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NON-ORTHOGONALISABLE VECTOR FIELDS ON SPHERES

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A $(k-1)$-field on S^{n-1} may be given as a section φ of the fibre bundle

$$p_{n,k}: V_{n,k} \downarrow S^{n-1}$$

with fibre $V_{n-1,k-1}$, or, equivalently, as a *semi-orthogonal* map, i.e., a map

$$\varphi': \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

which is isometric in the second variable and such that for the basis vector $e_1 \in \mathbb{R}^k$ and every $x \in \mathbb{R}^n$

$$\varphi'(x, e_1) = x.$$

A section φ of $p_{n,k}$ induces a semi-orthogonal map by adjunction, i.e.,

$$\varphi'(x,v) = \begin{cases} 0 & x = 0 \\ \varphi(\|x\|^{-1} \cdot x)(\|x\| \cdot v) & x \neq 0. \end{cases}$$

A semi-orthogonal map μ determines a section

$$ad(\mu): S^{n-1} \longrightarrow V_{n,k}$$

of $p_{n,k}$ by adjunction and restriction [**12**].

A $(k-1)$-field φ on S^{n-1} is called *skew* iff

$$\varphi'(-x, v) = -\varphi'(x, v).$$

φ is called *orthogonal* iff φ' is an orthogonal multiplication, i.e., iff φ' is isometric in both variables. Obviously, orthogonal r-fields are skew. An r-field φ is called *orthogonalisable* iff it is regularly homotopic to an orthogonal one, and *non*-orthogonalisable otherwise.

Skew r-fields on spheres were studied by Milgram and Zvengrowski. They show that every r-field on a sphere is regularly homotopic to a skew r-field [**9**]. They also remark that, in general, not every vector field on a sphere is orthogonalisable. We give some precise conditions for the existence of such non-orthogonalisable vector fields.

Some more notational conventions: For a fibre bundle

$$p:E\downarrow B$$

the space of sections of p will be denoted by $\Gamma(p)$. Let

$$o_{n-1}, \iota_{n-1} \in \pi_{n-1}(S^{n-1})$$

denote the homotopy class of the trivial map, resp. the indentity, and

$$(p_{n,k*})^{-1}(o_{n-1}), \text{resp.} (p_{n,k*})^{-1}(\iota_{n-1}) \subseteq \pi_{n-1}(V_{n,k})$$

their preimages in the homotopy exact sequence associated to $p_{n,k}$. Notice that inclusion induces the maps

$$\sigma_0: \pi_{n-1}(V_{n-1,k-1}) \overset{\cong}{\to} (p_{n,k*})^{-1}(o_{n-1}) \subseteq \pi_{n-1}(V_{n,k})$$

$$\sigma_1: \pi_0(\Gamma(p_{n,k})) \to (p_{n,k*})^{-1}(\iota_{n-1}) \subseteq \pi_{n-1}(V_{n,k}).$$

Lemma 1. *If $\Gamma(p_{n,k})$ is non-empty, σ_1 is a bijection. As a consequence, there is a bijection*

$$\alpha: \pi_{n-1}(V_{n-1,k-1}) \to \pi_0(\Gamma(p_{n,k}))$$

such that

$$\sigma_1(\alpha(x)) = \sigma_0(x) + \sigma_1(\alpha(0)).$$

Proof. According to [7], Lemma 1.1, the map σ_0 is injective. As $p_{n,k}$ is a fibration, every homotopy into the base space S^{n-1} ending with the identity map can be lifted to a homotopy into the total space ending with a section.

According to Eckmann [4], every orthogonal multiplication

$$\mu: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$$

arises from a representation of the Clifford group C_{k-1}. If $k \not\equiv 0(4)$, there is a well-determined a_k-dimensional irreducible C_{k-1}-representation M_k; if $k \equiv 0(4)$, there are exactly two non-equivalent irreducible representations M_k^0, M_k^1. Here a_k is given by

$$a_1 = 1, a_2 = 2, a_3 = a_4 = 4, a_5 = a_6 = a_7 = a_8 = 8, \quad \text{and} \quad a_{k+8} = 16a_k.$$

Thus, for n, k as above, $n = i \cdot a_k$, $i > 0$, every n-dimensional C_{k-1}-representation is equivalent to

(a) $i \cdot M_k$, $k \not\equiv 0(4)$
(b) $a \cdot M_k^0 \oplus b \cdot M_k^1$, $a + b = i$, $k \equiv 0(4)$.

The corresponding orthogonal multiplications, which are well-determined up to $0(n)$-conjugation, will be denoted by $\mu(i)$, resp. $\mu(a,b)$. We are interested in the adjoints of representatives $\mu(i)$, $\mu(a,b)$:

$$ad(\mu(i)), ad(\mu(a,b)): S^{n-1} \to V_{n,k}.$$

Obviously, conjugation by a rotation does not alter the homotopy class of such a vector field, whereas a non-rotation might do so [6]. A simple counting argument shows:

Proposition 1. *If a_k divides n, and*

(a) $k \not\equiv 0(4)$ *and* $|\pi_{n-1}(V_{n-1,k-1})| > 2$ *or*
(b) $k \equiv 0(4)$ *and* $|\pi_{n-1}(V_{n-1,k-1})| > 2((n/a_k)+1)$,

then there is a non-orthogonalisable $(k-1)$-field on S^{n-1}.

In general, it is difficult to determine these homotopy groups. Therefore, we study the boundary homomorphism

$$\partial: \pi_{m-1}(V_{s,r}) \to \pi_{m-2}(S^{s-r-1})$$

corresponding to the fibre bundle

$$S^{s-r-1} \to V_{s,r+1} \downarrow V_{s,r}$$

in the cases $(s,r)=(n-1,k-1)$, resp. (n,k).

∂ will be applied to the adjoints

$$\mathrm{ad}(\mu): S^{n-1} \to V_{n,k}$$

of semi-orthogonal maps

$$\mu: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n.$$

Note that for an orthogonal map μ, there is also an adjunction

$$\mathrm{ad}'(\mu): S^{k-1} \to 0(n).$$

We also make use of the generalised J-homomorphisms [12]

$$J_r^s: \pi_m(V_{s,r}) \to \pi_{m+r}(S^s).$$

The following propositions are the tools in our argument:

Proposition 2. [12] *If $n \equiv k(2)$ and $n > 2k+1$, then*

$$2 \cdot \pi_{k-1}^s \subseteq \sum^\infty \partial(\pi_{n-1}(V_{n-1,k-1})) \subset \pi_{k-1}^s.$$

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Proposition 3. [6][11]. *For $r<s$, the diagram*

$$\begin{array}{ccc} \pi_m(V_{s,r}) & \xrightarrow{\ \partial\ } & \pi_{m-1}(S^{s-r-1}) \\ {\scriptstyle J_r^s}\searrow & & \swarrow{\scriptstyle \Sigma^{r+1}} \\ & \pi_{m+r}(S^s) & \end{array}$$

commutes.

Lemma 2. *For an orthogonal map*

$$\mu:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n,$$

the following relation holds:

$$J_k^n(\mathrm{ad}\,(\mu))=(-1)^{kn}J(\mathrm{ad}'(\mu))\in\pi_{n+k-1}(S^n).$$

Our main result is:

Proposition 4. *If a_k divides n, $2k+1<n$, and*

(a) $k\equiv 2(4)$ *and* $2\cdot\mathrm{coker}\,J_{k-1}\neq 0$ *or*

(b) $k\equiv 0(4)$ *and* $2\cdot\mathrm{coker}\,J_{k-1}\neq 0$ *or*

$$2\cdot\left(\frac{n}{a_k}+1\right)<|\mathrm{im}\,J_{k-1}|,$$

then there exists a non-orthogonalisable $(k-1)$-field on S^{n-1}.

Proof. Obviously, the diagram

$$\begin{array}{ccc} \pi_{n-1}(V_{n-1,k-1}) & & \\ \downarrow{\scriptstyle \sigma_0} & \searrow^{\partial} & \\ & & \pi_{n-2}(S^{n-k-1}) \\ & \nearrow_{\partial} & \\ \pi_{n-1}(V_{n,k}) & & \end{array}$$

commutes. Using Lemma 1 and Proposition 2, it follows that for any $\varphi_0\in\pi_0(\Gamma(p_{n,k}))$, serving as a base point,

$$\sum^\infty\partial(\sigma_1(\pi_0(\Gamma(p_{n,k}))) \supseteq \sum^\infty\partial\sigma_1(\varphi_0)+2\cdot\pi_{k-1}^s.$$

According to Lemma 2, we get for every orthogonal map

$$\mu:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$$

that

$$\partial(\mathrm{ad}\,(\mu)) = J_k^n(\mathrm{ad}\,(\mu)) = (-1)^{kn} J(\mathrm{ad}'\,(\mu)),$$

thus

$$\textstyle\sum^\infty \partial(\mathrm{ad}\,(\mu)) \in \mathrm{im}\,(J_{k-1}).$$

Thus, the proposition is already proved if $2\cdot\mathrm{coker}\,J_{k-1} \neq \emptyset$. In the last case, we use the calculation of $K0$-groups via Clifford forms in [2]. The authors show that, for $k < n$,

$$\mathrm{ad}'\,(\mu(a,b)) = (a-b)\cdot g \in \pi_{k-1}(0(n)) \cong \pi_{k-1}(0),$$

where $g \in \pi_{k-1}(0) \cong \mathbb{Z}$ is a generating element. According to Proposition 2,

$$\textstyle\sum^\infty \partial(\sigma_1(\pi_0(\Gamma(p_{n,k})))) \supseteq \frac{n}{a_k}\cdot J(g) + 2\cdot\mathrm{im}\,J_{k-1}.$$

On the other hand

$$\left\{\textstyle\sum^\infty \partial(\mathrm{ad}\,(\mu(a,b)) \,\middle|\, a+b = \frac{n}{a_k}\right\} = \left\{J(\mathrm{ad}'\,(\mu(a,b))) \,\middle|\, a+b = \frac{n}{a_k}\right\}$$

$$= \left\{i\cdot J(g) \,\middle|\, -\frac{n}{a_k} \leqq i \leqq \frac{n}{a_k};\, i \equiv \frac{n}{a_k} \bmod 2\right\}.$$

Thus, under the second condition in Proposition 4(b), ∂ applied to orthogonal forms cannot fill out the image of ∂ applied to semi-orthogonal forms.

Corollary. (a) *For* $m \leqq 46$, *there is a non-orthogonalisable* 3-*field on* S^{4m-1}.
(b) *For* q *odd, there exist non-orthogonalisable* 23-*fields on* $S^{q\cdot 2^{11}-1}$.

Proof. (a) $\pi_3(V_{3,3}) = \pi_3(0(3)) = \mathbb{Z}$; from Paechter's tables [10], we find that, $\pi_{4l-1}(V_{4l-1,3}) = \mathbb{Z}/24 \oplus \mathbb{Z}/4$, $l \geqq 2$. Apply Proposition 1.

(b) An inspection of Toda's tables [13], [14] and [1] shows that $\mathrm{coker}\,J_{23} \cong \mathbb{Z}/3$. Apply Proposition 4.

The same methods are useful to compare spaces of skew-linear, non-singular bilinear and orthogonal maps (see [8], [3]). Two skew-linear maps φ_0, φ_1 are called homotopic, if there is a continuous path in the space of all skew-linear maps from φ_0 to φ_1; etc.

Proposition 5. (a) *For* k, n *as in Proposition* 1 *or* 4 *there exists a skew-linear form*

$$\varphi: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

which is not homotopic to an orthogonal form.

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(b) If $2\cdot\operatorname{coker} J_{k-1} \neq 0$ *and* a_k *divides n, there exists* $N \geqq 0$ *and a non-singular bilinear form*

$$\varphi : \mathbb{R}^n \times \mathbb{R}^{k+N} \to \mathbb{R}^{n+N}$$

which is not homotopic to an orthogonal form.

Remark. Thus, in these cases, even the sets of components of the spaces of skew-linear, bilinear and orthogonal forms are different.

Proof. For a semi-orthogonal map

$$\varphi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n,$$

the Hopf-construction is given by

$$H(\varphi) = J_k^n(\operatorname{ad}(\varphi)) = \textstyle\sum^\infty \partial(\operatorname{ad}(\varphi)) \in \pi_{k-1}^s.$$

Thus, the Hopf-construction applied to all semi-orthogonal forms yields just the set $\sum^\infty \partial(\pi_{n-1}(V_{n,k}))$, whereas applying it to only orthogonal forms gives us just the subset described in the proof of Proposition 4.

(a) is now proved by representing every regular homotopy class of $(k-1)$-fields on S^{n-1} by a skew $(k-1)$-field φ [**9**] and regarding the skew-linear form φ'. In (b) we use Lam's trick [**8**] to find a non-singular bilinear form

$$\varphi'_N : \mathbb{R}^n \times \mathbb{R}^{k+N} \to \mathbb{R}^{n+N}$$

by Clifford suspensions applied to φ. This process does not alter the Hopf-construction. On the other hand, a result of H. Hefter [**5**] shows, that

$$H(\psi) \in \operatorname{im} J_{a+b-c-1}$$

for every orthogonal form

$$\psi : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c.$$

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