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A GENERAL HEWITT-YOSIDA DECOMPOSITION

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Introduction. In 1952, E. Hewitt and K. Yosida [**3**] proved that a bounded, finitely additive real-valued set function has a unique representation as the sum of a countably additive function and a "purely finitely additive" function.

Below, using a variation of the Carathéodory process we give a suitable generalization to s-bounded vector-valued set functions. In fact, since the methods do not rely on scalar multiplication, we give the result for commutative Hausdorff topological groups.

It is interesting to note that the Carathéodory process gives an elegant proof even for real-valued functions [**1**].

0. Notation. Throughout the paper,

$$\begin{aligned} &\mathbf{N} && \text{is the set of non negative integers,} \\ &A\backslash B &&= \{a \in A : a \notin B\}, \\ &\cup K &&= \textstyle\bigcup_{A \in K} A, \\ &K_\sigma &&= \{\textstyle\bigcup_n A_n : A \text{ is a sequence in } K\}. \end{aligned}$$

The symbol S denotes an abstract space and Y, a commutative Hausdorff topological group. We denote by $\mathscr{V}$ a base for the neighbourhoods of 0 in Y consisting of closed symmetric sets, and by H a ring of subsets of S with $S \in H_\sigma$.

1. Preliminaries. For the basic definitions and preliminary results concerning group-valued set functions we refer to Sion [**7**; **8**]. However, except for a few straight forward generalizations of the real-valued terminology, we explain all the measure-theoretic concepts that we use.

We recall first that a function ϕ on H to Y is *s-bounded* if and only if $\phi(A_n) \to 0$, whenever A is a disjoint sequence in H. If ϕ is finitely additive, an equivalent condition is that $\phi(A_n)$ be Cauchy whenever A is an increasing sequence in H (and hence, also, whenever A is a decreasing sequence in H).

We assume throughout that *ϕ is finitely additive and s-bounded on H to a complete subset of Y*. (The completeness condition is of minor importance since we may always complete Y if necessary. In any case, the condition is automatically satisfied whenever Y is a locally convex space in which closed bounded sets are complete, since an s-bounded finitely additive function to a locally convex space is bounded. The proof follows Rickart [**5**; 2.4].)

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For purposes of $\epsilon/2^n$-type arguments, given V in $\mathscr{V}$, we choose, by continuity of addition, neighbourhoods V_n in $\mathscr{V}$ such that for all n in $\mathbf{N}$

$$\sum_{i=0}^{n} V_i \subset V.$$

2. The Carathéodory process. We now describe the process of generating an (outer) measure from an s-bounded finitely additive function. We impose on ϕ and H the conditions of the preceding sections.

2.1. *Definitions.* Let $\mathscr{P}$ be the collection of countable disjoint subfamilies of H and, for $A \subset S$, let $\mathscr{P}_A$ be the collection of members of $\mathscr{P}$ covering A, directed by refinement. ($\mathscr{P}_A$ is never empty, since $S \in H_\sigma$.)

(1) *For P in $\mathscr{P}$, $\bar{\phi}(P) = \sum_{E \in P} \phi(E)$, whenever the sum converges unconditionally.*

(2) *For $A \subset S$, $\mu(A) = \lim_{P \in \mathscr{P}_A} \bar{\phi}(P)$, whenever the limit exists.*

We notice that, since ϕ is finitely additive, $\bar{\phi}(P)$ is the limit of $\phi(\bigcup \Delta)$ as Δ runs through the finite subsets of P directed upward by inclusion.

The fundamental theorem about μ is the following:

2.2. THEOREM. *Let ϕ be s-bounded and finitely additive on H to a complete subset of Y. Then the μ defined in* 2.1 *is an H_σ-outer measure on S to Y. That is, μ is defined on all subsets of S, is σ-additive on a σ-field containing H and is H_σ-outer regular, in the sense that for each V in $\mathscr{V}$ and $A \subset S$, there exists A' in H containing A with $\mu(B) \in \mu(A) + V$, whenever $A \subset B \subset A'$.*

The proof consists in establishing the following lemmas.

2.3. LEMMAS.

(1) *The sum $\sum_{E \in P} \phi(E \cap F)$ converges unconditionally, uniformly for F in H; in particular, $\bar{\phi}(P)$ exists for all P in $\mathscr{P}$.*

(2) *$\mu(A)$ exists for all $A \subset S$.*

2.4. LEMMAS.

(1) *μ is σ-additive on H_σ.*

(2) *μ is H_σ-outer regular.*

(3) *For any increasing sequence A of subsets of S, $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$.*

(4) *The family of μ-measurable sets is a σ-field containing H on which μ is σ-additive.*

The Lemmas 2.4 are, in fact, valid whenever μ exists on all subsets of S, even without the additivity and completeness assumptions.

Proof of 2.3. (1) In the contrary case, the partial sums would not form a Cauchy net uniformly in F. Thus, for some V in $\mathscr{V}$, we could extract a disjoint sequence of finite $\Delta_n \subset P$ and a sequence of elements $F_n \in H$ such that $\sum_{E \in \Delta_n} \phi(E \cap F_n) \notin V$. Since ϕ is finitely additive, this means that $\phi(F_n \cap \bigcup \Delta_n) \notin V$, contradicting s-boundedness.

(2) If $\mu(A)$ does not exist, then $\bar{\phi}(P)$ does not form a Cauchy net as P runs over $\mathscr{P}_A$. Hence, for some V in $\mathscr{V}$, there exists a sequence P in $\mathscr{P}_A$ such that for each n in $\mathbf{N}$, P_{n+1} is finer than P_n and $\bar{\phi}(P_{n+1}) - \bar{\phi}(P_n) \notin V + V + V$. We now proceed to construct a decreasing sequence B in H with the property that for every n in $\mathbf{N}$, $\bar{\phi}(P_n) - \phi(B_n) \in V$, for then, by s-boundedness, eventually $\phi(B_n) - \phi(B_{n+1}) \in V$ so that

$$\begin{aligned}\bar{\phi}(P_n) - \bar{\phi}(P_{n+1}) &= \bar{\phi}(P_n) - \phi(B_n) + \phi(B_n) - \phi(B_{n+1}) \\ &\quad + \phi(B_{n+1}) - \bar{\phi}(P_{n+1}) \\ &\in V + V + V,\end{aligned}$$

contradicting the choice of P.

To construct the B_n, let V_n be the neighbourhoods described in § 1. Using (1), choose for each n in $\mathbf{N}$ a finite subset Δ_n of P_n such that for every F in H, $\phi(F \cap \cup \Delta \backslash \cup \Delta_n) \in V_n$, whenever Δ is a finite subset of P_n. (This implies also that $\phi(\cup \Delta_n) - \bar{\phi}(P_n) \in V_n$, since V_n is closed.) Put $A_n = \cup \Delta_n$ and define $B_n = \bigcap_{i=0}^n A_i$. To show that $\bar{\phi}(P_n) - \phi(B_n) \in V$, first note that for each $k \in \mathbf{N}$, $\phi(F \cap A_{k+1} \backslash A_k) \in V_k$ for all $F \in H$. Indeed, since P_{k+1} is a refinement of P_k, there exists a finite $\Delta \subset P_k$ such that $A_{k+1} = \cup \Delta_{k+1} \subset \cup \Delta$, whence $\phi(F \cap A_{k+1} \backslash A_k) = \phi(F \cap A_{k+1} \cap (\cup \Delta \backslash \cup \Delta_k)) \in V_k$. Now for each n in $\mathbf{N}$ we have

$$\begin{aligned}A_n &= B_n \cup (\textstyle\bigcap_{i=1}^n A_i \backslash A_0) \cup (\bigcap_{i=2}^n A_i \backslash A_1) \cup \ldots \cup (A_n \backslash A_{n-1}) \\ &= B_n \cup \textstyle\bigcup_{k=0}^{n-1} (\bigcap_{i=k+1}^n A_i \backslash A_k),\end{aligned}$$

so that

$$\bar{\phi}(P_n) - \phi(B_n) = \bar{\phi}(P_n) - \phi(A_n) + \phi(A_n) - \phi(B_n) \in V_n + \textstyle\sum_{k=0}^{n-1} V_k \subset V.$$

Thus the B_n satisfy the required property and the lemma is proved.

Proof of 2.4. The proof of (1) depends upon the usual properties of interchanging limits and summation for unconditional convergence. These can be proved in the usual manner, using the neighbourhoods V_n described in section 1.

Let A be a disjoint sequence in H_σ; say, $A_n = \cup Q_n$, where $Q_n \in \mathscr{P}$ for all n in $\mathbf{N}$. Put $Q = \bigcup_n Q_n$ and $B = \bigcup_n A_n$. Now, $\mu(B) = \lim_P \bar{\phi}(P)$ as P runs over the elements of $\mathscr{P}_B$ finer than Q, and $\mu(A_n) = \lim_{P'} \bar{\phi}(P')$ as P' runs over those elements of $\mathscr{P}_{A_n}$ finer than Q_n. These latter are of the form $P' = \{E \cap A_n : E \in P\}$, for P in $\mathscr{P}_B$ finer than Q. Hence,

$$\begin{aligned}\mu(B) &= \lim_P \textstyle\sum_{E \in P} \phi(E) \\ &= \lim_P \textstyle\sum_{n \in \mathbf{N}} \sum_{E \in P} \phi(A_n \cap E) \\ &= \textstyle\sum_{n \in \mathbf{N}} \lim_P \sum_{E \in P} \phi(A_n \cap E) \\ &= \textstyle\sum_{n \in \mathbf{N}} \mu(A_n).\end{aligned}$$

(2) Given $A \subset S$ and V in $\mathscr{V}$, let $P \in \mathscr{P}_A$ be such that $\bar{\phi}(P') - \mu(A) \in V$ for all $P' \in \mathscr{P}_A$ finer than P. Put $A' = \cup P$. Now if $A \subset B \subset A'$, and

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$P' \in \mathscr{P}_B$ is finer than P, we have $P' \in \mathscr{P}_A$, so $\bar{\phi}(P') \in \mu(A) + V$. Taking the limit as P' runs in $\mathscr{P}_B$ we have $\mu(B) \in \mu(A) + V$.

The proofs of (3) and (4) are essentially contained in section 5 of Sion [**6**]; we therefore omit them. They are also given in Sion [**7**].

3. The Hewitt-Yosida decomposition. In this section, in addition to the assumptions of sections 0 and 1, we require that H be a field, i.e., that $S \in H$. (This condition may be removed, for putting $\phi(S \setminus A) = -\phi(A)$ for A in the ring H extends ϕ to an s-bounded additive function on the field generated by H.)

We proceed to describe the Hewitt-Yosida decomposition of the s-bounded, finitely additive function ϕ on H to Y. The real-valued version was proved by Hewitt-Yosida [**3**] (see also [**1**; **2**, p. 163]).

We need two definitions.

3.1. *Definitions.* For ψ and ν, set functions on H to topological groups Y and Z, respectively,

(1) *ψ and ν are (topologically) singular, $\psi \perp_t \nu$, if and only if given neighbourhoods V and W of the origins in Y and Z, respectively, there exists such an A in H that, for all E in H,*

$$\psi(E \cap A) \in V \quad \text{and} \quad \nu(E \setminus A) \in W;$$

(2) *ψ is purely finitely additive if and only if ψ is finitely additive and $\psi \perp_t \nu$ for every σ-additive s-bounded ν on H to a commutative topological group.*

The general Hewitt-Yosida theorem is the following.

3.2. THEOREM. *Every s-bounded finitely additive set function ϕ on the field H with values in a complete subset of a topological group Y can be uniquely represented in the form $\phi = \phi_\sigma + \phi_p$, where ϕ_σ is σ-additive and s-bounded and ϕ_p is purely finitely additive.*

Proof. To prove the theorem, let μ be the Carathéodory measure generated by ϕ and H as in 2.1 and put

$$\phi_\sigma = \mu | H, \qquad \phi_p = \phi - \phi_\sigma.$$

Since μ is σ-additive on H_σ, we have that ϕ_σ is s-bounded. The theorem is now a consequence of the following lemma.

3.3. LEMMA.

(1) *ϕ is σ-additive if and only if $\phi_\sigma = \phi$;*

(2) *ϕ is purely finitely additive if and only if $\phi_\sigma = 0$.*

(Indeed, since ϕ_σ is σ-additive, $\phi_{\sigma\sigma} = \phi_\sigma$, by 3.3(1). This implies that $(\phi_p)_\sigma = (\phi - \phi_\sigma)_\sigma = 0$, so that ϕ_p is purely finitely additive, by 3.3(2). The lemma also establishes uniqueness, for if $\phi = \psi + \psi'$ were another such decomposition, then $\phi_\sigma = \psi_\sigma + \psi' = \psi$.)

Proofs.

(1) Necessity is an immediate consequence of the definitions; sufficiency, of Theorem 2.2.

(2) If ϕ is purely finitely additive, then $\phi \perp_t \phi_\sigma$. Thus, given any V in $\mathscr{V}$, there exists A in H such that $\phi(E \cap A) \in V$ and $\phi_\sigma(E\backslash A) \in V$ for all E in H. But if $P \in \mathscr{P}_{E\cap A}$ and P is finer than $\{E \cap A\}$, we have

$$\bar{\phi}(P) = \lim_\Delta \phi(E \cap A \cap \bigcup \Delta)$$

as Δ runs over the finite subsets of P, so $\phi(P)$ belongs to V. Taking the limit as P runs in $\mathscr{P}_{E\cap A}$ we have $\phi_\sigma(E \cap A) \in V$. Thus

$$\phi_\sigma(E) = \phi_\sigma(E \cap A) + \phi_\sigma(E\backslash A) \in V + V.$$

But V was arbitrary, so $\phi_\sigma(E) = 0$.

On the other hand, suppose $\phi_\sigma = 0$ and let ν be any σ-additive s-bounded function on H to a commutative topological group Z. If ϕ and ν are not singular, there exists V in $\mathscr{V}$ and a neighbourhood W of 0 in Z, such that whenever $A \in H$ and $\nu(E \cap A) \in W$ for all $E \in H$, $S\backslash A$ contains some $B \in H$ with $\phi(B) \notin V + V$. We first show that $\phi(B) \notin V + V$ implies that for any neighbourhood W' of 0 in Z,

(*) B contains some $B' \in H$ such that $\phi(B') \notin V$ and $\nu(B' \cap E) \in W'$, for all E in H.

Indeed, since $\phi_\sigma(B) = 0$, there exists $P \in \mathscr{P}_B$ with $\bigcup P = B$ such that $\bar{\phi}(P) \in V_0$. Thus P contains a finite Δ such that $\phi(\bigcup \Delta) \in \bar{\phi}(P) + V_0 \subset V_0 + V_0 \subset V$. Since ν is σ-additive and s-bounded, Δ may be chosen so that $\nu(E \cap B\backslash \bigcup \Delta) \in W'$, for all E in H (see Lemma 2.3(1)). Put $B' = B\backslash \bigcup \Delta$. Then for all E in H, $\nu(E \cap B') \in W'$ and $\phi(B') \notin V$, for otherwise $\phi(B) = \phi(B') + \phi(\bigcup \Delta) \in V + V$.

Now, choose for $n \in \mathbf{N}$, neighbourhoods W_n of 0 in Z such that $\sum_{i=0}^n W_n \subset W$. Since $\nu(E \cap \emptyset) = 0 \in W$, for all E in H, S contains some B in H with $\phi(B) \notin V + V$, and hence, by (*) a member A_0 of H such that $\phi(A_0) \notin V$ and $\nu(E \cap A_0) \in W_0$, for all E in H. Recursively, suppose $A_0, \ldots, A_n \in H$ have been chosen disjoint such that for each $i = 0, \ldots, n$, $\phi(A_i) \notin V$ and $\nu(E \cap A_i) \in W_i$ for all E in H. Then for all E in H, $\nu(E \cap \bigcup_{i=0}^n A_i) \in \sum_{i=0}^n W_i \subset W$, so $S\backslash \bigcup_{i=0}^n A_i$ contains some B in H with $\phi(B) \notin V + V$, and hence some $A_{n+1} \in H$ such that $\phi(A_{n+1}) \notin V$ and $\nu(E \cap A_{n+1}) \in W_{n+1}$, for all E in H. We have thus constructed a disjoint sequence A in H such that $\phi(A_n)$ does not converge to 0, contradicting s-boundedness of ϕ.

4. Remarks. (*The development of the general Carathéodory process*). In [**6**] M. Sion presented the first construction of a group-valued outer measure, starting with a σ-additive function on a ring of sets under a monotone convergence condition which is equivalent to s-boundedness.

In [**9**] we showed that a simple modification of the Phillips-Rickart integration process [**4**] makes the indefinite integral into a group-valued outer

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measure. M. Sion showed [**7**; **8**] that this construction yields a reasonable generalization of the Carathéodory process, even when one begins with a function, not necessarily additive, with values in a commutative topological semigroup with identity. (The construction necessarily coincides with that of Sion [**6**] for σ-additive functions defined on a ring.)

The important contribution of Phillips in this process was to remove the necessity of considering unconditional convergence, using a certain double limit instead of an iterated limit. In the present article, however, we did not need the full strength of these methods, since a strong form of unconditional convergence was guaranteed by s-boundedness (Lemma 2.3).

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