



# Zeros of symmetric Laurent polynomials of type $(BC)_n$ and Koornwinder–Macdonald polynomials specialized at $t^{k+1}q^{r-1} = 1$

Masahiro Kasatani

## ABSTRACT

A characterization of the space of symmetric Laurent polynomials of type  $(BC)_n$ , which vanish on a certain set of submanifolds, is given by using the Koornwinder–Macdonald polynomials. A similar characterization was given previously for symmetric polynomials of type  $A_n$  by using the Macdonald polynomials. We use a new method which exploits the duality relation. The method simplifies a part of the proof in the  $A_n$  case.

## 1. Introduction

Let  $k, r, n$  be positive integers. We assume that  $n \geq k + 1$  and  $r \geq 2$ . In [FJMM03],  $n$ -variable symmetric polynomials satisfying certain zero conditions are characterized by using the Macdonald polynomials [Mac95] specialized at

$$t^{k+1}q^{r-1} = 1. \tag{1}$$

To be precise, the paper [FJMM03] works in the following setting. Denote by  $m$  the greatest common divisor of  $k + 1$  and  $r - 1$ . Let  $\omega$  be an  $m$ th primitive root of unity. Then, the variety given by  $t^{(k+1)/m}q^{(r-1)/m} = \omega$  is an irreducible component of (1). It is uniformized as follows. Let  $\omega_1 \in \mathbb{C}$  be such that  $\omega_1^{(r-1)/m} = \omega$ . We consider the specialization of  $t, q$  in terms of the uniformization parameter  $u$ ,

$$t = u^{(r-1)/m}, \quad q = \omega_1 u^{-(k+1)/m}. \tag{2}$$

The following result was obtained in [FJMM03].

**THEOREM 1.1.** *For a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  satisfying*

$$\lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq n - k), \tag{3}$$

*the Macdonald polynomial  $P_\lambda \in \mathbb{C}(t, q)[x_1, \dots, x_n]^{\mathfrak{S}_n}$  has no pole at (2), and when it is specialized at (2), it vanishes on the submanifold given by*

$$x_i/x_{i+1} = tq^{s_i} \quad \text{for } 1 \leq i \leq k \tag{4}$$

*for each choice of non-negative integers  $s_i$  such that  $\sum_{i=1}^k s_i \leq r - 1$ . Moreover, a basis of the space of symmetric polynomials  $P \in \mathbb{C}(u)[x_1, \dots, x_n]^{\mathfrak{S}_n}$  satisfying the above condition is given by the Macdonald polynomials  $P_\lambda$  specialized at (2) where  $\lambda$  satisfies (3).*

The condition that a polynomial vanishes on the submanifold (4) is called the wheel condition corresponding to the submanifold (4) and a partition  $\lambda$  satisfying the condition (3) is

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called a  $(k, r, n)$ -admissible partition. Note that if we set  $s_{k+1} = r - 1 - \sum_{i=1}^k s_i$ , it follows that  $x_{k+1}/x_1 = tq^{s_{k+1}}$  from (4) and (1).

In this paper, we obtain a similar result in the case of  $n$ -variable symmetric Laurent polynomials of type  $(BC)_n$ . Here we say a Laurent polynomial in the variables  $x_1, \dots, x_n$  is of type  $(BC)_n$  if and only if it is symmetric and invariant for the change of the variable  $x_1$  to  $x_1^{-1}$ . The original case in [FJMM03] corresponds to  $A_n$ . We use the Koornwinder–Macdonald polynomials  $P_\lambda$  of type  $(BC)_n$  (see [Koo92]) in order to characterize the space of symmetric Laurent polynomials of type  $(BC)_n$  satisfying the wheel conditions. The Koornwinder–Macdonald polynomials depend on six parameters  $t, q, a, b, c, d$ .

We set  $W_n := \mathfrak{S}_n \times (\mathbb{Z}_2)^n$ . Our main result is the following.

**THEOREM 1.2.** *Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a  $(k, r, n)$ -admissible partition. Then, the Koornwinder–Macdonald polynomial  $P_\lambda \in \mathbb{C}(t, q, a, b, c, d)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n}$  has no pole at (2), and when it is specialized at (2), it satisfies the wheel conditions corresponding to (4). Moreover, a basis of the space of symmetric Laurent polynomials of type  $(BC)_n$  in  $\mathbb{C}(u, a, b, c, d)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n}$  satisfying the wheel conditions is given by the Koornwinder–Macdonald polynomials  $P_\lambda$  specialized at (2) where  $\lambda$  are  $(k, r, n)$ -admissible partitions.*

We explain a motivation to study the wheel condition in terms of representation theory of double affine Hecke algebra. In [Sah99], Sahi introduced the double affine Hecke algebra  $\mathcal{H}$  of type  $(C_n^\vee, C_n)$ . The algebra  $\mathcal{H}$  has six parameters  $t, q, a, b, c$  and  $d$ . He defined the  $n$ -variable Laurent polynomial representation  $\mathcal{R}$  of the algebra  $\mathcal{H}$  by inducing up the Noumi representation [Nou95] of affine Hecke algebra. In  $\mathcal{R}$ , the element  $X_i \in \mathcal{H}$  ( $1 \leq i \leq n$ ) is the multiplicative action of the variable  $x_i$  and the element  $Y_i \in \mathcal{H}$  ( $1 \leq i \leq n$ ) is a pairwise-commuting  $q$ -difference operator called  $q$ -Dunkl operator. Sahi also showed that  $\mathcal{R}$  is irreducible for generic parameters.

Let  $J^{(k,r)}$  be the space of  $(BC)_n$ -symmetric polynomials satisfying the wheel condition. Then we have that  $J^{(k,r)}$  is invariant under (a) the multiplicative action of  $(BC)_n$ -symmetric polynomials and (b) the action of  $q$ -difference operators  $\{D_i; i = 1, \dots, n\}$  defined in § 2. The invariance by  $\{D_i\}$  can easily be checked because the basis of  $J^{(k,r)}$  is given by  $\{P_\lambda\}$  which are simultaneous eigenfunctions with respect to  $\{D_i\}$ .

In the Laurent polynomial representation  $\mathcal{R}$ , two actions (a) and (b) are  $(BC)_n$ -symmetrized actions of  $X_i$  and  $Y_i$ , respectively. Namely, the ideal  $J^{(k,r)}$  is a subrepresentation of some subalgebra of  $\mathcal{H}$  which acts on  $\mathcal{R}^{W_n}$ .

This observation suggests that for specialized parameters at  $t^{k+1}q^{r-1} = 1$ , there exists a subrepresentation  $\mathcal{I} \subset \mathcal{R}$  of the algebra  $\mathcal{H}$  defined by some vanishing condition such as the wheel condition. (Actually, for the double affine Hecke algebra of type  $GL_n$ , there exist such subrepresentations. See [Kas05] for more details.)

Although the statement of Theorem 1.2 is quite analogous to that of Theorem 1.1, our proof of Theorem 1.2 is different from that of Theorem 1.1 given in [FJMM03]. In fact, our method gives an alternative proof simpler than that given in [FJMM03] for the  $A_n$  case. We use the duality relation for the Koornwinder–Macdonald polynomials  $P_\lambda$ . In [KMSV04], we obtain a further result by an application of the method used in this paper.

Let us explain the duality relation and the method of our proof. We denote by  $P_\lambda^*$  the dual Koornwinder–Macdonald polynomial [Van96] defined by dual parameters  $t, q, a^*, b^*, c^*, d^*$ :

$$\begin{aligned} a^* &= -a^{1/2}b^{1/2}c^{1/2}d^{1/2}q^{-1/2}, & b^* &= -a^{1/2}b^{1/2}c^{-1/2}d^{-1/2}q^{1/2}, \\ c^* &= -a^{1/2}b^{-1/2}c^{1/2}d^{-1/2}q^{1/2}, & d^* &= -a^{1/2}b^{-1/2}c^{-1/2}d^{1/2}q^{1/2}. \end{aligned} \tag{5}$$

For any partition  $\mu = (\mu_1, \dots, \mu_n)$  and  $f \in \mathbb{C}(t, q, a, b, c, d)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , we define specializations  $u_\mu(f), u_\mu^*(f)$  of  $f$  by

$$\begin{aligned} u_\mu(f) &:= f(t^{n-1}q^{\mu_1}a^*, t^{n-2}q^{\mu_2}a^*, \dots, q^{\mu_n}a^*), \\ u_\mu^*(f) &:= f(t^{n-1}q^{\mu_1}a, t^{n-2}q^{\mu_2}a, \dots, q^{\mu_n}a). \end{aligned} \tag{6}$$

In particular, we have

$$\begin{aligned} u_0(f) &= f(t^{n-1}a^*, t^{n-2}a^*, \dots, a^*), \\ u_0^*(f) &= f(t^{n-1}a, t^{n-2}a, \dots, a). \end{aligned}$$

The duality relations reads as

$$\frac{u_\mu^*(P_\lambda)}{u_0^*(P_\lambda)} = \frac{u_\lambda(P_\mu^*)}{u_0(P_\mu^*)}. \tag{7}$$

To prove the two statements, (i)  $P_\lambda$  has no pole at (2) and (ii)  $P_\lambda$  specialized at (2) satisfies the wheel conditions corresponding to (4), we use the duality relation with special choices of  $\mu$ . Here, we explain only the latter assuming that the former is already proved. The details of the proofs are given in the main body of the paper.

In order to study the values of  $P_\lambda$  on the submanifold (4), we use (7) by choosing  $\mu = (\mu_1, \dots, \mu_n)$  in such a way that

$$\mu_i - \mu_{i+1} = s_i \quad \text{for } i = 1, \dots, k, \tag{8}$$

$$\mu_i - \mu_{i+1} > 2 \left\lceil \frac{n}{k+1} \right\rceil (r-1) \quad \text{for } i = k+1, \dots, n-1. \tag{9}$$

From the definition of the dual polynomial  $P_\mu^*$ , it has no pole at the specialization (2) if (9) is valid. Without specialization (2) we have an explicit formula for  $u_0^*(P_\lambda)$  and  $u_0(P_\mu^*)$ , and we can easily count the order of zeros (or poles) for them. Using (7), we can prove that  $u_\mu^*(P_\lambda)$  vanishes at (2). Since enough  $\mu$  satisfying the conditions (8) and (9) exist, the Laurent polynomial  $P_\lambda$  itself should vanish at (4).

This much is the proof of the first half of Theorem 1.2. Let  $J^{(k,r)}$  be the space of symmetric Laurent polynomials  $P$  of type  $(BC)_n$  satisfying the wheel conditions, and for a positive integer  $M$ , let  $J_M^{(k,r)}$  be its subspace consisting of  $P$  such that the degree of  $P$  in each variable  $x_i$  is less than  $M$ . Due to the invariance for  $x_i \leftrightarrow x_i^{-1}$ , the dimension of this subspace is finite. From the first half of the proof, we have a lower estimate of the dimension of  $J_M^{(k,r)}$ . We give an upper estimate of the dimension of the same space by considering its dual space. This is a standard technique that originated in the papers by Feigin and Stoyanovsky [FS93, FS94]. Showing that these two estimates are equal, we finish the proof of Theorem 1.2.

## 2. Properties of the Koornwinder–Macdonald polynomials

Let  $n$  be the number of variables. We denote by  $W_n$  the group generated by permutations and sign flips ( $W_n \cong \mathfrak{S}_n \times (\mathbb{Z}_2)^n$ ). We consider a  $W_n$ -symmetric Laurent polynomial ring

$$\bar{\Lambda}_n = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n}. \tag{10}$$

We denote by  $\pi_n$  the set of partitions of length  $n$ ,  $\lambda = (\lambda_1, \dots, \lambda_n)$ . We denote by  $\hat{m}_\lambda$  a monomial  $W_n$ -symmetric Laurent polynomial:

$$\hat{m}_\lambda(x) := \sum_{\nu \in W_n \lambda} \prod_i x_i^{\nu_i}.$$

Let  $\Lambda_n = \bar{\Lambda}_n \otimes \mathbb{C}(t, q, a, b, c, d)$ . The Koornwinder–Macdonald polynomial  $P_\lambda(x)$  corresponding to  $\lambda$  is a simultaneous eigenfunction of the difference operators  $\{D_r; 1 \leq r \leq n\}$  (see [Van96]). The corresponding eigenvalues  $E_\lambda^{(r)}$  are of the form

$$E_\lambda^{(r)} := u_\lambda(\widehat{m}_{1r}) + \sum_{0 \leq s < r} a_{r,s} u_\lambda(\widehat{m}_{1s})$$

where  $u_\lambda$  is that in (5) and  $a_{r,s} \in \mathbb{C}[t^{\pm 1}, q^{\pm 1}, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, (a^*)^{\pm 1}]$ .

To be precise,

$$D_r := \sum_{\substack{J \subset \{1, \dots, n\}, 0 \leq |J| \leq r \\ \epsilon_j = \pm 1, j \in J}} U_{J^c, r-|J|}(x) V_{\epsilon J, J^c}(x) T_{\epsilon J, q}$$

$$V_{\epsilon J, K}(x) := \prod_{j \in J} a^* \frac{1 - ax_j^{\epsilon_j}}{1 - x_j^{\epsilon_j}} \frac{1 - bx_j^{\epsilon_j}}{1 + x_j^{\epsilon_j}} \frac{1 - cx_j^{\epsilon_j}}{1 - q^{1/2}x_j^{\epsilon_j}} \frac{1 - dx_j^{\epsilon_j}}{1 + q^{1/2}x_j^{\epsilon_j}} \prod_{j, j' \in J, j < j'} t^{-1} \frac{1 - tx_j^{\epsilon_j} x_{j'}^{\epsilon_{j'}}}{1 - x_j^{\epsilon_j} x_{j'}^{\epsilon_{j'}}} \frac{1 - tqx_j^{\epsilon_j} x_{j'}^{\epsilon_{j'}}}{1 - qx_j^{\epsilon_j} x_{j'}^{\epsilon_{j'}}}$$

$$\times \prod_{j \in J, k \in K} t^{-1} \frac{1 - tx_j^{\epsilon_j} x_k}{1 - x_j^{\epsilon_j} x_k} \frac{1 - tx_j^{\epsilon_j} x_k^{-1}}{1 - x_j^{\epsilon_j} x_k^{-1}}$$

$$U_{K,p}(x) := (-1)^p \sum_{\substack{L \subset K, |L|=p \\ \epsilon_l = \pm 1, l \in L}} \prod_{l \in L} a^* \frac{1 - ax_l^{\epsilon_l}}{1 - x_l^{\epsilon_l}} \frac{1 - bx_l^{\epsilon_l}}{1 + x_l^{\epsilon_l}} \frac{1 - cx_l^{\epsilon_l}}{1 - q^{1/2}x_l^{\epsilon_l}} \frac{1 - dx_l^{\epsilon_l}}{1 + q^{1/2}x_l^{\epsilon_l}}$$

$$\times \prod_{l, l' \in L, l < l'} t^{-1} \frac{1 - tx_l^{\epsilon_l} x_{l'}^{\epsilon_{l'}}}{1 - x_l^{\epsilon_l} x_{l'}^{\epsilon_{l'}}} \frac{1 - tq^{-1}x_l^{-\epsilon_l} x_{l'}^{-\epsilon_{l'}}}{1 - q^{-1}x_l^{-\epsilon_l} x_{l'}^{-\epsilon_{l'}}} \prod_{l \in L, k \in K \setminus L} t^{-1} \frac{1 - tx_l^{\epsilon_l} x_k}{1 - x_l^{\epsilon_l} x_k} \frac{1 - tx_l^{\epsilon_l} x_k^{-1}}{1 - x_l^{\epsilon_l} x_k^{-1}}$$

and

$$a_{r,s} := (-1)^{r-s} \sum_{r \leq l_1 \leq \dots \leq l_{r-s} \leq n} \prod_{i=1}^{r-s} (t^{n-l_i} a^* + t^{-n+l_i} (a^*)^{-1}),$$

where  $T_{\epsilon J, q} := \prod_{j \in J} T_{\epsilon_j j, q}$ , and

$$(T_{\pm j, q} f)(x_1, \dots, x_n) := f(x_1, \dots, x_{j-1}, q^{\pm 1} x_j, x_{j+1}, \dots, x_n).$$

For an indeterminate  $X$ , by taking the linear combination of  $\{D_r\}$ , we can define the operator  $D(X)$

$$D(X) := \sum_{i=0}^n D'_i X^{n-i},$$

where  $\{D'_i; 0 \leq i \leq n\}$  are defined inductively as follows

$$D'_0 = 1$$

$$D'_i = D_i - \sum_{j < i} a_{i,j} D'_j.$$

Then the eigenvalue  $E_\lambda(X)$  of the operator  $D(X)$  is given by

$$D(X)P_\lambda = E_\lambda(X)P_\lambda$$

$$E_\lambda(X) := \prod_{i=1}^n (X + t^{n-i} q^{\lambda_i} a^* + t^{-n+i} q^{-\lambda_i} (a^*)^{-1}).$$

We use the dominance ordering  $\lambda > \mu$  for partitions  $\lambda$  and  $\mu$ . We have the following.

LEMMA 2.1. Let  $c_{\lambda\mu}$  be

$$P_\lambda =: \widehat{m}_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} \widehat{m}_\mu.$$

If no  $\nu < \lambda$  exists such that  $E_\lambda(X) = E_\nu(X)$  at a certain specialization of parameters, then for any  $\mu < \lambda$ ,  $c_{\lambda\mu}$  has no pole at the same specialization.

*Proof.* It is clear from the defining equality of  $P_\lambda$

$$P_\lambda := \left( \prod_{\mu < \lambda} \frac{D(X) - E_\mu(X)}{E_\lambda(X) - E_\mu(X)} \right) m_\lambda. \quad \square$$

Remark 2.2. In order to distinguish eigenvalues in the specialization (15), the second-order operator  $D_1$  alone is not enough. The eigenvalue  $E_\lambda^{(1)}$  of  $D_1$  is given by

$$E_\lambda^{(1)} = \sum_{i=1}^n (t^{n-i} q^{\lambda_i} a^* + t^{-n+i} q^{-\lambda_i} (a^*)^{-1}).$$

For example, let  $n = 4$ ,  $k = 3$ , and  $r = 3$ . Then  $t^2 q = -1$ . Hence,  $E_{(3,3,2,0)}^{(1)} = E_{(4,3,3,0)}^{(1)}$ , although  $E_{(3,3,2,0)}(X) \neq E_{(4,3,3,0)}(X)$ . This is why we use the operator  $D(X)$ .

We define a dual Koornwinder–Macdonald polynomial  $P_\lambda^*$  by dual parameters  $a^*, b^*, c^*, d^*$  given in (5).

In [Sah99], we have the following relation.

PROPOSITION 2.3 (Duality). For all  $\lambda, \mu \in \pi_n$ , the Koornwinder–Macdonald polynomial  $P_\lambda$  and the dual Koornwinder–Macdonald polynomials  $P_\mu^* \in \Lambda_n$  satisfy the following duality relation:

$$\frac{u_\mu^*(P_\lambda)}{u_0^*(P_\lambda)} = \frac{u_\lambda(P_\mu^*)}{u_0(P_\mu^*)}. \quad (11)$$

Here, the definitions of  $u_\mu^*$  and  $u_\lambda$  are those in (6).

In [Van96], it is shown that the duality relation (11) implies the following evaluation formula.

PROPOSITION 2.4. We have

$$u_0^*(P_\lambda) = P_\lambda^{\text{sum}} \times P_\lambda^{\text{diff}} \times P_\lambda^{\text{single}},$$

$$P_\lambda^{\text{sum}} := \prod_{i < j} t^{-(\lambda_i + \lambda_j)/2} \frac{(t^{2n+1-i-j} (a^*)^2; q)_{\lambda_i + \lambda_j}}{(t^{2n-i-j} (a^*)^2; q)_{\lambda_i + \lambda_j}}, \quad (12)$$

$$P_\lambda^{\text{diff}} := \prod_{i < j} t^{-(\lambda_i - \lambda_j)/2} \frac{(t^{j-i+1}; q)_{\lambda_i - \lambda_j}}{(t^{j-i}; q)_{\lambda_i - \lambda_j}}, \quad (13)$$

$$P_\lambda^{\text{single}} := \prod_i a^{-\lambda_i} \frac{(t^{n-i} (a^*)^2, t^{n-i} a^* b^*, t^{n-i} a^* c^*, t^{n-i} a^* d^*; q)_{\lambda_i}}{(t^{n-i} a^*, -t^{n-i} a^*, t^{n-i} a^* q^{1/2}, -t^{n-i} a^* q^{1/2}; q)_{\lambda_i}}. \quad (14)$$

Here,  $(a; q)_l := \prod_{i=0}^{l-1} (1 - aq^i)$  and  $(a_1, a_2, \dots, a_p; q)_l := \prod_{i=1}^p (a_i; q)_l$ .

Remark 2.5. Note that in (13), there appear only factors of the form  $(1 - t^x q^y)$ ,  $x, y \in \mathbb{Z}_{\geq 0}$ . In (12), there appear only factors of the form  $(1 - t^x q^y (a^*)^2)$ ,  $x, y \in \mathbb{Z}_{\geq 0}$ . In (14), there appear only factors of the form  $(1 - t^x q^y (a^*)^2)$ ,  $(1 - t^x q^y a^* b^*)$ ,  $(1 - t^x q^y a^* c^*)$ ,  $(1 - t^x q^y a^* d^*)$ ,  $x, y \in \mathbb{Z}_{\geq 0}$ .

### 3. The spaces $I_M^{(k,r)}$ and $J_M^{(k,r)}$

In this section, we describe zero conditions and construct symmetric Laurent polynomials satisfying the zero conditions.

First, we describe a specialization of the parameters. Let  $k, r$  be integers such that  $1 \leq k \leq n - 1$  and  $r \geq 2$ . Let  $m$  be the greatest common divisor of  $(k + 1)$  and  $(r - 1)$ . Let  $\omega$  be a primitive  $m$ th root of unity. Let  $\omega_1 \in \mathbb{C}$  be such that  $\omega_1^{(r-1)/m} = \omega$ .

DEFINITION 3.1. For an indeterminate  $u$ , we consider the specialization of  $t$  and  $q$ :

$$t = u^{(r-1)/m}, \quad q = \omega_1 u^{-(k+1)/m}. \tag{15}$$

Then for integers  $x, y \in \mathbb{Z}$ ,  $t^x q^y = 1$  if and only if  $x = (k + 1)l$ ,  $y = (r - 1)l$  for some  $l \in \mathbb{Z}$ . Moreover, the multiplicity of  $(t^{(k+1)/m} q^{(r-1)/m} - \omega)$  in  $(t^{(k+1)l} q^{(r-1)l} - 1)$  is 1.

We define the subject of our study. We denote by  $\Lambda'_n$  the corresponding space  $\Lambda'_n := \bar{\Lambda}_n \otimes \mathbb{C}(u, a, b, c, d)$ .

DEFINITION 3.2. A sequence  $(s_1, \dots, s_{k+1})$  ( $s_1, \dots, s_{k+1} \in \mathbb{Z}_{\geq 0}$ ) is called a *wheel sequence* if  $s_1 + \dots + s_{k+1} = r - 1$ . For  $f \in \Lambda'_n$ , we consider the following *wheel condition*:

$$f = 0, \quad \text{if } x_{i+1} = tq^{s_i}x_i \quad (1 \leq i \leq k) \text{ for all wheel sequences } (s_1, \dots, s_{k+1}). \tag{16}$$

We consider the subspace  $J^{(k,r)} \subseteq \Lambda'_n$ ,

$$J^{(k,r)} := \{f \in \Lambda'_n; f \text{ satisfies (16)}\}. \tag{17}$$

We denote by  $\Lambda'_{n,M}$  the subspace consisting of  $f \in \Lambda'_n$  such that the degree of  $f$  in each  $x_i$  is less than  $M$ . We set  $J_M^{(k,r)} := J^{(k,r)} \cap \Lambda'_{n,M}$ .

Remark 3.3. For any partition  $\mu \in \pi_n$ ,  $u_\mu^*(x_1)/u_\mu^*(x_{k+1}) = t^k q^{\mu_1 - \mu_{k+1}}$ . Hence, the condition  $\mu_1 - \mu_{k+1} \leq r - 1$  corresponds to the existence of the wheel sequence:  $s_{k+1} = r - 1 - (\mu_1 - \mu_{k+1}) \geq 0$ . The wheel conditions for  $f(x) \in \Lambda'_n$  correspond to  $u_\mu^*(f) = 0$  at the specialization (15) for any partition  $\mu \in \pi_n$  such that  $\mu_1 - \mu_{k+1} \leq r - 1$ .

Remark 3.4. In [VS98], it is proved that  $u_\mu^*(P_\lambda) = 0$  for  $\mu_1 \leq N$  and  $\lambda_1 > N$  under the specialization of parameters  $t^{n-1} a b q^N = 1$ . As it were, this is the zero condition on the finite set  $\{(u_\mu^*(x_1), \dots, u_\mu^*(x_n)); \mu \in \pi_n, \mu_1 \leq N\}$ . On the other hand, in this paper, the wheel condition is on the infinite set, because it determines at most the ratio of variables.

For  $f(t, q, a, b, c, d) \in \mathbb{C}[t, q, a, b, c, d]$ , we use a specialization mapping  $\varphi$

$$\begin{aligned} \varphi : \mathbb{C}[t, q, a, b, c, d] &\rightarrow \mathbb{C}(u, a, b, c, d) \\ f(t, q, a, b, c, d) &\mapsto f(u^{(r-1)/m}, \omega_1 u^{-(k+1)/m}, a, b, c, d), \end{aligned}$$

and we extend  $\varphi$  to those elements of the field  $\mathbb{C}(t, q, a, b, c, d)$  for which the specialized denominator does not vanish.

LEMMA 3.5. If  $\lambda \in \pi_n$  satisfies

$$\lambda_i - \lambda_{i+k+1} > 2 \left\lfloor \frac{n}{k+1} \right\rfloor (r - 1) \quad \text{for } 1 \leq i \leq n - k - 1,$$

then  $P_\lambda$  and  $P_\lambda^*$  have no pole at the specialization (15).

Proof. First, we discuss  $P_\lambda$ . Suppose that there exists  $\mu$  such that  $\varphi(E_\mu(X)) = \varphi(E_\lambda(X))$ , that is

$$\{\varphi(t^{n-i} q^{\mu_i} a^* + t^{-n+i} q^{-\mu_i} (a^*)^{-1}); 1 \leq i \leq n\} = \{\varphi(t^{n-i} q^{\lambda_i} a^* + t^{-n+i} q^{-\lambda_i} (a^*)^{-1}); 1 \leq i \leq n\}.$$

Since  $u$  and  $a^*$  are generic, it must be satisfied that

$$\begin{aligned} & \{\varphi(t^{n-(k+1)l-i}q^{\mu_{(k+1)l+i}}); l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\} \\ & = \{\varphi(t^{n-(k+1)l-i}q^{\lambda_{(k+1)l+i}}); l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\} \end{aligned}$$

for  $1 \leq i \leq k+1$ . Hence,

$$\begin{aligned} & \{(r-1)l + \mu_{(k+1)l+i}; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\} \\ & = \{(r-1)l + \lambda_{(k+1)l+i}; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\} \end{aligned}$$

for  $1 \leq i \leq k+1$ .

Then for any  $1 \leq i \leq k+1$ , there exists  $l_i \geq 0$  such that  $(r-1)l_i + \mu_{(k+1)l_i+i} = \lambda_i$  and there exists  $l'_i \geq 0$  such that  $(r-1)l'_i + \lambda_{(k+1)l'_i+i} = \mu_i$ . If  $l'_i \neq 0$ , then by the hypothesis,

$$\begin{aligned} \mu_i - \mu_{(k+1)l_i+i} &= (r-1)l'_i + \lambda_{(k+1)l'_i+i} - \lambda_i + (r-1)l_i \\ &< (r-1)(l'_i + l_i) - 2\left\lfloor \frac{n}{k+1} \right\rfloor (r-1)l'_i \\ &\leq 2\left\lfloor \frac{n}{k+1} \right\rfloor (r-1)(1-l'_i) \\ &\leq 0. \end{aligned}$$

Hence,  $l'_i$  must be equal to 0, namely  $\lambda_i = \mu_i$ . Inductively, we have  $\lambda_{(k+1)l+i} = \mu_{(k+1)l+i}$  for all  $l \geq 0$ . It follows that  $\lambda = \mu$ . Therefore, from Lemma 2.1,  $P_\mu$  has no pole at the specialization (15).

For  $P_\lambda^*$ , its eigenvalue  $E_\lambda^*(X)$  is given by replacing  $a^*$  with  $a$  in  $E_\lambda(X)$ . Hence, we can similarly show that  $P_\lambda^*$  has no pole at the specialization (15).  $\square$

We are going to construct a basis of  $J_M^{(k,r)}$ .

DEFINITION 3.6.  $\lambda \in \pi_n$  is called  $(k, r, n)$ -admissible if

$$\lambda_i - \lambda_{i+k} \geq r \quad (\text{for all } 1 \leq i \leq n-k). \tag{18}$$

Our main result is the following.

THEOREM 3.7. For any  $(k, r, n)$ -admissible  $\lambda$ , Koornwinder–Macdonald polynomial  $P_\lambda$  has no pole at the specialization (15). Moreover, for any positive integer  $M$ , we have

$$I_M^{(k,r)} = J_M^{(k,r)}.$$

Here, we define a subspace  $I^{(k,r)}$  of  $\Lambda'_n$

$$I^{(k,r)} := \text{span}_{\mathbb{C}(u,b,c,d)}\{\varphi(P_\lambda); \lambda \text{ is } (k, r, n)\text{-admissible}\},$$

and we set

$$I_M^{(k,r)} := \text{span}_{\mathbb{C}(u,b,c,d)}\{\varphi(P_\lambda); \lambda \text{ is } (k, r, n)\text{-admissible and } \lambda_1 \leq M\}.$$

First, we prepare some propositions and lemmas.

DEFINITION 3.8. For  $p \in \mathbb{C}(t, q, a, b, c, d)$ , we denote by  $Z(p) \in \mathbb{Z}$  the multiplicity of  $(t^{(k+1)/m}q^{(r-1)/m} - \omega)$  in  $p$ . That is,

$$p = (t^{k+1}q^{r-1} - 1)^{Z(p)}p',$$

where the factor  $p' \in \mathbb{C}(t, q, a, b, c, d)$  has neither pole nor zero at (15).

PROPOSITION 3.9. For any partition  $\lambda \in \pi_n$ , we have

$$Z(u_0^*(P_\lambda)) = Z(u_0(P_\lambda^*)) = \#\{(i, l) \in \mathbb{Z}_{>0}^2; \lambda_i - \lambda_{i+(k+1)l-1} \geq (r-1)l + 1\} - \#\{(i, l) \in \mathbb{Z}_{>0}^2; \lambda_i - \lambda_{i+(k+1)l} \geq (r-1)l + 1\}.$$

*Proof.* Recall Remark 2.5. The factor  $P_\lambda^{\text{diff}}$  has the factors of the form  $(1 - t^x q^y)$  ( $x, y \in \mathbb{Z}_{\geq 0}$ ).

If  $j - i + 1 = (k + 1)l$  and  $\lambda_i - \lambda_j \geq (r - 1)l + 1$ , then  $u_0^*(P_\lambda)$  has the factor  $(1 - t^{(k+1)l} q^{(r-1)l})$  in the numerator of  $P_\lambda^{\text{diff}}$ . If  $j - i = (k + 1)l$  and  $\lambda_i - \lambda_j \geq (r - 1)l + 1$ , then  $u_0^*(P_\lambda)$  has the factor  $(1 - t^{(k+1)l} q^{(r-1)l})$  in the denominator of  $P_\lambda^{\text{diff}}$ . Otherwise, there does not exist the factor  $(1 - t^{(k+1)l} q^{(r-1)l})$  in  $P_\lambda^{\text{diff}}$ .

On the other hand,  $P_\lambda^{\text{sum}}$  and  $P_\lambda^{\text{single}}$  have neither pole nor zero at the specialization (15).

The specialization  $u_0(P_\lambda^*)$  is given by replacing parameters with dual ones in  $u_0^*(P_\lambda)$ . Since the specialization (15) is invariant under acting  $*$  on parameters, we have  $Z(u_0(P_\lambda^*)) = Z(u_0^*(P_\lambda))$ .  $\square$

COROLLARY 3.10. For any  $(k, r, n)$ -admissible  $\lambda$ , we have  $Z(u_0^*(P_\lambda)) = [n/(k + 1)]$ .

*Proof.* Since  $\lambda_i - \lambda_{i+k} \geq r$ ,

$$\begin{aligned} Z(u_0^*(P_\lambda)) &= \#\{(i, l) \in \mathbb{Z}_{>0}^2; i + (k + 1)l - 1 \leq n\} - \#\{(i, l) \in \mathbb{Z}_{>0}^2; i + (k + 1)l \leq n\} \\ &= \sum_{l \geq 1} \max\{n - (k + 1)l + 1, 1\} - \sum_{l \geq 1} \max\{n - (k + 1)l, 1\} \\ &= \left\lfloor \frac{n}{k + 1} \right\rfloor. \end{aligned} \quad \square$$

Remark 3.11. For  $g \in \Lambda_n$ , we take an integer  $N$  such that the degree of  $g$  in each variable  $x_i$  is less than  $N/2$ . Then to prove that  $g = 0$  (respectively,  $g$  has no pole) at the specialization (15), it is sufficient to show that there exist  $n$  subsets  $C_1, \dots, C_n \subseteq \mathbb{C}(a, b, c, d)[q^{\pm 1}, t^{\pm 1}]$ , which satisfy the following two conditions:

- (i) for each  $i$ ,  $\#\{\varphi(C_i)\} \geq N$  in  $\mathbb{C}(u, a, b, c, d)$ ;
- (ii) for all choices of  $c_i \in C_i$ ,  $Z(g(c_1, \dots, c_n)) > 0$  (respectively  $\geq 0$ ).

Motivated by the observation above, we define certain sets of partitions.

DEFINITION 3.12. A partition  $\eta$  is called thick if  $\eta_i \gg \eta_{i+1} \gg 0$  for all  $i$ . For a thick partition  $\eta \in \pi_n$ , a set of  $N^n$  partitions is defined by  $\pi_{\eta, N} := \{\mu \in \pi_n; \mu_i = \eta_i + d_i \text{ for all } i \text{ where } 0 \leq d_i \leq N - 1\}$ .

For a thick partition  $\eta \in \pi_{n-k}$ , we define  $\pi'_{\eta, N} := \{\mu \in \pi_n; \mu_1 - \mu_{k+1} < r, \mu_i = \eta_{i-k} + d_{i-k} \text{ for } k + 1 \leq i \leq n \text{ where } 0 \leq d_i \leq N - 1\}$ .

When we use these sets  $\pi_{\eta, N}$  and  $\pi'_{\eta, N}$ , we choose a sufficiently large  $N$  such that  $N \gg M$  and any thick partition  $\eta$  such that

$$\eta_i - \eta_{i+1} \gg \max\left(M, 2 \left\lfloor \frac{n}{k + 1} \right\rfloor (r - 1)\right), \quad \eta_i \gg \max\left(M, 2 \left\lfloor \frac{n}{k + 1} \right\rfloor (r - 1)\right).$$

We do not specify  $N$  and  $\eta$  in the following.

LEMMA 3.13. For  $\mu \in \pi_{\eta, N}$  or  $\mu \in \pi'_{\eta, N}$ ,  $P_\mu$  and  $P_\mu^*$  have no pole at the specialization (15). Moreover

$$Z(u_0^*(P_\mu)) = Z(u_0(P_\mu^*)) = \begin{cases} \left\lfloor \frac{n}{k + 1} \right\rfloor & \text{if } \mu \in \pi_{\eta, N} \ (\eta \in \pi_n), \\ \left\lfloor \frac{n}{k + 1} \right\rfloor - 1 & \text{if } \mu \in \pi'_{\eta, N} \ (\eta \in \pi_{n-k}). \end{cases}$$



*Proof.* If  $\mu$  is an element of  $\pi_{\eta,N}$  or  $\pi'_{\eta,N}$ , then  $\mu_i \gg \mu_{i+k+1}$  for  $1 \leq i \leq n - k - 1$ . Hence, from Lemma 3.5, we see that  $P_\mu$  and  $P_\mu^*$  have no pole at (15).

If  $\mu \in \pi_{\eta,N}$ , then for each  $1 \leq l \leq \lfloor n/(k+1) \rfloor$ ,  $\mu_i \gg \mu_{i+(k+1)l-1}$  ( $1 \leq i \leq n - (k+1)l + 1$ ) and  $\mu_i \gg \mu_{i+(k+1)l}$  ( $1 \leq i \leq n - (k+1)l$ ). Hence, from Proposition 3.9,  $Z(u_0^*(P_\mu)) = Z(u_0(P_\mu^*)) = \lfloor n/(k+1) \rfloor$ .

If  $\mu \in \pi'_{\eta,N}$ , then  $\mu_1 - \mu_{k+1} \leq r - 1$ . Hence, from Proposition 3.9,  $(i, l) = (1, 1)$  is the only different situation from the case  $\mu \in \pi_{\eta,N}$ . Therefore,  $Z(u_0^*(P_\mu)) = Z(u_0(P_\mu^*)) = \lfloor n/(k+1) \rfloor - 1$ .  $\square$

Now we are ready to prove a part of Theorem 3.7.

**THEOREM 3.14.** *For any  $(k, r, n)$ -admissible  $\lambda$ , Koornwinder–Macdonald polynomial  $P_\lambda$  has no pole at the specialization (15) and  $\varphi(P_\lambda)$  satisfies the wheel condition (16).*

*Proof.* Since  $\lambda$  is  $(k, r, n)$ -admissible,  $Z(u_0^*(P_\lambda)) = \lfloor n/(k+1) \rfloor$  from Corollary 3.10.

Let  $N \gg |\lambda|$  and let  $\mu \in \pi_{\eta,N}$  where  $\eta \in \pi_n$ . Then from Lemma 3.13,  $P_\mu^*$  has no pole at the specialization (15) and  $Z(u_0(P_\mu^*)) = \lfloor n/(k+1) \rfloor$ . From the duality relation (11),

$$u_\mu^*(P_\lambda) = \frac{u_\lambda(P_\mu^*)}{u_0(P_\mu^*)} u_0^*(P_\lambda).$$

Therefore,  $Z(u_\mu^*(P_\lambda)) \geq 0$ .

Since this holds for all  $\mu \in \pi_{\eta,N}$ , from Remark 3.11, we see that  $P_\lambda$  has no pole at the specialization (15).

Let  $\mu \in \pi'_{\eta,N}$  ( $\eta \in \pi_{n-k}$ ). Then from Lemma 3.13,  $P_\mu^*$  has no pole at the specialization (15) and  $Z(u_0(P_\mu^*)) = \lfloor n/(k+1) \rfloor - 1$ . From the duality relation (11), through the same argument as the above,  $Z(u_\mu^*(P_\lambda)) \geq 1$ .

We have shown  $u_\mu^*(P_\lambda) = 0$  at the specialization (15) for all  $\mu \in \pi'_{\eta,N}$ . Hence, from Remarks 3.3 and 3.11, we conclude that  $\varphi(P_\lambda)$  satisfies the wheel condition (16).  $\square$

**COROLLARY 3.15.** *The space  $I^{(k,r)}$  and  $I_M^{(k,r)}$  are well defined for any positive integer  $M$ , and we have  $J_M^{(k,r)} \supseteq I_M^{(k,r)}$ .*

#### 4. Estimate of $\dim J_M^{(k,r)}$

We have already constructed the polynomials satisfying the zero conditions. In this section, we show that  $J_M^{(k,r)} = I_M^{(k,r)}$  by giving an upper estimate of the dimension of  $J_M^{(k,r)}$ .

Fix  $g'_0, g'_1, g'_2, g'_3 \gg 1$ . We take the limit  $t \rightarrow 1, q \rightarrow \tau, a \rightarrow \tau g'_0, b \rightarrow -\tau g'_1, c \rightarrow \tau g'_2 + 1/2, d \rightarrow -\tau g'_3 + 1/2$ , where  $\tau$  is a primitive  $(r - 1)$ th root of unity. In this limit the wheel condition (16) reduces to

$$f = 0 \quad \text{if } x_i = \tau^{p_i} x_0 \quad (1 \leq i \leq k + 1) \tag{19}$$

for all  $p_1, \dots, p_{k+1} \in \mathbb{Z}$  and  $x_0 \in \mathbb{C}$ . We denote by  $\bar{J}^{(k,r)} \subseteq \bar{\Lambda}_n$  the space of  $(BC)_n$ -symmetric polynomials satisfying (19). We define

$$\bar{J}_M^{(k,r)} = \{f \in \bar{J}^{(k,r)}; \deg_{x_1} f \leq M\}.$$

Note that  $\dim_{\mathbb{C}(u,a,b,c,d)} J_M^{(k,r)} \leq \dim_{\mathbb{C}} \bar{J}_M^{(k,r)}$ .

We consider the commutative ring  $R_M := \mathbb{C}[e_0, e_1, e_2, \dots, e_M]$  for indeterminates  $\{e_i\}$ . We count the weight of  $e_i$  as 1 and the degree of  $e_i$  as  $i$ . We set  $e_\lambda := \prod_{i=1}^n e_{\lambda_i}$  for  $\lambda \in \pi_n$ . We denote by  $R_{M,n} \subseteq R_M$  the space spanned by the monomials  $e_\lambda$  such that  $\lambda \in \pi_n$  and  $\lambda_1 \leq M$ .

We use the dual language (see [FJMM03]). There is a non-degenerate coupling:

$$\begin{aligned} R_{M,n} \times \bar{\Lambda}_{n,M} &\rightarrow \mathbb{C}; \\ \langle e_\lambda, \widehat{m}_\mu \rangle &= \delta_{\lambda,\mu}. \end{aligned} \tag{20}$$

We introduce an abelian current

$$e(z) := \sum_{i=1}^M e_i(z^i + z^{-i}) + e_0.$$

It satisfies

$$\langle e(z_1)e(z_2)\cdots e(z_n), f \rangle = f(z_1, z_2, \dots, z_n) \quad \text{for } f \in \bar{\Lambda}_{n,M}.$$

Then for any  $f \in \bar{J}_M^{(k,r)}$ , we have

$$\langle e(\tau^{p_1} z) \cdots e(\tau^{p_{k+1}} z) e(z_{k+2}) \cdots e(z_n), f \rangle = 0 \quad \text{for all } (p_1, \dots, p_{k+1}) \in \mathbb{Z}^{k+1}.$$

Hence, the space

$$\text{span}_{\mathbb{C}}\{e(\tau^{p_1} z) \cdots e(\tau^{p_{k+1}} z) e(z_{k+2}) \cdots e(z_n); z, z_{k+2}, \dots, z_n \in \mathbb{C}, p_1, \dots, p_{k+1} \in \mathbb{Z}\} \tag{21}$$

is the orthogonal complement of  $\bar{J}_M^{(k,r)}$  with respect to the coupling  $\langle \cdot, \cdot \rangle$ . For  $p = (p_1, \dots, p_{k+1}) \in \mathbb{Z}^{k+1}$ , let  $r_d^p$  be the coefficient of  $z^d$  in

$$e(\tau^{p_1} z) \cdots e(\tau^{p_{k+1}} z) = \sum_d r_d^p z^d.$$

By the symmetry of exchanging  $z \leftrightarrow z^{-1}$  in the current  $e(z)$ , we have  $r_d^p = r_{-d}^p$ . We denote by  $\mathcal{J}_M$  the ideal of  $R_M$  generated by the elements  $r_d^p$ . Set  $\mathcal{J}_{M,n} := \mathcal{J}_M \cap R_{M,n}$ . Then the space (21) coincides with  $\mathcal{J}_{M,n}$ . Since  $\dim R_{M,n}/\mathcal{J}_{M,n} = \dim \bar{J}_M^{(k,r)}$ , the condition (19) is equivalent to the relations in the quotient space

$$r_d^p = 0 \quad \text{for all } p = (p_1, \dots, p_{k+1}) \in \mathbb{Z}^{k+1} \text{ and } d \geq 0.$$

PROPOSITION 4.1. *The image of the set  $\{e_\lambda; \lambda \in \pi_n \text{ is } (k, r, n)\text{-admissible, } \lambda_1 \leq M\}$  spans the quotient space  $R_{M,n}/\mathcal{J}_{M,n}$ .*

*Proof.* We introduce a total ordering for partitions and monomials. For two partitions  $\lambda$  and  $\mu$  such that  $|\lambda| > |\mu|$ , we define  $\lambda \succ \mu$ . For two partitions  $\lambda$  and  $\mu$  such that  $|\lambda| = |\mu|$ , we define  $\lambda \succ \mu$  if  $\lambda_1 > \mu_1$  or  $\lambda_1 = \mu_1, \lambda_2 > \mu_2$  or  $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \lambda_3 > \mu_3$ , etc. We define for the corresponding monomials  $e_\lambda$  and  $e_\mu$ ,  $e_\lambda \succ e_\mu$ .

Let us calculate  $r_d^p$ .

$$\begin{aligned} e(\tau^{p_1} z) \cdots e(\tau^{p_{k+1}} z) &= \prod_{j=1}^{k+1} \sum_{i_j=-M}^M e_{|i_j|}(\tau^{p_j} z)^{i_j} \\ &= \sum_{d \in \mathbb{Z}} z^d \left( \sum_{\substack{i_1 + \dots + i_{k+1} = |d| \\ i_j \geq 0}} \prod_{j=1}^{k+1} e_{i_j} \tau^{p_j i_j} + \sum_{\substack{\lambda \in \pi_{k+1} \\ |\lambda| > |d|}} c_{\lambda, |d|} e_\lambda \right). \end{aligned}$$

Hence, for any non-negative integer  $d$ ,

$$r_d^p = \sum_{\substack{i_1 + \dots + i_{k+1} = d \\ i_j \geq 0}} \prod_{j=1}^{k+1} e_{i_j} \tau^{p_j i_j} + \sum_{\substack{\lambda \in \pi_{k+1} \\ |\lambda| > d}} c_{\lambda, d} e_\lambda.$$

We define  $R_{M,k+1}^{(d)}$  by

$$R_{M,k+1}^{(d)} := \bigoplus_{\lambda \in \pi_{k+1}, |\lambda| \geq d, \lambda_1 \leq M} \mathbb{C}e_\lambda,$$

and we consider a quotient space

$$R_{M,k+1}^{(d)} / \left( R_{M,k+1}^{(d+1)} + \sum_{p \in \mathbb{Z}^{k+1}} \mathbb{C}r_d^p \right).$$

In this space,

$$\begin{aligned} 0 = r_d^p &= \sum_{\substack{i_1 + \dots + i_{k+1} = d \\ i_j \geq 0}} \prod_{j=1}^{k+1} e_{i_j} \tau^{p_j i_j} \\ &= \sum_{\substack{\nu \in \mathbb{Z}_{\geq 0}^{k+1} \\ \nu_j \leq r-2}} \tau^{p_1 \nu_1 + \dots + p_{k+1} \nu_{k+1}} \left( \sum_{\substack{\mu \in \mathbb{Z}_{\geq 0}^{k+1}, \sum_j \mu_j = d \\ \mu_j = \nu_j + (r-1)\kappa_j, \kappa_j \in \mathbb{Z}_{\geq 0}}} \prod_j e_{\mu_j} \right) \\ &= \sum_{\substack{\lambda \in \pi_{k+1} \\ \lambda_1 \leq r-2}} \left( \sum_{\nu \in \mathfrak{S}_{k+1} \lambda} \tau^{p_1 \nu_1 + \dots + p_{k+1} \nu_{k+1}} \right) \left( \sum_{\substack{\mu \in \mathbb{Z}_{\geq 0}^{k+1}, \sum_j \mu_j = d \\ \mu_j = \lambda_j + (r-1)\kappa_j, \kappa_j \in \mathbb{Z}_{\geq 0}}} \prod_j e_{\mu_j} \right). \end{aligned}$$

We set  $\pi_{k+1,d} := \{ \lambda \in \pi_{k+1}; |\lambda| = d \}$ . For a sequence of non-negative integers  $m := (m_0, \dots, m_{r-2})$  such that  $\sum m_i = k + 1$ , we define a subset  $\pi_{k+1,d}(m)$  by

$$\pi_{k+1,d}(m) := \{ \mu \in \pi_{k+1,d}; \#\{i; \mu_i \equiv a \pmod{r-1}\} = m_a \text{ for every } 0 \leq a \leq r-2 \}.$$

We denote by  $m_i^{(\lambda)}$  the multiplicity of  $i$  in  $\lambda$ . Define  $m(\lambda) := (m_0^{(\lambda)}, \dots, m_{r-2}^{(\lambda)})$ . Then,

$$\begin{aligned} r_d^p &= \sum_{\substack{\lambda \in \pi_{k+1} \\ \lambda_1 \leq r-2}} \left( \sum_{\mu \in \pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) \left( \sum_{\nu \in \mathfrak{S}_{k+1} \lambda} \tau^{p_1 \nu_1 + \dots + p_{k+1} \nu_{k+1}} \right) \\ &= \sum_{\substack{\lambda \in \pi_{k+1} \\ \lambda_1 \leq r-2}} \left( \sum_{\mu \in \pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) m_\lambda(\tau^{p_1}, \dots, \tau^{p_{k+1}}). \end{aligned}$$

Here,  $c_{\lambda,\mu} = \prod_i m_i^{(\lambda)}! / \prod_i m_i^{(\mu)}!$  and  $m_\lambda$  is the monomial  $\mathfrak{S}_{k+1}$ -symmetric polynomial (not Laurent).

Since  $\lambda_1 \leq r - 2$ , the degree of

$$\sum_{\substack{\lambda \in \pi_{k+1} \\ \lambda_1 \leq r-2}} \left( \sum_{\mu \in \pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) m_\lambda(x_1, \dots, x_{k+1}) \tag{22}$$

in each variable  $x_i$  is less than  $r - 2$ . On the other hand, we can choose the values of  $x_i$  from  $\tau^0, \tau^1, \dots, \tau^{r-2}$  independently. Hence, the expression (22) is identically zero in the quotient space  $R_{M,k+1}^{(d)} / (R_{M,k+1}^{(d+1)} + \sum_p \mathbb{C}r_d^p)$ . Since monomial symmetric polynomials are linearly independent, it follows that

$$\sum_{\mu \in \pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu = 0$$

in  $R_{M,k+1}^{(d)} / (R_{M,k+1}^{(d+1)} + \sum_p \mathbb{C}r_d^p)$ . Note that  $\mathcal{J}_{M,k+1} = \sum_{d=0}^{M^{k+1}} \sum_p \mathbb{C}r_d^p$ . Therefore, in  $R_{M,k+1} / \mathcal{J}_{M,k+1}$ , we have

$$\sum_{\mu \in \pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu = \sum_{\mu \in \pi_{k+1}, |\mu| \geq d+1, \mu_1 \leq M} c_\mu e_\mu.$$

For any  $(k, r, k + 1)$ -non-admissible partition  $\lambda \in \pi_{k+1}$  such that  $\lambda_1 \leq M$ , there exist some  $d$  and  $m$  so that  $\lambda \in \pi_{k+1,d}(m)$ . Moreover, the set  $\pi_{k+1,d}(m)$  contains at most one  $(k, r, k + 1)$ -non-admissible partition  $\lambda$ , and for all  $\mu \in \pi_{k+1,d}(m)$  such that  $\mu \neq \lambda$ , we have  $\mu \succ \lambda$ . Therefore,  $e_\lambda$  can be written in  $R_{M,k+1}/\mathcal{J}_{M,k+1}$  as follows:

$$e_\lambda = \sum_{\mu \succ \lambda, \mu_1 \leq M} c'_\mu e_\mu.$$

Let  $\lambda \in \pi_n$  be a  $(k, r, n)$ -non-admissible partition such that  $\lambda_1 \leq M$ . Then there exists  $i$  such that  $\lambda_i - \lambda_{i+k} < r$ . We set  $\mu := (\lambda_i, \dots, \lambda_{i+k}) \in \pi_{k+1}$  and  $\nu := (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+k+1}, \dots, \lambda_n)$ . Since  $\mu$  is  $(k, r, k + 1)$ -non-admissible, from the above argument, we can rewrite  $e_\mu$  as a linear combination of greater monomials  $\{e_{\mu'}; \mu' \succ \mu\}$  in  $R_{M,k+1}/\mathcal{J}_{M,k+1}$ . Hence,  $e_\lambda$  can be rewritten in  $R_{M,n}/\mathcal{J}_{M,n}$  as follows:

$$\begin{aligned} e_\lambda &= e_\mu e_\nu \\ &= \left( \sum_{\mu' \succ \mu, \mu'_1 \leq M} c_{\mu'} e_{\mu'} \right) e_\nu \\ &= \sum_{\lambda' \succ \lambda, \lambda'_1 \leq M} c_{\lambda'} e_{\lambda'}. \end{aligned}$$

Here, in the last equality, we set  $\lambda' := \mu' \cup \nu$ .

If  $e_{\lambda'}$  is still  $(k, r, n)$ -non-admissible for some  $\lambda'$ , we further rewrite  $e_{\lambda'}$  as a linear combination of greater monomials. Since  $\{\lambda \in \pi_n; \lambda_1 \leq M\}$  is a finite set, this procedure stops in finite times.  $\square$

COROLLARY 4.2. We have  $\dim J_M^{(k,r)} \leq \#\{\lambda \in \pi_n; \lambda \text{ is } (k, r, n)\text{-admissible and } \lambda_1 \leq M\}$ .

By Corollaries 3.15 and 4.2, we complete the proof of Theorem 3.7.

### 5. Application to Macdonald symmetric polynomials

We can apply the method in Section 3 to a proof of Theorem 1.1.

In [Mac95], symmetry relations [Mac95, ch. VI, (6.6)] and special values [Mac95, ch. VI, (6.11')] of Macdonald symmetric polynomials have been given. By a combinatorial argument similar to that employed in this paper, we see that for any  $(k, r, n)$ -admissible partition  $\lambda$ , the multiplicity of the factor  $(1 - t^{k+1}q^{r-1})$  in the right-hand side of (6.11') is  $[n/(k + 1)]$ . Moreover, for  $\mu \in \pi_{\eta,N}$  or  $\pi'_{\eta,N}$ , the same results as Lemma 3.13 also follow. Hence, from symmetry relations, through the same argument as Theorem 3.14, we conclude that the Macdonald symmetric polynomial is well defined and satisfies the wheel conditions if  $\lambda$  is  $(k, r, n)$ -admissible.

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Masahiro Kasatani [kasatani@math.kyoto-u.ac.jp](mailto:kasatani@math.kyoto-u.ac.jp)

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan