Since $u_3(t) > 0$ on $[t_2, t_3)$, $u_3(t_3) = 0$, and $r_{p3}(t) > t_3$, $u_3'(t_3) < 0$. Now let $\omega(x) = \det\,(u_i^{(j)}(x))$, $i = 1, 2, 3$, $j = 0, 1, 2$. Since $\omega(t_2) > 0$ and $\omega(t_3) < 0$, $\omega(x)$ has a zero at some point $\xi \in (t_2, t_3)$. It follows that there is a (nontrivial) linear combination of $u_i(x)$, $i = 1, 2, 3$ with a triple zero at ξ. But this same linear combination has a zero of order p at t_1 which contradicts $r_{p3}(t) > t_3$. Hence

$$r_{p12}(t) \geq \min [r_{(p+1)11}(t), r_{p21}(t), r_{p3}(t)].$$

The other three inequalities in this theorem are proved similarly.

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