BULL. AUSTRAL. MATH. SOC. 18-02, 18A15, 18C10, 18E20
VOL. 7 (1972), 1-76.

Aspects of topoi

Peter Freyd

After a review of the work of Lawvere and Tierney, it is shown that every topos may be exactly embedded in a product of topoi each with 1 as a generator, and near-exactly embedded in a power of the category of sets. Several metatheorems are then derived. Natural numbers objects are shown to be characterized by exactness properties, which yield the fact that some topoi can not be exactly embedded in powers of the category of sets, indeed that the "arithmetic" arising from a topos dominates the exactness theory. Finally, several, necessarily non-elementary, conditions are shown to imply exact embedding in powers of the category of sets.

The development of elementary topoi by Lawvere and Tierney strikes this writer as the most important event in the history of categorical algebra since its creation. The theory of abelian categories served as the "right" generalization for the category of abelian groups. So topoi serve for - no less - the category of sets. For each the motivating examples were categories of sheaves, abelian-valued sheaves for the first, set-valued sheaves for the second. But topoi are far richer than abelian categories (surely foreshadowed by the fact that abelian-valued sheaves are just the abelian-group objects in the category of set-valued sheaves). Whereas abelian categories, nice as they are, appear in various contexts only with the best of luck, topoi appear at the very foundation of mathematics. The theory of topoi provides a method to "algebraicize" much

Received 29 December 1971. Prepared from a series of lectures delivered at the Conference on Categorical Algebra, University of New South Wales, 13-17 December 1971. The author was in Australia as a Fulbright Fellow.

of mathematics.

In this work, we explore the exact embedding theory of topoi. Again the subject is richer than that for abelian categories. It is not the case that every small topos can be exactly embedded in a power of the category of sets or even a product of ultra-powers (Theorem 5.23). It is the case that every topos can be exactly embedded in a product of well-pointed topoi (Theorem 3.23) ("well-pointed" means "1 is a generator"), and near-exactly embedded in a power of the category of sets (Theorem 3.24) ("near-exact" means "preserves finite limits, epimorphisms, and coproducts"). We thus gain several metatheorems concerning the exactness theory of topoi (Metatheorem 3.31).

The obstructions to the existence of exact functors lie in the "arithmetic" of topoi (Proposition 5.33, Theorem 5.52). No set of elementary conditions can imply exact embeddability into a power of the category of sets (Corollary 5.15), but rather simple, albeit non-elementary, conditions do allow such (§5.6). The easiest to state: a countably complete topos may be exactly embedded in a power of the category of sets.

The most impressive use of the metatheorems is that certain exactness conditions imply that something is a natural numbers object (Theorem 5.43). A consequence is that a topos has a natural numbers object iff it has an object A such that $1 + A \simeq A$ (Theorem 5.44).

We begin with a review of the work of Lawvere and Tierney (through Corollary 2.63). All the definitions and theorems are theirs, though some of the proofs are new. Sections 4.1 and 5.1 and Propositions 5.21, 5.22 are surely also theirs. It is easy to underestimate their work: it is not just that they proved these things, it's that they dared believe them provable.

1. Cartesian closed categories

A *cartesian closed category* is a finitely bicomplete category such that for every pair of objects A, B the set-valued functor $(-\times A, B)$ is representable. This is the non-elementary (in the technical sense of "elementary") definition. We shall throughout this work tend to give first

a definition in terms of the representability of some functor and then show how such can be replaced with an elementary definition - usually, indeed, with an essentially algebraic definition.

The translation from representability to elementary is the Mac Lane-notion of "universal element". For cartesian-closedness (forgive, oh Muse, but "closure" is just not right) we obtain the following:

For every A, B *there is an object* B^A *and a map* $B^A \times A \to B$ *(the "evaluation map") such that for any* $X \times A \to B$ *there exists a unique* $f : X \to B^A$ *such that*

$$\begin{tikzcd} & X\times A \arrow[dl, "f\times 1"'] \arrow[dr] & \\ B^A\times A \arrow[rr] & & B \end{tikzcd} .$$

Holding A fixed, B^A becomes a functor on B: given $f : B \to C$, $f^A : B^A \to C^A$ is the unique map such that

$$B^A \times A \xrightarrow{f^A\times 1} C^A \times A \longrightarrow C = B^A \times A \longrightarrow B \xrightarrow{f} C .$$

Holding B fixed, B^A becomes a contravariant functor on A: given $g : A' \to A$, $B^g : B^A \to B^{A'}$ is the unique map such that

$$B^A \times A' \xrightarrow{B^g\times 1} B^{A'} \times A' \longrightarrow B = B^A \times A' \xrightarrow{1\times g} B^A \times A \longrightarrow B .$$

One can easily verify that

$$B^A \xrightarrow{f^A} C^A \xrightarrow{C^g} C^{A'} = B^A \xrightarrow{B^g} B^{A'} \xrightarrow{f^{A'}} C^{A'} ,$$

that is, B^A is a bifunctor.

Recall that a group may be defined either as a semigroup satisfying a couple of elementary conditions or as a model of a purely algebraic theory (usually with three operators: multiplication, unit, and inversion). It is important that groups may be defined either way: .there are times when groups are best viewed as special kinds of semigroups, and there are times when they thrive as models of an algebraic theory. So it is with lattices,

and every elaboration of the notion of lattice; and so it is with cartesian-closed categories.

To begin with, a category may be viewed as a model of a two-sorted partial algebraic theory. The two sorts are "maps" and "objects". We are given two unary operators from maps to objects, "domain" and "codomain", and one coming back, "identity map"; and we are given a partial binary operator from maps to maps, *the domain of which is given by an equation in the previous operators.* Partial algebraic theories for which such is the case, namely those such that the partial operators may be ordered and the domain of each is given by equations on the previous, are better than just partial algebraic theories. We shall call them *essentially algebraic.* A critical feature of essentially algebraic theories is that their models are closed, in the nicest way, under direct limits.

Finite bicompleteness becomes algebraic. The terminal object (better called the "terminator") is a constant, 1, together with a uninary operation, t, from objects to maps, such that

$$\text{domain}(t_A) = A ,$$

$$\text{codomain}(t_A) = 1 ,$$

$$t_1 = 1_1 ,$$

$$A \xrightarrow{f} B \xrightarrow{t} 1 = t_A .$$

The equations that stipulate domain and codomain are conventionally absorbed in the notation, thus: $t(A) = A \to 1$.

For binary products we have four binary operations, one from objects to objects denoted $A_1 \times A_2$, two from objects to maps denoted

$$A_1 \times A_2 \xrightarrow{p_1} A_1 , \quad A_1 \times A_2 \xrightarrow{p_2} A_2 ,$$

one from maps to maps denoted

$$A_1 \times A_2 \xrightarrow{f_1 \times f_2} B_1 \times B_2 ;$$

and one unary operation from objects to maps: $A \xrightarrow{\Delta} A \times A$. The

equations:

$$A_1 \times A_2 \xrightarrow{f_1 \times f_2} B_1 \times B_2 \xrightarrow{p_i} B_i = A_1 \times A_2 \xrightarrow{p_i} A_i \xrightarrow{f_i} B_i \ , \quad i = 1, 2 \ ,$$

$$A \xrightarrow{\Delta} A \times A \xrightarrow{p_i} A = 1_A \ , \quad i = 1, 2 \ ,$$

$$A \xrightarrow{g} B_1 \times B_2 = A \xrightarrow{\Delta} A \times A \xrightarrow{(gp_1) \times (gp_2)} B_1 \times B_2 \ .$$

For equalizers we have partial binary operators from maps, one to objects, one to maps. The domain of each operator is given by the equations

$$\text{domain}(f) = \text{domain}(g) \ , \quad \text{codomain}(f) = \text{codomain}(g) \ .$$

We may denote the pairs f, g in the domain by $A \xrightarrow{f,g} B$. The object-valued operator is denoted $E(f, g)$, the map-valued operator is denoted $E(f, g) \to A$.

We have a third partial operator from triples $\langle h, f, g \rangle$ of maps to maps. The equations that define the domain are:

$$\text{codomain}(h) = \text{domain}(f) = \text{domain}(g) \ ,$$

$$\text{codomain}(f) = \text{codomain}(g) \ ,$$

$$hf = hg \ .$$

Given $X \xrightarrow{h} A \xrightarrow{f,g} B$ the value of this operator is denoted $X \xrightarrow{\overline{h}} E(f, g)$. The equations:

$$E(f, g) \to A \xrightarrow{f} B = E(f, g) \to A \xrightarrow{g} B \ ,$$

$$X \xrightarrow{\overline{h}} E(f, g) \to A = h \ ,$$

$$X \xrightarrow{k} E(f, g) = X \xrightarrow{\overline{(X \xrightarrow{k} E(f,g) \to A)}} E(f, g) \ .$$

We can make cartesian-closedness essentially algebraic by taking two binary operators: one from objects to objects denoted B^A , one from objects to maps denoted $B^A \times A \to B$; and a quaternary partial operator from quadruples $\langle X, A, B, f \rangle$ such that $\text{domain}(f) = X \times A$, $\text{codomain}(f) = B$, valued as a map $X \xrightarrow{\overline{f}} B^A$. The equations:

$$X \times A \xrightarrow{\overline{f}\times 1} B^A \times A \to B = f \ ,$$

$$X \xrightarrow{g} B^A = \overline{\langle X,\ A,\ B,\ X \times A \xrightarrow{g\times 1} B^A \times A \to B\rangle} \ .$$

Such is the nearest approximation to the previous elementary definitions. It is usually more convenient to ask that B^A is a bifunctor covariant in the lower variable, contravariant in the upper, equipped with natural transformations

$$e_{A,B} : B^A \times A \to B \ , \quad e^*_{A,B} : B \to (B{\times}A)^A$$

such that

$$B \times A \xrightarrow{e^*_{A,B}\times 1} (B{\times}A)^A \times A \xrightarrow{e_{A,B\times A}} B \times A = 1 \ ,$$

$$B^A \xrightarrow{e^*_{A,B^A}} (B^A{\times}A)^A \xrightarrow{(e_{A,B})^A} B^A = 1 \ .$$

The existence of elementary definitions should not, in itself, oblige us to give elementary proofs. The great technical *tour de force* in Gödel's incompleteness proof, namely that primitive recursive functions (a second-order notion if there ever was one) are all first-order definable (indeed Gödel-recursive) does not oblige us, but *allows* us to stop worrying about primitive recursive functions. Certainly it is worth knowing when things are elementary - we shall use the elementary nature of topoi (for example, Corollary 5.15, Theorem 5.23) and their essentially algebraic nature (Theorem 3.21). First-order logic is surely an artifice, albeit one of the most important inventions in human thought. But none of us thinks in a first-order language. The predicates of natural dialectics are order-insensitive (one moment's individuals are another's equivalence-class) and our appreciation of mathematics depends on our ability to *interpret* the words of mathematics. The interpretation itself is not first-order.

The reduction of a subject to an elementary one - in other than the formal method of set theory - usually marks a great event in mathematics. The elementary axioms of topoi are a testament to the ingenuity and insight of human genius. I will refuse to belittle this triumph of mind over matter by taking it as evidence that mind is matter.

PROPOSITION 1.11 for cartesian-closed categories. *The following natural maps are isomorphisms:*

$$0 \simeq 0 \times A \ ,$$

$$(A{\times}C) + (B{\times}C) \simeq (A{+}B) \times C \ ,$$

$$1^A \simeq 1 \ ,$$

$$(A{\times}B)^C \simeq A^C \times B^C \ ,$$

$$A^0 \simeq 1 \ ,$$

$$A^{B+C} \simeq A^B \times A^C \ ,$$

$$A \simeq A^1 \ ,$$

$$A^{B\times C} \simeq (A^B)^C \ .$$

Proof. Each is a result of the existence of an adjoint. For $-{\times}A$ to have a right-adjoint $(-^A)$, it must be cocontinuous, hence it preserves coproducts. Dually $(-^A)$, having a left-adjoint, preserves limits. A^- is adjoint to itself on the right, hence carries colimits to limits.

Elementary proofs would go like this: for each X there exists unique $0 \times A \to X$ because there exists unique $0 \to X^A$, hence $0 \times A$ is a coterminator. □

Caution: 0^A need not be 0. (Indeed $0^0 \simeq 1$.)

PROPOSITION 1.12 for cartesian-closed categories. *If* $A \to 0$ *exists then* $A \simeq 0$.

Proof. The existence of $A \to 0$ yields a map $A \to A \times 0$, and we obtain

$$A \to \underset{\substack{\wr\!\wr \\ 0}}{A \times 0} \to A = 1_A \ .$$

As always, $0 \to A \to 0 = 1_0$. □

A *degenerate* category is one with just one object and map. (Note that finite bicompleteness implies non-empty.)

PROPOSITION 1.13. *A cartesian-closed category is degenerate iff there exists* $1 \to 0$. □

PROPOSITION 1.14. *For any small* $\mathbb{A}$ *the category of set-valued contravariant functors* $S^{\mathbb{A}^{op}}$ *is cartesian-closed.*

Proof. $S^{\mathbb{A}^{op}}$ has a generating set, the representable functors (herein denoted H_A). For any T , $-\times T : S^{\mathbb{A}^{op}} \to S^{\mathbb{A}^{op}}$ is easily seen to preserve colimits because such are constructable "object-wise" and $-\times TA : S \to S$ preserves colimits. Hence for any T, S , $(-\times T, S)$ carries colimits into limits and the special adjoint functor theorem says that it is representable.

We can, of course, construct S^T more directly. We know that $S^T(A) \simeq \left(H_A, S^T\right) \simeq (H_A \times T, S)$ and we could use the latter as the definition of $S^T(A)$. The evaluation map $S^T \times T \to S$ is easily constructed object-wise: given $\langle \eta, x \rangle \in (S^T \times T)(A)$, that is, a transformation $\eta : H_A \times T \to S$ and an element $x \in T(A)$, define $e(\eta, x) = \eta_A(1_A, x)$. For the co-evaluation map $S \to (S \times T)^T$, let $x \in S(A)$ and define $e^*(x) \in (S \times T)^T(A)$ as the transformation $\eta \times 1 : H_A \times T \to S \times T$ where η is such that $\eta_A(1_A) = x$. All the equations are directly verifiable. □

Note that if $\mathbb{A}$ has finite products, then $S^{H_A} \simeq S(A\times -)$, and hence $S^T(A)$ is the set of transformations from T to $S(A\times -)$.

If $\mathbb{A}$ is a monoid M , $S^{\mathbb{A}^{op}}$ may be viewed as the discrete representations of M and S^T is the set of homomorphisms from $M \times T$ to S , where M is used to denote the "regular" representation.

If M is a group, then a homomorphism $f : M \times T \to S$ is determined by $f(1, x)$, $x \in T$, and given any function $g : T \to S$ we may define $f(\alpha, x) = \alpha g(\alpha^{-1}x)$. Thus S^T is the set of all functions from T to

S . αg is the function $(\alpha g)(x) = \alpha\left(g\left(\alpha^{-1}x\right)\right)$. The forgetful functor $S^M \to S$ preserves exponentiation.

1.2 Heyting algebras.

A *Heyting algebra* is a cartesian-closed category in which for every A, B , $(A, B) \cup (B, A)$ has at most one element. The latter condition says, of course, that we are dealing with a partially ordered set. The finite bicompleteness says that it is a lattice with 0 and 1 . The cartesian-closedness says that there is an operation on the objects such that

$$\left(A,\ B^C\right) \neq \emptyset \quad \text{iff} \quad (A\times C,\ B) \neq \emptyset \ .$$

We switch notation: the objects are lower-case $x, y, z, \ldots$, the existence of a map from x to y is stated with $x \leq y$, the product of x and y is denoted $x \wedge y$, the coproduct as $x \vee y$, the "Heyting operation" as $x \to y$. We recall that the following equations give us a lattice with $x \leq y$ defined as $x = x \wedge y$:

$$\begin{array}{ll} 1 \wedge x = x \ , & 0 \vee x = x \ , \\ x \wedge x = x \ , & x \vee x = x \ , \\ x \wedge y = y \wedge x \ , & x \vee y = y \vee x \ , \\ x \wedge (y\wedge z) = (x\wedge y) \wedge z \ , & x \vee (y\vee z) = (x\vee y) \vee z \ , \end{array}$$
$$x \wedge (y\vee x) = x = (x\wedge y) \vee x \ .$$

Non-equationally, $x \to y$ is characterized by $z \leq (x \to y)$ *iff* $z \wedge x \leq y$. That is, $x \to y$ is the largest element whose intersection with x is dominated by y .

PROPOSITION 1.21 for Heyting algebras.

$$\begin{gathered} 0 = 0 \wedge x \ , \\ (x\wedge y) \vee (x\wedge z) = x \wedge (y\vee z) \ , \\ x \to 1 = 1 \ , \\ x \to (y\wedge z) = (x \to y) \wedge (x \to z) \ , \\ 0 \to x = 1 \ , \\ (x\vee y) \to z = (x \to z) \wedge (y \to z) \ , \\ x = 1 \to x \ , \\ (x\wedge y) \to z = x \to (y \to z) \ . \end{gathered}$$

Proof. Translate Proposition 1.11. $\square$

PROPOSITION 1.22 **for Heyting algebras.** *We can characterize* $x \to y$ *with the following equations:*

$$x \to x = 1 ,$$
$$x \wedge (x \to y) = x \wedge y ,$$
$$y \wedge (x \to y) = y ,$$
$$x \to (y \wedge z) = (x \to y) \wedge (x \to z) .$$

Proof. Given $z \le (x \to y)$, that is, $z = z \wedge (x \to y)$, then $z \wedge x = z \wedge (x \to y) \wedge x = z \wedge x \wedge y \le y$. Conversely, given $z \wedge x \le y$, we note first that the fact that $f(u) = x \to u$ preserves intersections implies that it preserves order, and $x \to (z \wedge x) \le x \to y$. On the other hand $x \to (z \wedge x) = (x \to x) \wedge (x \to z) = x \to z$ and $z \wedge (x \to z) = z$, that is, $z \le x \to z$. Hence $z \le x \to z = x \to (x \wedge z) \le x \to y$. $\square$

We define the *negation* of an element, denoted $\neg x$, as $x \to 0$. Note that $z \le \neg x$ *iff* $z \wedge x = 0$, that is, $\neg x$ is the largest element disjoint from x .

A *complement* of x is an element y such that $x \wedge y = 0$ and $x \vee y = 1$. In a Heyting algebra, if x has a complement it must be $\neg x$: because $x \wedge y = 0 \Rightarrow y \le \neg x$ and

$$x \vee y = 1 \Rightarrow \neg x = \neg x \wedge (x \vee y) = (\neg x \wedge x) \vee (\neg x \wedge y)$$
$$= \neg x \wedge y \Rightarrow \neg x \le y .$$

PROPOSITION 1.23 **for Heyting algebras.** $x \le \neg\neg x$; *if* $x \le y$ *then* $\neg y \le \neg x$; $\neg x = \neg\neg\neg x$.

Proof. The first two statements are immediate. $(\neg x) \le \neg\neg(\neg x)$ by the first statement, $\neg(\neg\neg x) \le \neg x$ by the second applied to the first. $\square$

A *boolean algebra* is a Heyting algebra which satisfies the further equation $x = \neg\neg x$.

In a boolean algebra, negation is thus an order-reversing involution and De Morgan's laws are easy consequences: $\neg(x \wedge y) = \neg x \vee \neg y$. (Note that in any Heyting algebra

$$\neg(x \vee y) = (x \vee y) \to 0 = (x \to 0) \wedge (y \to 0) = \neg x \wedge \neg y .)$$

Hence $x \vee \neg x = \neg\neg(x \vee \neg x) = \neg(\neg x \wedge x) = \neg 0 = 1$ and every element

has a complement. Conversely if for all x , $x \vee \neg x = 1$ then
$\neg\neg x = \neg\neg x \wedge (x \vee \neg x) = (\neg\neg x \wedge x) \vee (\neg\neg x \wedge \neg x) = \neg\neg x \wedge x = x$.

It is easy to verify that in a boolean algebra $x \rightarrow y = \neg x \vee y$.

A complete lattice has a Heyting algebra structure, by the adjoint functor theorem, iff $f(n) = x \wedge n$ preserves all conjunctions, that is, if for any set $\{y_i\}_{i \in I}$, $x \wedge \bigvee y_i = \bigvee (x \wedge y_i)$. Such is usually called a complete distributive lattice. The lattice of open sets in a topological space is therefore a Heyting algebra. Consider the unit interval, and let:

$$1 = \vdash\!\!\dashv ,$$

$$x = \mapsto ,$$

$$\neg x = \leftarrow\!\!\dashv ,$$

$$x \vee \neg x = \mapsto\!\!\leftarrow\!\!\dashv ,$$

$$\neg\neg x = x ,$$

$$\neg (x \vee \neg x) = 0 ,$$

$$\neg\neg (x \vee \neg x) = 1 ,$$

$$(x \vee \neg x) \neq \neg\neg (x \vee \neg x) .$$

In any space, negation yields the interior of the set-theoretic complement. Double negation yields the interior of the closure.

Another ready example of a complete Heyting algebra is the lattice of left-ideals in any monoid. For a non-complete example, take any linearly ordered set with 0 and 1 but otherwise not complete and define

$$(x \rightarrow y) = \begin{cases} 1 & \text{if } x \leq y , \\ y & \text{if } y < x . \end{cases}$$

In any Heyting algebra, define $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. Then $z \leq (x \leftrightarrow y)$ *iff* $z \wedge x = z \wedge y$, that is, $x \leftrightarrow y$ is the largest element which meets x and y in the same way. We can reverse things to obtain the *symmetric definition of Heyting algebras*, namely the operations and equations of a lattice together with a binary operator satisfying:

$$1 \leftrightarrow x = x = x \leftrightarrow 1 \qquad\qquad (1 \text{ is a unit}),$$

$$x \leftrightarrow x = 1 \qquad (x \text{ is an inverse of } x),$$

$$(x \leftrightarrow y) \wedge z = (x \wedge z \leftrightarrow y \wedge z) \wedge z \quad \text{(almost distributive)}.$$

Then $(x \leftrightarrow y) \wedge x = (x \leftrightarrow x \wedge y) \wedge x = (1 \leftrightarrow y) \wedge x = y \wedge x$. Similarly $(x \leftrightarrow y) \wedge y = x \wedge y$. Hence if $z \leq (x \leftrightarrow y)$ then

$$z \wedge x = z \wedge (x \leftrightarrow y) \wedge x = z \wedge y \wedge x$$

and

$$z \wedge y = z \wedge (x \leftrightarrow y) \wedge y = z \wedge x \wedge y ,$$

that is $z \wedge x = z \wedge y$.

Conversely, if $z \wedge x = z \wedge y$, then

$$(x \leftrightarrow y) \wedge z = (x \wedge z \leftrightarrow y \wedge z) \wedge z = 1 \wedge z = z ,$$

that is, $z \leq (x \leftrightarrow y)$.

We may then define $x \to y$ as $x \leftrightarrow x \wedge y$ and verify $z \leq (x \to y)$ iff $z \wedge x \leq y$.

$\leftrightarrow$ is a symmetric binary operation with a unit and inverses. It is not associative in general, for $\neg x = x \leftrightarrow 0$ and $\neg\neg x = (x \leftrightarrow 0) \leftrightarrow 0$. Associativity implies that $\neg\neg x = x \leftrightarrow (0 \leftrightarrow 0) = x \leftrightarrow 1 = x$.

Conversely, in boolean algebras, $\leftrightarrow$ is associative. An orthogonal comment: $\leftrightarrow$ is a loop operation only in a boolean algebra, for given x if we can find y such that $y \leftrightarrow 0 = x$ then $x = \neg y = \neg\neg\neg y = \neg\neg x$.

Given a congruence $\equiv$ on a Heyting algebra it is easy to see that the set $F = \{x | x \equiv 1\}$ has the properties:

$$1 \in F ,$$

$$x \in F \Rightarrow x \vee y \in F ,$$

$$x, y \in F \Rightarrow x \wedge y \in F .$$

We call such a set a *filter*. We can recover the congruence from F : $x \equiv y$ *iff* $(x \leftrightarrow y) \in F$ because if $x \equiv y$ then $(x \leftrightarrow x) \equiv (x \leftrightarrow y)$ and $(x \leftrightarrow y) \in F$; if $(x \leftrightarrow y) \equiv 1$ then

$$x \equiv x \wedge (x \leftrightarrow y) = y \wedge (x \leftrightarrow y) \equiv y .$$

Moreover:

PROPOSITION 1.24 for Heyting algebras. *If* F *is a filter then the relation* $\equiv$ *defined by* $x \equiv y$ *iff* $(x \leftrightarrow y) \in F$ *is a congruence.*

Proof. $\equiv$ is clearly reflexive and symmetric. For transitivity suppose $(x \leftrightarrow y), (y \leftrightarrow z) \in F$. It suffices to show that $(x \leftrightarrow y) \wedge (y \leftrightarrow z) \leq (x \leftrightarrow z)$. (This is the only use of the fact that F is closed under intersection.) And for that inequality we need only show $(x \leftrightarrow y) \wedge (y \leftrightarrow z) \wedge x = (x \leftrightarrow y) \wedge (y \leftrightarrow z) \wedge z$, an easy matter.

For $x \equiv y \Rightarrow x \wedge z \equiv y \wedge z$ it suffices to show $(x \leftrightarrow y) \leq (x \wedge z) \leftrightarrow (y \wedge z)$, that is,

$$(x \leftrightarrow y) \wedge (x \wedge z) = (x \leftrightarrow y) \wedge (y \wedge z) ,$$

an easy matter. For $x \equiv y \Rightarrow x \vee z \equiv y \vee z$ we must show $(x \leftrightarrow y) \wedge (x \vee z) = (x \leftrightarrow y) \wedge (y \vee z)$, which because of distributivity is again an easy matter.

Finally for $x \equiv y \Rightarrow (x \leftrightarrow z) \equiv (y \leftrightarrow z)$ it suffices to show $x \leftrightarrow y \leq (y \leftrightarrow z) \leftrightarrow (x \leftrightarrow z)$, that is,

$$(x \leftrightarrow y) \wedge (x \leftrightarrow z) = (x \leftrightarrow y) \wedge (y \leftrightarrow z) .$$

Using the third defining equation of $\leftrightarrow$, the left-hand side is $(x \leftrightarrow y) \wedge [(x \leftrightarrow y) \wedge x \leftrightarrow (x \leftrightarrow y) \wedge z]$ and the right-hand side is $(x \leftrightarrow y) \wedge [(x \leftrightarrow y) \wedge y \leftrightarrow (x \leftrightarrow y) \wedge z]$, clearly equal since $(x \leftrightarrow y) \wedge x = (x \leftrightarrow y) \wedge y$. $\square$

1.3 Adjoint functors arising from cartesian-closed categories.

PROPOSITION 1.31. *Let* A *be a cartesian-closed category and* $A' \subset A$ *a full reflective subcategory,* $R : A \to A'$ *the reflector. Then* R *preserves products iff for all* $B \in A'$, B^A *is in* A'.

Proof. Suppose $R(A \times C) \simeq RA \times RC$, all A, C. We wish to show for $B \in A'$, that for any $C \in A$, $(RC, B^A) \simeq (C, B^A)$. (We are invoking the "Kelly view" of full reflective subcategories, namely that A' consists of all those objects X such that $(RY, X) \simeq (Y, X)$. This can be seen by letting $Y = X$. We obtain from $(RX, X) \twoheadrightarrow (X, X)$ a map $RX \to X$ such that $X \to RX \to X = 1$. From $(RX, X) \rightarrowtail (X, X)$ we obtain $RX \to X \to RX = 1$. The Kelly view is that every full reflective subcategory

 https://doi.org/10.1017/S0004972700044828

is definable by stipulating a class of maps $\mathcal{C}$ and then looking at all X such that (f, X) is an isomorphism for all $f \in \mathcal{C}$.)

To return:

$$(RC, B^A) \simeq (RC\times A, B) \simeq (R(RC\times A), B) \simeq$$
$$(RC\times RA, B) \simeq (R(C\times A), B) \simeq (C\times A, B) \simeq (C, B^A) .$$

Conversely, if for all $B \in \mathcal{A}'$, $B^A \in \mathcal{A}'$, then we wish to show that $R(A\times C)$ is, as defined in $\mathcal{A}'$, a product of RA, RC .

It suffices, then, to show that $\mathcal{A}'(RA\times RC, -) \simeq \mathcal{A}'(R(A\times C), -)$, that is, for all $B \in \mathcal{A}'$, $(RA\times RC, B) \simeq (R(A\times C), B)$. But $(R(A\times C), B) \simeq (A\times C, B)$ and

$$(RA\times RC, B) \simeq (RA, B^{RC}) \simeq (A, B^{RC}) \simeq (A\times RC, B) \simeq (RC, B^A)$$
$$\simeq (C, B^A) \simeq (A\times C, B) . \quad \square$$

For any category $\mathcal{A}$ and $B \in \mathcal{A}$, $\mathcal{A}/B$ denotes the category whose objects are $\mathcal{A}$-maps of the form $A \to B$, and whose maps are triangles

$$\begin{tikzcd}[column sep=small] A \arrow[rr] \arrow[dr] & & A' \arrow[dl] \\ & B & \end{tikzcd} .$$

$\Sigma_B : \mathcal{A}/B \to \mathcal{A}$ denotes the forgetful functor. Note that $\mathcal{A}/1 \to \mathcal{A}$ is an isomorphism. The naive construction of colimits in $\mathcal{A}/B$ works, that is, given colimits in $\mathcal{A}$. The naive construction of equalizers in $\mathcal{A}/B$ works; and it is the purest of tautologies that the $\mathcal{A}/B$-product of $A \to B$ and $A' \to B$ is their $\mathcal{A}$-pullback. Note that the terminator of $\mathcal{A}/B$ is $B \xrightarrow{1} B$.

PROPOSITION 1.32. $\Sigma_B : \mathcal{A}/B \to \mathcal{A}$ *preserves and reflects colimits, equalizers, pullbacks and monomorphisms.* $\square$

If $\mathcal{A}$ has finite products, we can define $\times B : \mathcal{A} \to \mathcal{A}/B$ by $A \mapsto (A\times B \xrightarrow{p} B)$.

PROPOSITION 1.33. Σ_B *is the left-adjoint of* $\times B$. $\square$

The next proposition says for $\mathcal{A}$ cartesian-closed, that $\times B$ has a right-adjoint. First, note the necessity of cartesian-closedness: if $\times B$

has a right-adjoint Π_B , then

$$(-\times B,\ A) = \left(\Sigma_B(\times B(-)),\ A\right) \simeq \left(\times B(-),\ \times B(A)\right) \simeq \left(-,\ \Pi_B(\times B(A))\right) ,$$

that is, given Π_B we can construct A^B as $\Pi_B(\times B(A))$.

PROPOSITION 1.34. *For cartesian-closed* $\mathbf{A}$ *,* $\times B : \mathbf{A} \to \mathbf{A}/B$ *has a right-adjoint.*

Proof. Given $(A \xrightarrow{f} B) \in \mathbf{A}/B$ define $\Pi_B(A \xrightarrow{f} B)$ by the pullback

$$\begin{tikzcd}
\Pi_B(A \xrightarrow{f} B) \arrow[r] \arrow[d] & A^B \arrow[d, "f^B"] \\
1 \arrow[r] & B^B
\end{tikzcd} ,$$

where $1 \to B^B$ corresponds to $B \xrightarrow{1} B$. For any $C \in \mathbf{A}$ we obtain a pullback in the category of sets

$$\begin{tikzcd}
(C,\ \Pi_B(A \to B)) \arrow[r] \arrow[d] & (C,\ A^B) \arrow[d, "{(C,\ f^B)}"] \\
(C,\ 1) \arrow[r] & (C,\ B^B)
\end{tikzcd} .$$

Three of these sets are naturally equivalent to other things and we obtain a pullback

$$\begin{tikzcd}
(C,\ \Pi_B(A \to B)) \arrow[r] \arrow[d] & (C\times B,\ A) \arrow[d, "{(C\times B,\ f)}"] \\
1 \arrow[r] & (C\times B,\ B)
\end{tikzcd} ,$$

where the bottom map sends 1 to $C \times B \xrightarrow{p} B$. Viewing $(C,\ \Pi_B(A \to B))$ as a subset of $(C\times B,\ A)$, we note that it can be described as $\{g : C \times B \to A \mid C \times B \xrightarrow{g} A \xrightarrow{f} B = C \times B \xrightarrow{p} B\}$, which is precisely the description of

$$\mathbf{A}/B(C \times B \xrightarrow{p} B,\ A \xrightarrow{f} B) = \mathbf{A}/B(\times B(C),\ A \xrightarrow{f} B) . \qquad \square$$

We note here, in anticipation of the next chapter's development, that for any $B_1 \xrightarrow{f} B_2$ we can define $f^\# : A/B_2 \to A/B_1$ by pulling back along f, $\Sigma_f : A/B_1 \to A/B_2$ by composing with f. On the other hand $B_1 \xrightarrow{f} B_2$ can be viewed as an object in A/B_2, and we could consider

$$\Sigma_{(B_1 \xrightarrow{f} B_2)} : (A/B_2)/(B_1 \to B_2) \to A/B_2 ,$$

$$\times(B_1 \to B_2) : A/B_2 \to (A/B_2)/(B_1 \to B_2) .$$

But $(A/B_2)/(B_1 \to B_2)$ is isomorphic to A/B_1, and the isomorphism reveals $\Sigma_{(B_1 \to B_2)}$ as Σ_f, $\times(B_1 \to B_2)$ as $f^\#$. Therefore $f^\#$ has a right adjoint, each f, iff A/B is cartesian-closed, each B.

1.4 Modal operators in Heyting algebras.

In any partially-ordered set, viewed as a category, the full reflective subcategories are in one-to-one correspondence with the order-preserving inflationary idempotents, that is, the functions f such that

$$x \le y \Rightarrow f(x) \le f(y) ,$$

$$x \le f(x) ,$$

$$f(f(x)) = f(x) .$$

Clearly, the reflector of a full reflective subcategory is such. Conversely, given such f then $\mathrm{Image}(f)$ is reflective as follows:

Given $y \in \mathrm{Im}(f)$ then $x \le y \Rightarrow f(x) \le f(y) = y$ and $f(x) \le y \Rightarrow x \le f(x) \le y$.

As a corollary of Proposition 1.31 we obtain

PROPOSITION 1.41. *Let H be a Heyting algebra with a unary operator denoted $\bar{x}$ such that*

$$\bar{\bar{x}} = \bar{x} ,$$

$$x \le \bar{x} \quad (\text{or} \quad x \wedge \bar{x} = x),$$

$$x \le y \Rightarrow \overline{x} \le \overline{y} \quad (or \quad \overline{x} = \overline{x} \wedge \overline{(x \vee y)}\ ).$$

Then $\forall_{x,y}(\overline{x \wedge y} = \overline{x} \wedge \overline{y})$ *iff* $\forall_{x,y}\big(x = \overline{x} \Rightarrow (y \to x) = \overline{(y \to x)}\big)$. □

We shall say that $\overline{x}$ is a *closure operator* if

$$\overline{\overline{x}} = \overline{x} ,$$

$$x \le \overline{x} \quad (\text{or} \quad x \wedge \overline{x} = x\ ),$$

$$\overline{x \wedge y} = \overline{x} \wedge \overline{y} .$$

Note that the last equation implies $x \le y \Rightarrow \overline{x} \le \overline{y}$.

By the above remarks, the image of a closure operator is a reflective sub-Heyting-algebra with the property that $x = \overline{x} \Rightarrow (y \to x) = \overline{(y \to x)}$.

PROPOSITION 1.42 for Heyting algebras. *$\neg\neg x$ is a closure operator. In particular* $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$.

Proof. We have already noted in Proposition 1.23 that $x \le \neg\neg x$ and $\neg\neg\neg\neg x = \neg\neg x$. From $x \le y \Rightarrow \neg y \le \neg x$ we easily obtain that $x \le y \Rightarrow \neg\neg x \le \neg\neg y$. By Proposition 1.41 it suffices to show that if $x = \neg\neg x$ then $y \to x = \neg\neg(y \to x)$. For purposes of clarity, let $z = \neg x$. Then $x = \neg\neg x$ yields $x = \neg z$. We use only this last equation from now on.

$$x = \neg z = (z \to 0) ,$$

$$y \to x = y \to (z \to 0) = (y \wedge z) \to 0 = \neg(y \wedge z) ,$$

$$\neg\neg(y \to x) = \neg\neg\neg(y \wedge z) = \neg(y \wedge z) = y \to x \qquad □$$

(We have proved that the defining equation of a Heyting algebra implies the equation $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$. There is, therefore, an entirely equational proof. My attempts to find one (essentially by translating the proof herein) have yielded only the most unbelievably long expressions I've ever seen.) The image of double negation is easily seen to be a boolean algebra.

1.5 Scattered comments.

Besides topoi and Heyting algebras important examples of cartesian-closed categories are the categories of small categories and various modifications of categories of topological spaces, which modifications

exist precisely to gain cartesian-closedness. The chief such modifications are k-spaces and Spanier's quasi-topological spaces.

For any category $\mathbb{A}$ define the pre-ordered set $\mathbb{A}'$ with $\mathbb{A}$-objects as elements and $A \le B$ iff $\mathbb{A}(A, B) \ne \emptyset$. Define $P(\mathbb{A})$ to be the skeleton of $\mathbb{A}'$. P is a reflection from the category of categories to the category of posets. P clearly preserves products and hence the category of posets is cartesian-closed.

Moreover, if $\mathbb{A}$ is cartesian-closed then $P(\mathbb{A})$ is, and $\mathbb{A} \to P(\mathbb{A})$ preserves $\times$, $+$ and exponentiation.

Let M be a monoid, S^M its category of representations. The Heyting algebra $P(S^M)$ changes wildly depending on M . For M a single point, $P(S^M) = P(S) = \{0, 1\}$. For M a group, $P(S^M)$ is a set (and can be described in terms of sets of subgroups of M). For M the natural numbers $P(S^M)$ is huge but does satisfy a transfinite descending chain condition, a fact which requires a long proof and can be found - together with a description of $P(S^M)$ - in a paper by (of all people) me ([2], 225-229), in a section entitled "When does petty imply lucid?".

For M almost anything else, $P(S^M)$ fails the transfinite chain condition.

2. The fundamentals of topoi

Let $\mathbb{A}$ be any category with pullbacks. Given $A \in \mathbb{A}$ define $\mathrm{Sub}(A)$ to be the set of subobjects of A (so assume that $\mathbb{A}$ is well-powered). We can make Sub into a contravariant functor by pulling back.

A *topos* is a cartesian-closed category for which Sub is representable; that is, there is an object Ω and a natural equivalence $(-, \Omega) \to \mathrm{Sub}$. Recall that any $\eta : (-, A) \to T$ is determined by knowing $\eta_A(1_A) \in TA$, hence there exists $\Omega' \rightarrowtail \Omega$ such that if we define $\eta : (A, \Omega) \to \mathrm{Sub}(A)$ by $\eta(f) = (\mathrm{Sub}(f))(\Omega' \rightarrowtail \Omega)$ then η is an isomorphism; that is, for every $A' \rightarrowtail A$ there exists a unique $A \to \Omega$ such that there exists $A' \to \Omega'$ such that $\begin{array}{ccc} A' & \to & A \\ \downarrow & & \downarrow \\ \Omega' & \to & \Omega \end{array}$ is a pullback. The

representability of Sub is thus revealed as an elementary condition. But we can do better. Note that in particular for any A there exists a unique $A \to \Omega$ such that there exists $A \to \Omega'$ such that $\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow & & \downarrow \\ \Omega' & \longrightarrow & \Omega \end{array}$ is a pullback. Given any $A \to \Omega'$ then for $A \to \Omega = A \to \Omega' \to \Omega$ it is the case that $\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow & & \downarrow \\ \Omega' & \longrightarrow & \Omega \end{array}$ is a pullback. The uniqueness condition then says that (A, Ω') has precisely one element. In other words Ω' is a terminator. Thus we could define a topos as a cartesian-closed category together with an object Ω and a map $1 \xrightarrow{t} \Omega$ such that for any $A' \rightarrowtail A$ there exists unique $A \to \Omega$ such that $\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ is a pullback. (We needn't quantify $A' \to 1$ because 1 is a terminator.) Of course, this elementary condition *implies* well-poweredness.

PROPOSITION 2.11. *For any small* $\mathbf{A}$, $S^{\mathbf{A}^{op}}$ *is a topos.*

Proof. We showed in Proposition 1.14 that $S^{\mathbf{A}^{op}}$ is cartesian-closed. For the Ω-condition we again assume the result to discover the proof. If Ω exists, then $\Omega(A) = (H_A, \Omega) = \mathrm{Sub}(H_A)$. A subfunctor of H_A is called an *A-crible*, alternatively described as a collection of maps $\mathbf{C}$ into A such that $B \to A \in \mathbf{C} \Rightarrow B' \to B \to A \in \mathbf{C}$. Defining $\Omega(A)$ as the set of A-cribles, we make Ω into a contravariant functor, again by "pulling back": given $A' \to A$ and an A-crible $\mathbf{C}$ define $\mathbf{C}'$ as the set of maps $B \to A'$ such that $B \to A' \to A \in \mathbf{C}$.

Any $1 \to \Omega$ is a choice for each A of an element in ΩA. Define $t : 1 \to \Omega$ to correspond to the maximal crible, each A. (The maximal A-crible is the set of all maps into A).

Given $T' \subset T$ define $T \to \Omega$ by sending $x \in TA$ to the A-crible of all maps $B \xrightarrow{f} A$ such that $(Tf)(x) \in T'(B)$ and verify that $\begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ is a pullback. The uniqueness of $T \to \Omega$ can be directly verified. □

PROPOSITION 2.12 for topoi. *Every monomorphism is an equalizer.*

Proof. Given $A' \rightarrowtail A$ let $\begin{array}{ccc} A' & \to & A \\ \downarrow & & \downarrow \\ 1 & \to & \Omega \end{array}$ be a pullback. Then A' is the equalizer of $A \times 1 \xrightarrow{p} A \to \Omega$ and $A \times 1 \xrightarrow{p} 1 \to \Omega$. But $A \times 1 \simeq A$. □

COROLLARY 2.13 for topoi. *If* $A \to B$ *is both mono and epi, it is iso.*

Proof. It can be the equalizer only of $f, f : B \to C$. □

In any finitely bicomplete category we define the "regular image" of $A \xrightarrow{f} B$ as the equalizer of the cokernel-pair, that is, the equalizer of $x, y : B \to C$ where

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "x"] \\ B \arrow[r, "y"'] & C \end{tikzcd}$$

is a pushout. It is a routine exercise to verify that the regular image is the smallest regular subobject through which f factors ("regular" is equal to "appears as an equalizer"). Because all subobjects are regular in a topos, the regular image is the smallest subobject allowing a factorization of $A \xrightarrow{f} B$. We shall call it the *image* of f, denoted $\mathrm{Im}(f)$

PROPOSITION 2.14 for topoi. $A \to \mathrm{Im}(f)$ *is epi.*

If $A \twoheadrightarrow C \rightarrowtail B = f$ *then there exists unique* $\mathrm{Im}(f) \simeq C$ *such that*

$$\begin{tikzcd} & \mathrm{Im}(f) \arrow[dr] \arrow[d] & \\ A \arrow[ur] \arrow[r] & C \arrow[r] & B \end{tikzcd}.$$

Given $\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D \end{tikzcd}$ *there exists unique* $\mathrm{Im}(f) \to \mathrm{Im}(g)$ *such that*

$$\begin{array}{ccccc} A & \to & \mathrm{Im}(f) & \to & B \\ \downarrow & & \downarrow & & \downarrow \\ C & \to & \mathrm{Im}(g) & \to & D \end{array}.$$

Proof. By construction, $\mathrm{Im}(f) = B$ iff f is epic. Hence, for the first statement it suffices to show $\mathrm{Im}\big(A \to \mathrm{Im}(f)\big) = \mathrm{Im}(f)$; but the minimality of $\mathrm{Im}(f)$ does just that.

The other two statements are immediate from the characterization of $\mathrm{Im}(f)$ as the smallest subobject through which f factorizes, together with Corollary 2.13. □

$\mathrm{Sub}(1)$ is a Heyting algebra. It is clearly a lattice (because images allow us to construct unions of subobjects) and for any $U \subset 1$, $(A, U^B) \simeq (A \times B, U)$ has at most one element, hence the map $U^B \to 1$ is a monomorphism. The subobjects of 1, in fact, form a full reflective subcategory (the reflector constructed by taking images: $X \mapsto \mathrm{Im}(X \to 1)$).

The subobjects of 1 in $S^{\mathsf{A}^{\mathrm{op}}}$ may be seen to correspond to collections of A-objects with the property that if A is in the collection and there exists $B \to A$ then B is in the collection. Hence if A is strongly connected (for example, a monoid) then 1 has only the two subobjects it always must. If A is $\bullet \to \bullet \to \cdots \to \bullet$ then the subobjects of 1 in $S^{\mathsf{A}^{\mathrm{op}}}$ form the linearly ordered set one bit longer than A.

Returning to Ω in $S^{\mathsf{A}^{\mathrm{op}}}$, suppose A is a monoid M. We may view $S^{\mathsf{A}^{\mathrm{op}}}$ as the left-representations of M, that is, sets X together with $M \times X \to X$ such that $1 \cdot x = x$, $\alpha(\beta \cdot x) = (\alpha\beta) \cdot x$. Then Ω is the set of left-ideals in M. The action of M on Ω is *not* multiplication but division: $\alpha \cdot \underline{\underline{A}} = \{\beta \mid \beta\alpha \in \underline{\underline{A}}\}$. $1 \xrightarrow{t} \Omega$ is the unit ideal. If M is a group then $\Omega \simeq 1 + 1$; and, for monoids, conversely.

2.2 The representability of partial maps.

A *partial map* from A to B is a map from a subobject of A to B. Formally, we consider pairs $\langle A' \rightarrowtail A, A' \to B \rangle$, define $\langle A' \rightarrowtail A, A' \to B \rangle \equiv \langle A'' \rightarrowtail A, A'' \to B \rangle$ if there exists an isomorphism $A' \to A''$ such that

$$\begin{tikzcd} & & A \\ A' \arrow[r] \arrow[urr, tail] \arrow[drr] & A'' \arrow[ur, tail] \arrow[dr] & \\ & & B \end{tikzcd},$$

and define $\mathrm{Par}(A, B)$ as the set of equivalence classes. Fixing B, $\mathrm{Par}(-, B)$ is a contravariant functor, by pulling back.

A *relation* from A to B is a subobject of $A \times B$. Calling the set of such $\mathrm{Rel}(A, B)$ and fixing B, $\mathrm{Rel}(-, B)$ becomes a contravariant functor, again by pulling back. $\mathrm{Rel}(-, B)$ is representable, namely by Ω^B.

Every map from A to B can be viewed as a partial-map and we obtain a transformation $(-, B) \to \mathrm{Par}(-, B)$. Every partial-map $\langle A' \rightarrowtail A, A' \to B\rangle$ yields a relation, $A' \rightarrowtail A \times B$ (its "graph") and we obtain a transformation $\mathrm{Par}(-, B) \to \mathrm{Rel}(-, B)$. Both transformations are monic.

The transformation $(-, B) \to \mathrm{Rel}(-, B)$ must come from a monomorphism $B \to \Omega^B$, the *singleton map*. $B \to \Omega^B$ may be computed to correspond to $B \times B \to \Omega$ where $\begin{array}{ccc} B & \xrightarrow{\Delta} & B\times B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ is a pullback.

We shall show that $\mathrm{Par}(-, B)$ is representable. First:

PROPOSITION 2.21 for topoi (unique existentiation). *Given* $C \to A$ *there exists* $Q \rightarrowtail A$ *such that* $\begin{array}{ccc} Q & \xrightarrow{1} & Q \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \end{array}$ *is a pullback and for any* $X \to A$ *such that* $\begin{array}{ccc} X & \xrightarrow{1} & X \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \end{array}$ *is a pullback, there exists* $X \to Q \to A = X \to A$.

Proof. Define $(-, A) \to \mathrm{Rel}(-, C)$ by sending $X \to A$ to the pullback $\begin{array}{ccc} R & \to & X \\ \downarrow & & \downarrow \\ C & \to & A \end{array}$ (R to be viewed as a subobject of $X \times C$). This transformation is induced by a map $A \to \Omega^C$ (which can be computed as $A \to \Omega^A \xrightarrow{\Omega^g} \Omega^C$).

Define Q by the pullback $\begin{array}{ccc} Q & \rightarrowtail & A \\ \downarrow & & \downarrow \\ C & \to & \Omega^C \end{array}$. For any X, $\begin{array}{ccc} (X,Q) & \to & (X,A) \\ \downarrow & & \downarrow \\ (X,C) & \to & (X,\Omega^C) \end{array}$ is a pullback. Viewing (X, Q) as a subset of (X, A) we see that $f \in (X, Q)$ iff in pullback $\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ C & \to & A \end{array}$, $X' \subset X \times C$ as a relation from $X \to C$,

describes a map from X to C, that is, $X' \to X$ is an isomorphism. □

Given a relation from A to B described by $\begin{array}{ll} C \to A \\ \downarrow & \\ B & \end{array}$ let $Q \rightarrowtail A$ be as described above. We obtain a partial-map, $\langle Q \rightarrowtail A, Q \to C \to B \rangle$ in $\mathrm{Par}(A, B)$. This operation, $\mathrm{Rel}(-, B) \to \mathrm{Par}(-, B)$ is natural by the above lemma. Moreover $\mathrm{Par}(-, B) \to \mathrm{Rel}(-, B) \to \mathrm{Par}(-, B)$ is the identity. The idempotent transformation $\mathrm{Rel}(-, B) \to \mathrm{Par}(-, B) \to \mathrm{Rel}(-, B)$ must come from an idempotent $\Omega^B \xrightarrow{g} \Omega^B$, and we define $\Omega^B \to \tilde{B}$, $\tilde{B} \to \Omega^B$ as a splitting of g. ($\tilde{B} \to \Omega^B$ can be defined as the equalizer of 1 and g.) Clearly $(-, \Omega^B) \to (-, \tilde{B})$, $(-, \tilde{B}) \to (-, \Omega^B)$ splits $(-, g)$ and:

PROPOSITION 2.22 for topoi. *$\mathrm{Par}(-, B)$ is representable.* □

$\Omega^B \xrightarrow{g} \Omega^B$ corresponds to a map $B \times \Omega^B \to \Omega$ (not the evaluation map) which corresponds to a subobject of $B \times \Omega^B$ which can be computed to be $B \xrightarrow{\langle 1, s \rangle} B \times \Omega^B$ where s is the singleton map. In a telling sense $B \to \tilde{B}$ is a generalization of $1 \to \Omega$, to wit:

PROPOSITION 2.23. *For any partial map $\langle A' \rightarrowtail A, A' \to B \rangle$ there exists unique $A \to \tilde{B}$ such that* $\begin{array}{ccc} A' & \to & A \\ \downarrow & & \downarrow \\ B & \to & \tilde{B} \end{array}$ *is a pullback.*

Proof. The transformation $\eta : (-, \tilde{B}) \to \mathrm{Par}(-, B)$ is determined by $\eta_{\tilde{B}}(1_{\tilde{B}}) \in \mathrm{Par}(\tilde{B}, B)$. Let $\langle B' \to \tilde{B}, B' \to B \rangle$ represent $\eta_{\tilde{B}}(1_{\tilde{B}})$. Then the fact that η is an equivalence says that for any $\langle A' \rightarrowtail A, A' \to B \rangle$ there is unique $A \to \tilde{B}$ such that there is $A' \to B'$ such that $\begin{array}{ccc} A' & \to & A \\ \downarrow & & \downarrow \\ B' & \to & \tilde{B} \end{array}$ is a pullback and $A' \to B' \to B = A \to B$. In particular, for any $A \to B$ there exists unique $A \to \tilde{B}$ such that there exists $A \to B'$ such that $\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \tilde{B} \end{array}$ is a pullback and $A \to B' \to B = A \to B$. If $B' \to B$ were other than an isomorphism, then we would obtain a contradiction. □

Note that $\mathrm{Sub}(A) = \mathrm{Par}(A, 1)$ and $\tilde{1} = \Omega$.

Given a map $B \to B'$ we can define $\mathrm{Par}(-, B) \to \mathrm{Par}(-, B')$ by

composition and obtain a map $\tilde{B} \to \tilde{B}'$. Alternatively, from the above proposition, there exists a unique map $\tilde{B} \to \tilde{B}'$ such that $\begin{array}{ccc} B & \to & \tilde{B} \\ \downarrow & & \downarrow \\ B' & \to & \tilde{B}' \end{array}$ is a pullback, and $\tilde{B}$ is revealed as a covariant functor, $B \to \tilde{B}$ as a natural transformation.

2.3. The fundamental theorem of topoi

A *logical-morphism* of topoi is a functor that preserves finite limits, colimits, Ω , and exponentiation.

THEOREM 2.31 for topoi. *For any topos* T *and* $B \in T$, T/B *is a topos.*

For $f : B_1 \to B_2$ *the functor* $f^{\#} : T/B_1 \to T/B_2$ *defined by pulling back along* f , *has a left-adjoint* Σ_f *and a right-adjoint* Π_f , $f^{\#}$ *is bi-continuous and a logical morphism.*

Proof. We noted at the end of 1.3 that the second sentence follows from the first.

The Ω condition is easy:

$$\mathrm{Sub}_{T/B}(A \to B) \simeq \mathrm{Sub}_T(A) \simeq T(\Sigma_B(A \to B), \Omega) \simeq T/B(A \to B, \Omega\times B \to B) .$$

Given $A \xrightarrow{f} B$, $C \xrightarrow{g} B$ we wish to construct $(C \to B)^{(A\to B)}$ in T/B . Let $B \to \tilde{B}^A$ in T correspond to $k : A\times B \to \tilde{B}$, the unique map such that $\begin{array}{ccc} A & \xrightarrow{\langle 1,f\rangle} & A\times B \\ \downarrow & & \downarrow \\ B & \longrightarrow & \tilde{B} \end{array}$ is a pullback and let $\begin{array}{ccc} P & \to & \tilde{C}^A \\ \downarrow & & \downarrow \tilde{g}^A \\ B & \to & \tilde{B}^A \end{array}$ be a pullback. Then $P \to B$ is $(C \to B)^{(A\to B)}$ as follows:

Given $X \xrightarrow{h} B$,

$$\begin{array}{ccc} T/B(X \to B, P \to B) & \to & T(X, P) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & T(X, B) \end{array}$$

is a pullback, by definition of T/B . By definition of $P \to B$,

$$\begin{array}{ccccc} (X,\ P) & \to & (X,\ \tilde{C}^A) & \simeq & \mathrm{Par}(X{\times}A,\ C) \\ \downarrow & & \downarrow & & \downarrow \\ (X,\ B) & \to & (X,\ \tilde{B}^A) & \simeq & \mathrm{Par}(X{\times}A,\ B) \end{array}$$

is a pullback. Then

$$\begin{array}{ccc} T/B(X \to B,\ P \to B) & \to & \mathrm{Par}_T(X{\times}A,\ C) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{Par}_T(X{\times}A,\ B) \end{array}$$

is a pullback where $1 \to \mathrm{Par}(X{\times}A,\ B)$ corresponds to $X \times A \xrightarrow{h\times 1} B \times A \xrightarrow{k} \tilde{B}$. The element in $\mathrm{Par}(X{\times}A,\ B)$ is therefore the result of pulling back:

$$\begin{array}{ccc} \bullet & \to & X{\times}A \\ \downarrow & & \downarrow h{\times}1 \\ A & \to & B{\times}A \\ \downarrow & & \downarrow k \\ B & \to & \tilde{B} \end{array}$$

and that is the same as the pullback $\begin{array}{ccc} Q & \to & X \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$, that is, the product, in T/B of $X \to B$ and $A \to B$; that is, $T/B(X \to B,\ P \to B)$ is the set of T-partial maps from $X \times A$ to C such that when composed with $C \to B$ yield just what they should.

That $f^{\#}$ preserves exponentiation is reducible, as discussed at the end of 1.3, to seeing that $\times B : T \to T/B$ preserves exponentiation. We wish to compare $\left(-,\ \times B(C^A)\right)$ and $\left(-,\ \times B(C)^{\times B(A)}\right)$.

$$\left(D \to B,\ \times B(C^A)\right) \simeq \left(\Sigma_B(D \to B),\ C^A\right) \simeq (D,\ C^A) \simeq (D{\times}A,\ C) ,$$

$$\left(D \to B,\ \times B(C)^{\times B(A)}\right) \simeq \left((D \to B) \times \times B(A),\ \times B(C)\right) \simeq \left(\Sigma_B\left((D \to B) \times \times B(A)\right),\ C\right) .$$

It suffices to show $D \times A \simeq \Sigma_B\left((D \to B) \times \times B(A)\right)$.

$(D \to B) \times \times B(A)$ is the pullback $\begin{array}{ccc} P & \to & A{\times}B \\ \downarrow & & \downarrow p \\ D & \to & B \end{array}$ which can be directly verified as $\begin{array}{ccc} A{\times}D & \to & A{\times}B \\ p\downarrow & & \downarrow p \\ D & \to & B \end{array}$. Hence $\Sigma_B\left((D \to B) \times \times B(A)\right) = A \times D$. $f^{\#}$

preserves limits and colimits because it has both a left and right adjoint. □

COROLLARY 2.32 for topoi. *Pullbacks of epimorphisms are epimorphisms.*

Proof. Given $A \xrightarrow{f} B$, $C \twoheadrightarrow B$, we view $C \twoheadrightarrow B$ as an object in T/B . $f^{\#}(C \twoheadrightarrow B)$ is the pullback. Because $f^{\#}$ preserves epimorphisms and terminal objects, $f^{\#}(C \to B) \to f^{\#}(B \to B)$ is an epimorphism. □

COROLLARY 2.33 for topoi. *Given* $A \xrightarrow{f} B + C$ *there exists*

$$\begin{tikzcd} A_1 + A_2 \arrow[dr, "f_1+f_2"'] & \simeq & A \arrow[dl, "f"] \\ & B+C & \end{tikzcd}$$

Proof. View $B \to B + C$ and $C \to B + C$ as objects in $T/B{+}C$ and apply $f^{\#}$ to obtain $A_1 \to A$, $A_2 \to A$ in T/A . □

Given a filter F (as defined in 1.2) on the Heyting algebra Sub(1) we obtain a Serre-class of maps in the topos, namely those maps $f : A \to B$ such that there exists $U \in F$, $(U \subset 1)$ so that $T \to T/U$ sends f to an isomorphism. The result of inverting all such maps is a topos T/F which may be also constructed by $T/F(A, B) = \varinjlim(A{\times}U, B)$. The most insightful way to construct T/F is to take the direct limit of the topoi T/U , $U \in F$. Using the fact that all the induced maps $T/U \to T/V$ for $V \subset U$ are logical morphisms and that topoi are essentially algebraic, T/F is easily believed to be a topos. Finally, one must note that $\mathrm{Sub}_{T/F}(1) \simeq (\mathrm{Sub}_T(1))/F$. Strangely enough, we shall not use this construction.

Given relations $\begin{matrix} R_1 \to A \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} R_2 \to B \\ \downarrow \\ C \end{matrix}$ let $\begin{matrix} R & \to & R_1 \\ \downarrow & & \downarrow \\ R_2 & \to & B \end{matrix}$ be a pullback and define the composition $R_1 \circ R_2$ as $\mathrm{Im}(R \to A{\times}C)$.

PROPOSITION 2.34 for topoi. *Composition of relations is associative.*

Proof. Given $X \to A\times B$, $Y \to A\times B$ and $B' \to B$ let

$$\begin{array}{ccc} X' & \to & B' \\ \downarrow & & \downarrow \\ X & \to & B \end{array} \quad , \quad \begin{array}{ccc} Y' & \to & B' \\ \downarrow & & \downarrow \\ Y & \to & B \end{array}$$

be pullbacks. Corollary 2.32 says that if $\operatorname{Im}(X \to A\times B) = \operatorname{Im}(Y \to A\times B)$ then $\operatorname{Im}(X' \to A\times B') = \operatorname{Im}(Y' \to A\times B')$. That fact, together with the associativity of pullbacks provides the proof. Let

$$\begin{array}{ccccccc} R_6 & \to & R_4 & \to & R_1 & \to & A \\ \downarrow & & \downarrow & & \downarrow & & \\ R_5 & \to & R_2 & \to & B & & \\ \downarrow & & \downarrow & & & & \\ R_3 & \to & C & & & & \\ \downarrow & & & & & & \\ D & & & & & & \end{array}$$

be such that all squares are pullbacks. $\operatorname{Im}(R_4 \to A\times C) = R_1 \circ R_2$ and hence $\operatorname{Im}(R_6 \to A\times D) = (R_1 \circ R_2) \circ R_3$. Equally $\operatorname{Im}(R_6 \to A\times D) = R_1 \circ (R_2 \circ R_3)$. □

Given a topos T we obtain a *category of relations* $\operatorname{Rel}(T)$ and an embedding $T \to \operatorname{Rel}(T)$. In particular Rel is a bifunctor from T to sets, contravariant on the first variable, covariant on the second. Given $f : A \to B$ the transformation $\operatorname{Rel}(-, f) : \operatorname{Rel}(-, A) \to \operatorname{Rel}(-, B)$ yields a transformation $(-, \Omega^A) \to (-, \Omega^B)$ which must be induced by a map from Ω^A to Ω^B to be called $\exists_f$. The transformation $(-, -) \to \operatorname{Rel}(-, -)$ is natural and we obtain

$$\begin{array}{ccc} A & \to & \Omega^A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists_f} \\ B & \to & \Omega^B \end{array} .$$

$\operatorname{Rel}(A, B)$ is of course, a Heyting algebra. $\operatorname{Rel}(-, B)$ is a Heyting-algebra-valued functor, but $\operatorname{Rel}(A, -)$ is not. $\exists_f$ preserves unions but not intersections. We will use the fact that it preserves order.

2.4 The propositional calculus of a topos

PROPOSITION 2.41 for topoi. *For every B , the subobjects of B form a Heyting algebra and the operations of same are preserved by pulling*

back.

Proof. The subobjects of B are the subobjects of the terminator in T/B . □

We can therefore view Sub as a Heyting-algebra-valued functor. Necessarily, the object which represents Sub must be a Heyting algebra in T . That is, there exist maps $1 \xrightarrow{t} \Omega$, $1 \xrightarrow{f} \Omega$, $\Omega\times\Omega \xrightarrow{\cap} \Omega$, $\Omega\times\Omega \xrightarrow{\cup} \Omega$, $\Omega\times\Omega \xrightarrow{\rightarrow} \Omega$ which satisfy the equations of a Heyting algebra $(t = 1, f = 0)$, which maps yield the Heyting algebra structure on each $\mathrm{Sub}(B)$. $1 \xrightarrow{f} \Omega$ can be computed as the unique map such that
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "f"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$
is a pullback; $\Omega\times\Omega \xrightarrow{\cap} \Omega$ as the map such that
$$\begin{tikzcd} 1 \arrow[r, "\langle t, t\rangle"] \arrow[d] & \Omega\times\Omega \arrow[d, "\cap"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd}$$
is a pullback. Let

$$\Omega \vee \Omega = \mathrm{Im}\left(\Omega+\Omega \xrightarrow{\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}} \Omega\times\Omega\right) .$$

Then
$$\begin{tikzcd} \Omega\vee\Omega \arrow[r] \arrow[d] & \Omega\times\Omega \arrow[d, "\cup"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$
is a pullback. Finally, define $\Omega\times\Omega \xrightarrow{\leftrightarrow} \Omega$ to be the unique map such that
$$\begin{tikzcd} \Omega \arrow[r, "\Delta"] \arrow[d] & \Omega\times\Omega \arrow[d, "\leftrightarrow"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$
is a pullback. One may directly verify the equations of the symmetric definition of Heyting algebras for these maps so defined.

Note that for each B , Ω^B becomes a Heyting algebra in T . There are two Heyting algebras for B : $\mathrm{Sub}(B)$ which lives in the category of sets, Ω^B which lives in T .

A *boolean* topos is one in which Ω is boolean. $\Omega \xrightarrow{\neg} \Omega$ is the

unique map such that $\begin{tikzcd} 1 \arrow[r, "f"] \arrow[d] & \Omega \arrow[d, "\neg"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$ is a pullback. Hence T is boolean if

$\Omega \xrightarrow{\neg} \Omega \xrightarrow{\neg} \Omega = 1_\Omega$. Note that every Sub(B) is thus forced to be boolean, and given $B' \rightarrowtail B$ there exists a complement $B'' \rightarrowtail B$ such that $\begin{tikzcd} 0 \arrow[r] \arrow[d] & B' \arrow[d] \\ B'' \arrow[r] & B \end{tikzcd}$ is a pullback, $B'+B'' \to B$ is epi. In any topos, such implies that $B'+B'' \to B$ is an iso, just by showing that $B'+B'' \to B$ is mono, an easy matter using Corollary 2.33. Products of boolean topoi are easily seen to be boolean.

PROPOSITION 2.42 for topoi. *T is boolean iff* $1 + 1 \xrightarrow{\binom{t}{f}} \Omega$ *is an isomorphism.*

Proof. Clearly if $1 + 1$ works as Ω then $\Omega \xrightarrow{\neg} \Omega$ can be none other than the twist map on $1 + 1$, and $\neg\neg = 1$.

Conversely for T boolean, the complement of $1 \xrightarrow{t} \Omega$ is $1 \xrightarrow{f} \Omega$ and the remarks above yield $1 + 1 \simeq \Omega$. □

PROPOSITION 2.43. $S^{\mathsf{A}^{\mathrm{op}}}$ *is boolean iff* A *is a groupoid.*

Proof. If A is a groupoid then $S^{\mathsf{A}^{\mathrm{op}}}$ is a product of categories of the form S^G , G a group. We observed earlier that Ω in S^G is $1 + 1$.

Conversely, suppose $A \to B$ in A does not have a left inverse, that is, there is no $B \to A$ such that $B \to A \to B = 1_B$. $A \to B$ generates a B-crible, neither empty nor everything; that is $\Omega(B)$ has more than 2 elements. But if $1 + 1 \simeq \Omega$ then $\Omega(B) \simeq (1{+}1)(B) = 1 + 1$. □

By a *closure operation* on B we mean a closure operation as defined in 1.4 on Sub(B) , that is an intersection-preserving inflationary idempotent Sub(B) $\to$ Sub(B) . A *global closure operation* is a choice for each B of a closure operation that makes Sub $\to$ Sub natural. Necessarily such must be induced by a map $j : \Omega \to \Omega$, and j itself is a

closure operation in the internal sense, that is, $j^2 = j$,

$\Omega \xrightarrow{\langle 1,j\rangle} \Omega\times\Omega \xrightarrow{\cap} \Omega = 1$ and $\begin{array}{ccc} \Omega\times\Omega & \xrightarrow{\cap} & \Omega \\ {\scriptstyle j\times j}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega\times\Omega & \xrightarrow[\cap]{} & \Omega \end{array}$. Rather mysteriously:

PROPOSITION 2.44 for topoi.

$$\Omega \xrightarrow{\langle 1,j\rangle} \Omega\times\Omega \xrightarrow{\cap} \Omega = 1$$

iff

$$1 \xrightarrow{t} \Omega \xrightarrow{j} \Omega = 1 \xrightarrow{t} \Omega .$$

Proof. First a less algebraic proof:

Given $j : \Omega \to \Omega$ consider $(B, \Omega) \xrightarrow{(B,j)} (B, \Omega)$. If $1 \xrightarrow{t} \Omega \xrightarrow{j} \Omega = 1 \xrightarrow{t} \Omega$ then for $\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ a pullback, the result of applying (B, j) yields

$$\begin{array}{ccc} \overline{B}' & \to & B \\ \downarrow & & \downarrow \\ \Omega' & \to & \Omega \\ \downarrow & & \downarrow{\scriptstyle j} \\ 1 & \to & \Omega \end{array} ,$$

and there exists $1 \to \Omega'$ such that $1 \to \Omega' \to \Omega = 1 \xrightarrow{t} \Omega$, hence $\begin{array}{ccc} B' & \to & \overline{B}' \\ \downarrow & & \downarrow \\ 1 & \to & \Omega' \end{array}$ is a pullback and (B, j) is inflationary.

More directly, let $\begin{array}{ccc} U & \longrightarrow & \Omega \\ \downarrow & & \downarrow{\scriptstyle \langle 1,\, j\rangle} \\ 1 & \xrightarrow[\langle t,t\rangle]{} & \Omega\times\Omega \end{array}$ be a pullback. Because $\langle 1, j\rangle$ is monic, $U \to 1$ is. If $1 \xrightarrow{t} \Omega \xrightarrow{j} \Omega = 1 \xrightarrow{t} \Omega$ then there exists $1 \to U$ such that $1 \to U \to \Omega = 1 \xrightarrow{t} \Omega$ and $U \to 1$ is an isomorphism; that is $\begin{array}{ccc} 1 & \xrightarrow{t} & \Omega \\ \downarrow & & \downarrow{\scriptstyle \langle 1,\, j\rangle} \\ 1 & \xrightarrow[\langle t,t\rangle]{} & \Omega\times\Omega \end{array}$ is a pullback. Hence

$$\begin{array}{ccc} 1 & \xrightarrow{t} & \Omega \\ & & \downarrow \langle 1, j \rangle \\ \Big\downarrow & & \Omega\times\Omega \\ & & \downarrow \cap \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

is a pullback and the uniqueness condition on Ω implies the result. □

As shown in Proposition 1.42, $\Omega \xrightarrow{\neg\neg} \Omega$ is a closure operation. Another example arises as follows: let $U \subseteq 1$ and define for each B, $\mathrm{Sub}(B) \to \mathrm{Sub}(B)$ by $(B' \to B) \mapsto (B' \cup (B\times U))$. (Notice that $B\times U \to B\times 1 \to B$ is a monomorphism.) Clearly this operation is inflationary and idempotent. To see that it is intersection-preserving consider $B', B'' \subset B$. Then

$$(B'\cup(B\times U)) \cap (B''\cup(B\times U)) = (B'\cap B'') \cup (B'\cap(B\times U)) \cup ((B\times U)\cap B'') \cup (B\times U) ,$$

by distributivity. The middle two terms are contained in $B \times U$, hence we obtain

$$(B'\cup(B\times U)) \cap (B''\cup(B\times U)) = (B'\cap B'') \cup (B\times U) .$$

The operation is natural because $\begin{array}{ccc} B\times U & \to & B \\ \downarrow & & \downarrow \\ U & \to & 1 \end{array}$ is a pullback.

2.5. Injective objects

PROPOSITION 2.51 for topoi. Ω *is injective.*

Proof. Given $A \rightarrowtail B$ and $A \to \Omega$ let $\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ be a pullback. Then

let $\begin{array}{ccc} A' & \longrightarrow & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ be a pullback. Then verify that $A \to B \to \Omega = A \to \Omega$. □

COROLLARY 2.52 for topoi. Ω^C *is injective.*

Proof. Given $A \rightarrowtail B$ we wish to show that $(B, \Omega^C) \to (A, \Omega^C)$ is epic. But

$$\begin{array}{ccc} (B, \Omega^C) & \to & (A, \Omega^C) \\ \wr\!\wr & & \wr\!\wr \\ (B\times C, \Omega) & \to & (A\times C, \Omega) \end{array}$$

and $A\times C \to B\times C$ is monic. □

Using the singleton map $B \to \Omega^B$ we obtain:

PROPOSITION 2.52 for topoi. *Every object may be embedded in an injective.* □

COROLLARY 2.53 for topoi. *Pushouts of monomorphisms are monomorphisms.*

Proof. Given a pushout $\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$ choose $C \rightarrowtail E$, E injective. There exists $\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r] & E \end{tikzcd}$, hence there exists $C \to D \to E = C \rightarrowtail E$ and $C \to D$ is monic. □

PROPOSITION 2.54 for topoi. *If* $\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] & D \end{tikzcd}$ *is a pullback, then* $\begin{tikzcd} \mathrm{Sub}(D) \arrow[r] \arrow[d] & \mathrm{Sub}(B) \arrow[d] \\ \mathrm{Sub}(C) \arrow[r] & \mathrm{Sub}(A) \end{tikzcd}$ *is a pushout.*

Proof. Just use the distributivity, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for Heyting algebras. □

COROLLARY 2.55 for topoi. *If* E *is injective then* $(-, E)$ *carries intersections (that is, monomorphic pullbacks) into pushouts.*

Proof. If E is injective, then $E \rightarrowtail \Omega^E$ splits and it suffices to show that $(-, \Omega^E)$ carries monomorphic pullbacks into pushouts. But $(-, \Omega^E) \simeq (-\times E, \Omega)$ and $-\times E$ preserves (as in any category, any E) monomorphic pullbacks, and Proposition 2.54 says that $(-, \Omega)$ is as desired. □

PROPOSITION 2.56 for topoi. *Given* $\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] & D \end{tikzcd}$, *it is a pullback iff for injective objects* E, $\begin{tikzcd} (D, E) \arrow[r] \arrow[d] & (C, E) \arrow[d] \\ (B, E) \arrow[r] & (A, E) \end{tikzcd}$ *is a pushout.*

Proof. The family $\{(-, E)\}$, E injective, is collectively faithful

by Proposition 2.52 and hence reflects isomorphisms. Any family that reflects isomorphisms reflects the limits it preserves (any category). □

2.6. Sheaves

Let $j : \Omega \to \Omega$ be a closure operation as defined in the last section, that is $j^2 = j$ and j is a homomorphism with regard to $\Omega\times\Omega \xrightarrow{\cap} \Omega$ and $1 \xrightarrow{t} \Omega$. Given $B' \rightarrowtail B$, we will write $\overline{B'}$ for the result of applying (B, j) to $\mathrm{Sub}(B)$.

We will say that $B' \rightarrowtail B$ is *j-closed* if $\overline{B'} = B'$ and *j-dense* if $\overline{B'} = B$.

We say that A is *j-separated* if for all j-dense $B' \rightarrowtail B$, $(B, A) \to (B', A)$ is monic.

We say that A is a *j-sheaf* if for all j-dense $B' \rightarrowtail B$, $(B, A) \to (B', A)$ is an isomorphism.

A functor is *exact* if it preserves all finite limits and colimits.

THEOREM 2.61. The fundamental theorem of sheaves. *The full subcategories of j-separated objects and j-sheaves are reflective and each is cartesian-closed.*

The full subcategory of j-sheaves is a topos and its reflector is exact.

Proof. We fix j and drop the prefix "j-". If $B' \rightarrowtail B$ is dense then so is $B'\times C \to B\times C$ because if $\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow g \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$ is a pullback, then density is equivalent to $B \xrightarrow{g} \Omega \xrightarrow{j} \Omega = B \to 1 \xrightarrow{t} \Omega$, and

$$\begin{array}{ccc} B'\times C & \to & B\times C \\ \Big\downarrow & & \downarrow p \\ & & B \\ & & \downarrow g \\ 1 & \to & \Omega \end{array}$$

is a pullback. Hence if A is separated (sheaf) and $B' \rightarrowtail B$ dense then

$$\begin{array}{ccc} (B, A^C) & \to & (B', A^C) \\ \wr & & \wr \\ (B\times C, A) & \to & (B'\times C, A) \end{array},$$

and the horizontal maps are mono (iso), and A^C is separated (sheaf).

Let Sep_j be the full subcategory of separated objects and Sh_j be the full subcategory of sheaves.

We have just seen that Sep_j and Sh_j are cartesian-closed (except possibly for finite completeness) and hence by Proposition 1.31, when we know that Sep_j and Sh_j are reflective, we will know that the reflector preserves products.

Define $\Omega_j \to \Omega$ as the equalizer of 1 and j. Because j is idempotent, there exists $\Omega \to \Omega_j$ such that $\Omega \to \Omega_j \to \Omega = j$, $\Omega_j \to \Omega \to \Omega_j = 1$. Ω_j is injective.

LEMMA 2.611. *Ω_j is a sheaf.*

Proof. Let $B' \rightarrowtail B$ be dense, and $B' \to \Omega_j$ given. The injectivity of Ω_j yields
$$\begin{tikzcd} B' \arrow[r] \arrow[dr] & B \arrow[d] \\ & \Omega_j \end{tikzcd}.$$
We need only the uniqueness condition. Suppose $B' \to B \xrightarrow{f} \Omega_j = B' \to B \xrightarrow{g} \Omega_j$. Let
$$\begin{tikzcd} B_1 \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ 1 \arrow[r, "t"'] & \Omega_j \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_2 \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ 1 \arrow[r, "t"'] & \Omega_j \end{tikzcd}$$
be pullbacks. Then B_1 and B_2 are both closed. But $B_1 \cap B' = B_2 \cap B'$ and we have $\overline{B_1 \cap B'} = \overline{B}_1 \cap \overline{B}' = B_1$, $\overline{B_2 \cap B'} = B_2$ and $f = g$. □

COROLLARY 2.612. *For any A, Ω_j^A is a sheaf.* □

The definition of Sep_j and Sh_j easily says that both are closed under limits, Sep_j under subobjects.

The idempotence of j says that for $B' \subset B$, B' is dense in $\overline{B}'$

Note that the closed subobjects of B are in natural correspondence

with (B, Ω_j) .

LEMMA 2.613. *A is separated iff $A \xrightarrow{\Delta} A\times A$ is a closed subobject.*

Proof. If A is separated, let $\bar{A} \subset A\times A$ be the closure of Δ . There is at most one $\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ & \searrow_1 & \downarrow \\ & & A \end{array}$. $A \to \bar{A} \to A\times A \xrightarrow{p_1} A = A \to \bar{A} \to A\times A \xrightarrow{p_2} A$, hence $\bar{A} \to A\times A \xrightarrow{p_1} A = \bar{A} \to A\times A \xrightarrow{p_2} A$, and hence $\bar{A} \subset \mathrm{Eq}(p_1, p_2) = \Delta$. Thus $A = \bar{A}$.

Conversely, suppose $A \xrightarrow{\Delta} A\times A$ is closed. Let $B' \rightarrowtail B$ be dense and $B' \to B \xrightarrow{f} A = B' \to B \xrightarrow{g} A$. The equalizer of f, g can be constructed as the pullback $\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow\langle f,g\rangle \\ A & \xrightarrow[\Delta]{} & A\times A \end{array}$, hence E is closed. But $B' \subset E$ and $\bar{B}' \subset \bar{E} \subset B$; thus $E = B$ and $f = g$. □

LEMMA 2.614. *A is separated iff $A \to \Omega^A = A \to \Omega^A \xrightarrow{j^A} \Omega^A$.*

Proof. j^A yields a transformation $j : \mathrm{Rel}(-, A) \to \mathrm{Rel}(-, A)$ alternatively described as taking closures of subobjects of $-\times A$. The equation of the lemma is equivalent to:

$$(-, A) \to \mathrm{Rel}(-, A) = (-, A) \to \mathrm{Rel}(-, A) \xrightarrow{j} \mathrm{Rel}(-, A)$$

which may be tested on $1_A \in (A, A)$. The left side yields $A \xrightarrow{\Delta} A\times A$; the right side yields the closure of Δ and the last lemma yields the proof. □

LEMMA 2.615. *For any A the image of $A \to \Omega^A \xrightarrow{j^A} \Omega^A$ is the reflection of A into Sep_j .*

Proof. Let $A \to \hat{A}$ be the image of $A \to \Omega^A \to \Omega^A$. j^A factors as $\Omega^A \twoheadrightarrow \Omega_j^A \to \Omega^A$ and $\hat{A}$ is a subobject of Ω_j^A . Lemma 2.611 said that $\Omega_j^A \in \mathrm{Sep}_j$ and Sep_j is clearly closed under subobjects, hence $\hat{A} \in \mathrm{Sep}_j$.

If $B \in \mathrm{Sep}_j$ then by the last lemma $B \to \hat{B}$ is an isomorphism. Given $f : A \to B$, $B \in \mathrm{Sep}_j$ we need only show that the outer rectangle of

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \Omega^A \arrow[r, "j^A"] \arrow[d, "\exists_f"] & \Omega^A \arrow[d, "\exists_f"] \\ B \arrow[r] & \Omega^B \arrow[r, "j^B"'] & \Omega^B \end{tikzcd}$$

commutes to obtain $\begin{tikzcd} A \arrow[r] \arrow[dr] & \hat{A} \arrow[d] \\ & B \end{tikzcd}$. Equivalently we consider

$$\begin{tikzcd} (-, A) \arrow[r] \arrow[d, "{(-, f)}"'] & \mathrm{Rel}(-, A) \arrow[r, "j"] \arrow[d, "{\mathrm{Rel}(-, f)}"] & \mathrm{Rel}(-, A) \arrow[d, "{\mathrm{Rel}(-, f)}"] \\ (-, B) \arrow[r] & \mathrm{Rel}(-, B) \arrow[r, "j"'] & \mathrm{Rel}(-, B) \end{tikzcd} .$$

The right-hand square does not commute. Given $R \subset X \times A$ and chasing clockwise we obtain $\mathrm{Im}(\overline{R} \to X \times B)$ and in the other direction $\overline{\mathrm{Im}(R \to X \times B)}$. Inverse, not direct, images preserve closures. But the fact for inverse images yields that $\mathrm{Im}(\overline{R} \to X \times B) \subset \overline{\mathrm{Im}(R \to X \times B)}$ and because direct images preserve order we also have $\mathrm{Im}(R \to X \times B) \subset \mathrm{Im}(\overline{R} \to X \times B)$. It suffices to show that when R is the graph of a map then $\mathrm{Im}(R \to X \times B) = \overline{\mathrm{Im}(R \to X \times B)}$. But if $R \subset X \times A$ is the graph of $g : X \to A$ then $\mathrm{Im}(R \to X \times B)$ is the graph of $X \xrightarrow{g} A \xrightarrow{f} B$ and the last lemma says precisely that graphs of maps into separated objects are closed. □

For the reflectivity of Sh_j it suffices to show it reflective in Sep_j .

LEMMA 2.616. *If* $A' \rightarrowtail A$ *is closed,* $A \in \mathrm{Sh}_j$ *then* $A' \in \mathrm{Sh}_j$.

Proof. Given $B \rightarrowtail B$ dense and $B' \to A$, let $\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$ commute and $\begin{tikzcd} B'' \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$, a pullback. Then B'' is closed, $B' \subset B''$ and hence $B'' = B$. □

We'll say that a separated object is *absolutely closed* if whenever it appears as a subobject it appears as a closed subobject.

LEMMA 2.617. *Sheaves are absolutely closed and separated. Absolutely closed objects are sheaves.*

Proof. Given $A \rightarrowtail B$, $A \in \mathrm{Sh}_j$, let $\bar{A} \rightarrowtail B$ be the closure. There exists $\bar{A} \to A$ such that
$$\begin{tikzcd} \bar{A} \arrow[d] \arrow[dr] & \\ A \arrow[r] & B \end{tikzcd}$$
, that is, $A \subset \bar{A} \subset A$.

Conversely, if A is separated then it appears as a subobject of $\Omega_j^A \in \mathrm{Sh}_j$ and if A is absolutely closed, then by the last lemma, A is a sheaf. □

LEMMA 2.618. Ω_j *satisfies the* Ω*-condition for* Sh_j. □

LEMMA 2.619. *Given* $A \rightarrowtail B$, B *a sheaf, then the closure of* A *in* B *is the reflection of* A *in* Sh_j.

Proof. Let $\bar{A} \to B$ be the closure. Lemma 2.616 says that $\bar{A} \in \mathrm{Sh}_j$. For any $C \in \mathrm{Sh}_j$, $(\bar{A}, C) \approx (A, C)$ because $A \to \bar{A}$ is dense. □

Because every separated A can be embedded in a sheaf (for example, Ω_j^A) we obtain that every separated A has a reflection in Sh_j. Composing the two reflections, the reflection of an arbitrary A is the closure of the image of $A \to \Omega_j^A$.

We saw at the beginning of the proof that Sh_j is closed under exponentiation. Lemma 2.618 says that Sh_j has an Ω, and we have just seen that it is reflective, hence finitely cocomplete. Thus Sh_j is a topos. Note that for E injective in Sh_j, that $E \rightarrowtail \Omega_j^E$ retracts and is injective in the ambient category T.

Since the reflection $R : T \to \mathrm{Sh}_j$ preserves products it suffices for exactness to show that it preserves equalizers. Given $f, g : A \to B$ in any category, the equalizer of f, g may be constructed as the pullback of

$$\begin{tikzcd} & A \arrow[d, "\langle 1,g \rangle"] \\ A \arrow[r, "\langle 1,f \rangle"'] & A \times B \end{tikzcd},$$

hence it suffices to show that $R : T \to \mathrm{Sh}_j$ preserves monomorphic pullbacks.

Given a pullback $\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & D \end{array}$ in T , it suffices according to Proposition 2.56 to show that for any injective $E \in \mathrm{Sh}_j$, it is the case that $\begin{array}{ccc} (RD, E) & \to & (RC, E) \\ \downarrow & & \downarrow \\ (RB, E) & \to & (RA, E) \end{array}$ is a pushout. But that diagram is isomorphic to $\begin{array}{ccc} (D, E) & \to & (C, E) \\ \downarrow & & \downarrow \\ (B, E) & \to & (A, E) \end{array}$ and since E is injective in T , Corollary 2.55 provides the finish. □

Consider $S^{\to}$. $\Omega = (3 \to 2)$. Let $U \subset 1$ be $(0 \to 1) \subset (1 \to 1)$ and j the closure operator that sends $(A' \to B') \subset (A \to B)$ to $(A' \to B) \subset (A \to B)$. Then the j-sheaves are of the form $(A \to 1)$. $\Omega_j = (2 \to 1)$. The reflection of $(A \to B)$ is $(A \to 1)$. The reflection of Ω is $(3 \to 1)$ *not* Ω_j .

Recall that $\Omega \xrightarrow{\neg\neg} \Omega$ is a closure operator.

PROPOSITION 2.62 for topoi. $\mathrm{Sh}_{\neg\neg}$ *is boolean.*

Proof. $1 + 1 \xrightarrow{\binom{t}{f}} \Omega$ is easily seen to be dense, hence becomes an isomorphism in $\mathrm{Sh}_{\neg\neg}$. □

COROLLARY 2.63. *Every non-degenerate topos has an exact co-continuous functor to a non-degenerate boolean topos.*

Proof. 0 is a $\neg\neg$-sheaf. □

Following the remarks at the end of Section 2.4, let $U \subset 1$ and consider $\Omega \xrightarrow{j} \Omega$ the closure operator so that for $B' \subset B$, $\overline{B'} = B' \cup (B \times U)$. The reflector $T \to \mathrm{Sh}_j$ sends $0 \to U$ tc an isomorphism, that is, U is sent to the zero-object. Sh_j is degenerate iff $U = 1$.

COROLLARY 2.64 for topoi. *Given* $x, y : 1 \to B$, $x \neq y$, *there exists a boolean topos* $\mathcal{B}$, *exact co-continuous* $F \to \mathcal{B}$ *that separates*

x, y .

Proof. Let $U \subset 1$ be the equalizer of x, y . Then for j as described above, $T \to \mathrm{Sh}_j$ sends $x, y : 1 \to B$ to a pair of maps with 0 as equalizer. Now apply Corollary 2.63. □

THEOREM 2.65 (the plentitude of boolean topoi). *For every* $f, g : A \to B$, $f \neq g$, *there exists a boolean topos* $\mathcal{B}$, *exact co-continuous* $T \to \mathcal{B}$ *that separates* f, g .

Every small topos can be exactly embedded in a boolean topos.

Proof. $T \to T/A$ sends f, g to
$$\begin{tikzcd} A\times A \arrow[rr, "f\times 1"] \arrow[dr] & & B\times A \arrow[dl] \\ & A & \end{tikzcd} , \quad \begin{tikzcd} A\times A \arrow[rr, "g\times 1"] \arrow[dr] & & B\times A \arrow[dl] \\ & A & \end{tikzcd} . \quad \begin{matrix} A \\ \downarrow \\ A \end{matrix}$$
is the terminator for T/A and

$$A \xrightarrow{\Delta} A\times A \xrightarrow{f\times 1} B\times A \neq A \xrightarrow{\Delta} A\times A \xrightarrow{g\times 1} B\times A .$$

Hence we can apply Corollary 2.64 to obtain $T/A \to \mathcal{B}$ as desired. □

Let H be a partially ordered set. We define $j : \Omega \to \Omega$ in $S^{H^{\mathrm{op}}}$ as follows: for $x \in H$, $\Omega(x) = \{A \subset H \mid u \leq v \in A \Rightarrow u \in A \Rightarrow u \leq x\}$; define $j_x(A)$ to be the set of all $u \in H$ such that there exists $A' \subset A$ with the property that $\forall_{z \in H}[\forall_{v \in A'}(v \leq z) \Rightarrow (v \leq u)]$. (If $\cup A'$ exists, then this simply says $u \leq \cup A'$.) j is easily seen to be idempotent and inflationary.

It is natural and it preserves intersections iff H is very distributive, for example, a Heyting algebra. For H a Heyting algebra $(1, \Omega_j)$ is its completion. Hence for H a complete Heyting algebra $(1, \Omega_j) \simeq H$ and $\Omega_j(x) = \{u \in H \mid u \leq x\}$. That is, $j_x(A) = \cup A$.

For H the lattice of non-empty open sets in a space X , $(1, \Omega_j)$ is the lattice of all open sets, Sh_j is the classically defined category of sheaves.

The definition of a Grothendieck topology on A is almost the definition of a closure operator in $S^{A^{\mathrm{op}}}$. Closure operators are a bit more general.

2.7. Insoluble topoi, or how topoi aren't as complete as you'd like

In a powerful way, topoi are "internally complete". One may, for example, define a map $\Omega^{\Omega^A} \to \Omega^A$ which acts as a union operator. Hence given any $B \subset \Omega^A$ we obtain a subobject $\cup B \subset A$ $\Big($via the functions $\mathrm{Sub}\left(\Omega^A\right) \simeq \left(1, \Omega^{\Omega^A}\right) \to \left(1, \Omega^A\right) = \mathrm{Sub}(A)\Big)$ which has all of the *properly stated* properties of a union.

There is a catch. $\cup B$ is not necessarily the least upper bound of the subobjects of A described by $(1, B) \subset \left(1, \Omega^A\right) \simeq \mathrm{Sub}(A)$. The type of completeness we often need is just that; that is, given $B \subset \Omega^A$, a least upper bound for $(1, B) \subset \mathrm{Sub}(A)$. $\cup B$ fails miserably. For the singleton map $A \to \Omega^A$, $\cup A = A$. But the least upper bound of $(1, A) \subset \mathrm{Sub}(A)$ would be the least subobject $\hat{A}$ of A such that $(1, \hat{A}) \simeq (1, A)$. To say that $\hat{A} = A$, all A, is equivalent to saying that 1 is a generator (that is, a well-pointed topos as defined in the next section).

If for every A there were such $\hat{A} \subset A$, then $\cup \hat{B}$ is the least upper bound of $(1, B) \subset \mathrm{Sub}(A)$. (We shall not use this construction and hence will not prove it.) The existence of $\hat{A} \subset A$ is elementary but independent of the axioms for topoi. Indeed, as we shall show, it is not equivalent to any essentially algebraic axioms.

We will call T *solvable* if for every A there exists $\hat{A} \subset A$ such that $(1, \hat{A}) \simeq (1, A)$ and for all $B \subset A$ such that $(1, B) \simeq (1, A)$ it is the case that $\hat{A} \subset B$.

PROPOSITION 2.71. *Not all topoi are solvable.*

Proof. We saw at the end of the last section that every complete Heyting algebra appears as $(1, \Omega)$. Let T_1 be such that $(1, \Omega)$ is the order-type of the unit interval and let $T_2 \subset T_1$ be a countable elementary submodel. In T_2, $(1, \Omega)$ is a dense ordering and countable, hence there exists $\{U_n \subset 1\}_n$, $U_n \subset U_{n+1}$ such that $\cup U_n$ does not exist. Let $V \subsetneqq 1$ be such that $U_n \subset V$ all n.

Consider the direct system $\mathcal{T}_2 / \prod_{i=1}^{n} (1+U_i)$ induced by the obvious projection maps $\prod^{n+1}(1+U_i) \to \prod^{n}(1+U_i)$ and let T be the direct limit. The essentially algebraic nature of topoi insures that T is a topos. We can give a more elementary description of T as follows:

objects: $\left\langle n,\ A \to \prod^{n}(1+U_i)\right\rangle$, $A \in T_2$;

maps: from $\left\langle n,\ A \xrightarrow{a} \prod^{n}(1+U_i)\right\rangle$ to $\left\langle m,\ B \xrightarrow{b} \prod^{m}(1+U_i)\right\rangle$,

are equivalence classes of pairs $\langle k, f\rangle$ where

$$f : A \times \prod_{i=n+1}^{k} (1+U_i) \to B \times \prod_{i=m+1}^{k} (1+U_i)$$

such that

$$A \times \prod_{i=n+1}^{k} (1+U_i) \xrightarrow{f} B \times \prod_{i=m+1}^{k} (1+U_i) \xrightarrow{b\times 1} \prod_{i=1}^{m}(1+U_i) \times \prod_{i=m+1}^{k} (1+U_i)$$
$$\tilde{\to} \prod_{i=1}^{k}(1+U_i) = A \times \prod_{i=n+1}^{k} (1+U_i) \xrightarrow{a\times 1} \prod_{i=1}^{n}(1+U_i) \times \prod_{i=n+1}^{k} (1+U_i) \tilde{\to} \prod_{i=1}^{k}(1+U_i) .$$

The equivalence relation is generated by $\langle k, f\rangle \equiv \langle k+1, f\times 1\rangle$.

The notation is eased by replacing $T_2 / \prod^{n}(1+U_i)$ with its image T/n in T. Hence $T/0 = T_2$, $\bigcup T/n = T$, $T(A, B) = \varinjlim T/n(A, B)$.

For $A \in T/n$, $B \in T/0$,

$$T(A, B) \simeq \varinjlim T/0\left(A \times \prod_{i=n+1}^{k} (1+U_i),\ B\right)$$

(an easy verification). Thus $T(1, 1+V) \simeq \varinjlim T/0\left(\prod_{i=1}^{k}(1+U_i),\ 1+V\right)$. Note that $\prod^{k}(1+U_i)$ is a coproduct of 2^k subobjects of 1, one of which *is*

one, one of which is U_k , the other smaller than U_k . Hence the union of the images of $T/0\left\{\prod^{k}(1+U_i),\ 1+V\right\}$ is $1 + U_k$.

Suppose $W \rightarrowtail 1+V \in T$ were such that $(1, W) \simeq (1+V)$. We shall show that there exists $W' \to 1+V$ with the same property, and such that $W \not\subset W'$. $W \to 1+V$ in T appears as $A \times \prod^{k}(1+U_i) \xrightarrow{g} 1+V$ some k . For all n , $1+U_n \subset \mathrm{Im}(g) \subset 1+V$. Let $B \subset V$ be such that $U_n \subset B$ all n , but $1+B \subsetneq \mathrm{Im}(g)$. Then $1+B \to 1+V$ as a subobject in T is contained in $W \to 1+V$. □

3. Well-pointed topoi

A topos is *well-pointed* if it is non-degenerate and if 1 is a generator.

PROPOSITION 3.11 for well-pointed topoi.

$(1, \Omega) = 2$;

$1 + 1 \simeq \Omega$;

$(1, -)$ *preserves coproducts, epimorphisms, epimorphic families, and pushouts of monomorphisms;*

$A \not\simeq 0 \Rightarrow A$ *is injective;*

$A \not\simeq 0, 1 \Rightarrow A$ *is a cogenerator.*

Proof. For $A \not\simeq 0$ there are at least two maps from A to Ω , hence there exists $1 \to A$.

Let $U \subset 1$. If $U \not\simeq 0$ then there exists $1 \to U$ and $1 \to U \to 1 = 1_1$ forces $U \to 1$ to be epic. Hence $(1, \Omega) = 2$.

$1 + 1 \to \Omega$ is always monic. Because $(1, 1+1) \approx (1, \Omega)$ and 1 reflects isomorphisms, $1 + 1 \simeq \Omega$.

For $(1, A) + (1, B) \simeq (1, A+B)$ use Corollary 2.33.

Given $A \twoheadrightarrow B$ and $1 \to B$ let $\begin{array}{ccc} C & \to & 1 \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$ be a pullback. By Corollary

2.32, $C \to 1$ is epic; thus $C \not\cong 0$ and there exists $1 \to C$ yielding

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[d] \\ A \arrow[r] & B \end{tikzcd} .$$

Given any family $\{A_i \to B\}$ collectively epimorphic, and $1 \xrightarrow{f} B$, we can view $\{A_i \to B\}$ as a collection of objects in T/B which collectively cover the terminal object. Applying $f^{\#}$ we obtain a collection of objects that do the same. Hence for some i, $f^{\#}(A_i \to B) \neq 0$ and $\{(1, A_i) \to (1, B)\}$ is collectively epimorphic.

Given a pushout $\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$ we use the booleanness to write $B = A + A'$ and we see that $\begin{tikzcd} A \arrow[r] \arrow[d] & A+A' \arrow[d] \\ C \arrow[r] & C+A \end{tikzcd}$ is a pushout, clearly preserved by $(1, -)$.

Given $A \rightarrowtail B$, $A \not\cong 0$, again write $B = A + A'$ and choose $1 \to A$. Then $B = A+A' \to A+1 \to A$ is a right-inverse for $A \to B$. □

PROPOSITION 3.12. *If* $\mathcal{B}$ *is a boolean topos then it is well-pointed iff for all* $A \in \mathcal{B}$, $A \not\cong 0$ *there exists* $1 \to A$.

Proof. Let $f, g : B \to C$, $f \neq g$. Let $E \subset B$ be the equalizer of f, g, $A \subset B$ the complement of E in B. Hence if there exists $1 \to A$ then there exists $1 \to B$ that distinguishes f, g. □

3.2. The plentitude of well-pointed topoi

A *logical morphism* of topoi is a functor that preserves all the structure.

THEOREM 3.21. *For every small boolean topos* $\mathcal{B}$ *and* $A \in \mathcal{B}$, $A \not\cong 0$ *there exists a well-pointed* $\hat{\mathcal{B}}$ *and logical* $T : \mathcal{B} \to \hat{\mathcal{B}}$, $TA \not\cong 0$. T *preserves epimorphic families.*

Proof. We first show:

LEMMA 3.211. *For every boolean topos* $\mathcal{B}$ *and* $A \in \mathcal{B}$, $A \not\cong 0$ *there exists a topos* $\mathcal{B}'$, *logical* $T : \mathcal{B} \to \mathcal{B}'$, $T(A) \not\cong 0$. T *preserves epimorphic families; and for all* $B \in \mathcal{B}$ *either* $TB \simeq 0$ *or there exists*

$1 \to TB$.

Proof of lemma. Well-order the objects of $\mathcal{B}$, taking A as first object. We construct an ordinal sequence of topoi and logical morphisms as follows:

$$\mathcal{B}_0 = \mathcal{B} .$$

If $T : \mathcal{B}_0 \to \mathcal{B}_\alpha$ *has the described property, terminate the sequence at* α .

If, on the other hand, there exist $B \in \mathcal{B}_0$ *such that* $TB \not\approx 0$ *and* $(1, TB) = \emptyset$ *then take* B *to be the first such and define* $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha/TB$.

If β *is a limit ordinal, and* $\mathcal{B}_\alpha$ *is defined for all* $\alpha < \beta$ *, then* $\mathcal{B}_\beta$ *is the colimit of the* $\mathcal{B}_\alpha$*'s* .

The essentially algebraic nature of topoi insures that $\mathcal{B}_\beta$ is a topos.

The functor $\mathcal{B}_\alpha \to \mathcal{B}_{\alpha+1}$ carries TB to an object with a map from 1 . Moreover for every $C \in \mathcal{B}_0$ such that $\mathcal{B}_\alpha(1, TC) \neq \emptyset$, $\mathcal{B}_\alpha(1, T'C) \neq \emptyset$ and eventually the sequence must terminate. $\square$

Now, for the theorem, define a sequence on the finite ordinals by

$$\mathcal{B}_0 = \mathcal{B} ,$$

$$\mathcal{B}_{n+1} = \mathcal{B}'_n ,$$

(as defined in the lemma) and $\hat{\mathcal{B}} = \varinjlim \mathcal{B}_n$.

$\hat{\mathcal{B}}$ is boolean because $\mathcal{B} \to \hat{\mathcal{B}}$ is logical and $1 + 1 \simeq \Omega$ in $\mathcal{B}$ implies the same in $\hat{\mathcal{B}}$. Proposition 3.12 says that $\hat{\mathcal{B}}$ is well-pointed.

That T preserves epimorphic families follows from the fact that colimits of such functors are such functors. $\square$

COROLLARY 3.22. *Every small boolean topos can be logically embedded in a product of well-pointed topoi, and the embedding preserves epimorphic families.*

Proof. For exact $T : \mathcal{B} \to \mathcal{B}'$ between boolean topoi, T is faithful iff $TA \simeq 0 \Rightarrow A \simeq 0$. □

Composing with Theorem 2.65 we obtain:

THEOREM 3.23. *Every small topos can be exactly embedded in a product of well-pointed topoi and the embedding preserves epimorphic families.*

Composing with $(1, -)$ and using Proposition 3.11 we obtain:

THEOREM 3.24. *For every small topos* $\mathcal{T}$ *there exists faithful* $T : \mathcal{T} \to \Pi\mathcal{S}$.

T preserves all finite limits, coproducts, epimorphisms, epimorphic families, and pushouts of monomorphisms.

3.3. Metatheorems

By the *universal theory of exactness* of a category we mean all true universally quantified sentences using the predicates of composition, finite limits and colimits. By the *universal Horn theory of exactness* we mean all the universally quantified Horn sentences in the predicates of exactness, that is, sentences of the form $A_1 \wedge A_2 \wedge \ldots \wedge A_n \Rightarrow A_{n+1}$ where each A_i says either that something commutes, or is a limit, or is a colimit. By theories of *near exactness* we mean those using the predicates of composition, finite limits, coproducts, epimorphisms, and pushouts of monomorphisms.

As easy corollaries of Theorem 3.21 through Theorem 3.24 we obtain:

METATHEOREM 3.31. *The universal Horn theory of near exactness true for the category of sets is true for any topos.*

The universal Horn theory of exactness true for all well-pointed topoi is true for all topoi.

The universal Horn theory of topoi true for all well-pointed topoi is true for all boolean topoi.

The universal theory of topoi true for all well-pointed topoi is true for all boolean topoi in which $(1, \Omega) = 2$.

We will show later that there do exist universal Horn sentences in

exactness predicates true for S but not true for all well-pointed topoi.

An *equivalence relation* on A is a relation $E \subset A{\times}A$ satisfying the usual axioms.

COROLLARY 3.32 for topoi. *Every equivalence relation is effective; that is, given* $E \subset A{\times}A$ *there exists* $A \to B$ *such that* $\begin{array}{ccc} E & \to & A \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$ *is a pullback.*

Proof. We get rid of the existential quantifier by defining $A \to B$ to be the coequalizer of the two maps from E to A. We note that the statement is in the universal Horn theory of exactness and it suffices to prove it in well-pointed topoi.

Accordingly, let $\begin{array}{ccc} E' & \to & A \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$ be a pullback. If $E \neq E'$ there exists $1 \to E'$ that can not be factored through $E \to E'$. Hence there exists $x, y : 1 \to A$ such that $1 \xrightarrow{x} A \to B = 1 \xrightarrow{y} A \to B$ but $1 \xrightarrow{\langle x,y \rangle} A{\times}A$ not in E. Let A' be the complement of $\mathrm{Im}(x) \cup \mathrm{Im}(y)$ and define $A \xrightarrow{g} 1{+}1{+}1$ by

$$A' \to A \xrightarrow{g} 1{+}1{+}1 = A' \to 1 \xrightarrow{u_1} 1{+}1{+}1 ,$$

$$1 \xrightarrow{x} A \xrightarrow{g} 1{+}1{+}1 = 1 \xrightarrow{u_2} 1{+}1{+}1 ,$$

$$1 \xrightarrow{y} A \xrightarrow{g} 1{+}1{+}1 = 1 \xrightarrow{u_3} 1{+}1{+}1 .$$

Then $E \rightrightarrows A \xrightarrow{g} 1{+}1{+}1$ equalizes and there must be $\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & 1{+}1{+}1 & \end{array}$, a contradiction. □

COROLLARY 3.33 for topoi. *Every epimorphism is a coequalizer.*

Proof. We again get rid of the existential quantifier by stating it as: every epimorphism is the coequalizer of its kernel-pair. (The kernel-pair of $A \to B$ is the pullback $\begin{array}{ccc} E & \to & A \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$.) It suffices to prove it in well-pointed topoi. Given $A \twoheadrightarrow B$ let $A \to B'$ be the coequalizer of

$E \rightrightarrows A$. It suffices to show that $B' \to B$ is monic. Let $1 \xrightarrow{x'} B' \to B = 1 \xrightarrow{y'} B' \to B$ and find $x, y : 1 \to A$ so that $1 \xrightarrow{x} A \to B' = x'$, $1 \xrightarrow{y} A \to B' = y'$. Then $1 \xrightarrow{\langle x,y \rangle} A{\times}A$ lies in $E \subset A{\times}A$ and $x' = y'$. □

COROLLARY 3.34 for topoi. *If* $\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & D \end{array}$ *is a pullback and* $C{+}B \to D$ *is epi, then it is a pushout.*

Proof. The statement lies in the universal Horn theory of near exactness and it suffices to prove it in S , an easy matter. □

3.4. Solvable topoi

Given a topos T and $A \in T$ we'll say that A is a *well-pointed object* if the maps from 1 to A are jointly epimorphic. Define $T_p \subset T$ to be the full subcategory of well-pointed objects. In Section 2.7 we defined a solvable topos, which definition is equivalent to the coreflectivity of T_p .

PROPOSITION 3.41 for topoi. T_p *is closed under finite products.*

Proof. Suppose $A, B \in T_p$. We may assume that neither A nor B is 0 . Let $f, g : A{\times}B \to C$, $f \neq g$. We wish to find $1 \to A{\times}B$ such that $1 \to A{\times}B \xrightarrow{f} C \neq 1 \to A{\times}B \xrightarrow{g} C$. Let $\bar{f}, \bar{g} : A \to C^B$ correspond to f, g , and let $1 \xrightarrow{x} A$ be such that $1 \xrightarrow{x} A \xrightarrow{\bar{f}} C^B \neq 1 \xrightarrow{x} A \xrightarrow{\bar{g}} C^B$. Because $B \nsim 0$ there exists $1 \to B$, and hence $B \to 1$ is epi, which yields:

$$1{\times}B \xrightarrow{x\times 1} A{\times}B \xrightarrow{\bar{f}\times 1} C^B{\times}B \neq 1{\times}B \xrightarrow{x\times 1} A{\times}B \xrightarrow{\bar{g}\times 1} C^B{\times}B .$$

By following with the evaluation $C^B{\times}B \to C$ we obtain

$$B \to A{\times}B \xrightarrow{f} C \neq B \to A{\times}B \xrightarrow{g} C .$$

Finally, let $1 \to B$ be such as to separate these two maps from B to C . □

By a *two-valued topos* we mean $(1, \Omega) = 2$.

PROPOSITION 3.42 for solvable two-valued topoi. *T_p is a well-pointed topos. Subobjects of T_p objects are in T_p, hence $T_p \to T$ is exact.*

Proof. Because T_p is closed under products and coreflective, exponentiation is effected by exponentiating in T and then coreflecting.

Given $f : A \to B$, $B \in T_p$, consider $f^{\#} : T/B \to T/A$. The collection of T/B-objects, $\{1 \to B\}$ are such that their maps to the terminator form a jointly epimorphic family. Hence $f^{\#}$ applied to that family yields a similar family in T/A. (Thus in any topos, the maps from subterminators to A form a jointly epimorphic family.) Because 1 has only two subobjects, $A \in T_p$.

Given $A \rightarrowtail B$ in T_p let $A' \to B$ be the negation of A as defined in T. We wish to show that $A \cup A' = B$, for such implies that $1 + 1$ satisfies the Ω condition in T_p. It suffices to show that for any $1 \to B$ there exists either $1 \to A \to B = 1 \to B$ or $1 \to A' \to B = 1 \to B$. If neither, then $(A' \cup \mathrm{Im}(1 \to B)) \cap A = 0$ and the maximality of A' yields a contradiction. $\square$

3.5. Topoi exactly embeddable in well-pointed topoi

We have shown that every topos is exactly embeddable in a product of well-pointed topoi, and a residual question presents itself: which are exactly embeddable in a single well-pointed topos? Because of the elementary nature of the latter (they are closed under ultra-products), we are asking which topoi have the universal exactness theory of well-pointed topoi. For example: $(U \rightarrowtail 1) \Rightarrow (0 \twoheadrightarrow U) \vee (U \twoheadrightarrow 1)$. That is, such topoi must be such that $(1, \Omega) = 2$.

It suffices to show for each n and

$$\left\langle A_1 \xrightarrow{f_1, g_1} B_1,\ A_2 \xrightarrow{f_2, g_2} B_2,\ \dots,\ A_n \xrightarrow{f_n, g_n} B_n \right\rangle,\quad f_i \neq g_i,\quad i = 1, 2, \dots, n$$

that there is a well-pointed topos T' and exact $T : T \to T'$ such that

$T(f_i) \neq T(g_i)$, $i = 1, 2, \ldots, n$. (Ultra-products again). Let

$A = A_1 \times A_2 \times \ldots \times A_n$ and $E_i \subset A$ the equalizer of $A \to A_i \xrightarrow{f_i, g_i} B_i$.

$E_i = A_1 \times \ldots \times A_{i-1} \times E_i' \times A_{i+1} \times \ldots \times A_n$ where E_i' is the equalizer of f_i, g_i . Let $E = E_1 \cup \ldots \cup E_n$. If $E \neq A$ we know that there exists $T : \mathcal{T} \to \mathcal{T}'$, $\mathcal{T}'$ well-pointed, such that $T(E) \neq T(A)$ and hence $T(f_i) \neq T(g_i)$.

It is a statement in the universal exactness theory of well-pointed topoi that if each $E_i' \neq A_i$ then $E \neq A$. Hence such is a necessary condition for exact embeddability into a well-pointed topos. We can make it elementary by noticing that the case for arbitrary n follows from the case $n = 2$:

If $(A' \times B) \cup (A \times B') = A \times B$ *then either* $A' = A$ *or* $B' = B$.

Further reductions can occur by replacing A with A/A' , the pushout of
$$\begin{array}{ll} A' & \to A \\ \downarrow & \\ 1 & \end{array} .$$

If $(1 \times B) \cup (A \times 1) = A \times B$ *then either* $A = 1$ *or* $B = 1$.

Together with $(1, \Omega) = 2$ this can be seen to be sufficient.

Finally, if one considers $1/\mathcal{T}$, that is, the category whose objects are of the form $1 \to A$ and whose maps are of the form
$$\begin{tikzcd} & A \arrow[d] \\ 1 \arrow[ur] \arrow[r] & B \end{tikzcd}$$
, define $(1 \to A) \vee (1 \to B)$ as the coproduct in $1/\mathcal{T}$, $(1 \to A) \wedge (1 \to B)$ as the cokernel of $(1 \to A) \vee (1 \to B) \to (1 \to A \times B)$, $(-\wedge-$ has a right-adjoint, namely exponentiation), then the condition for exact embeddability into a well-pointed topos is that the half-ring of isomorphism types with $\vee$ as addition and $\wedge$ as multiplication is without zero-divisors.

4. The first order calculus of a topos

Let L be a vocabulary of predicates and operators; that is, the objects of L are either pairs $\langle P, n \rangle$ or $\langle f, n \rangle$ where we call P an n-ary predicate, f an n-ary operator.

An *interpretation* of L in a topos T is an object $B \in T$, a subobject $\overline{P} \subset B^n$ for each $\langle P, n \rangle \in$ L, a map $\overline{f} : B^n \to B$ for each $\langle f, n \rangle \in$ L.

Given any derived n-ary predicate or operator using the vocabulary of L and the classical logical connections and quantifiers, we wish to stipulate a subobject of B^n. The definition is recursive. The rules for defining maps from expressions in the L-operators are well known. Given two n-ary predicates P, Q already assigned values in $\mathrm{Sub}(B^n)$, we can easily define $\overline{P \wedge Q}$, $\overline{P \vee Q}$, $\overline{P \to Q}$ as a subobject in B^n.

Given an n-ary predicate $P(x_1, \ldots, x_n)$ and operators $f_1, \ldots, f_n$ each m-ary, we define $\overline{P(f_1, \ldots, f_n)}$ as the pullback $\begin{array}{ccc} \square & \to & B^m \\ \downarrow & & \downarrow \\ \overline{P} & \to & B^n \end{array}$ where $B^m \to B^n$ is the obvious.

Given an n-ary $P(x_1, \ldots, x_n)$ modeled as $\overline{P} \subset B^n$, we define $\overline{\exists_{x_n} P(x_1, \ldots, x_n)}$ as the image of $\overline{P} \to B^n \to B^{n-1}$. For $\forall$ we need:

PROPOSITION 4.11 for topoi. *Given $g : A \to B$ and $A' \subset A$, there exists a maximal $B' \subset B$ such that $g^{-1}(B') \subset A'$. Such B' is called $\forall_g A'$.*

Proof. $B' \to B$ is $\Pi_g(A' \to A)$. $\square$

We define $\overline{\forall_{x_n} P(x_1, \ldots, x_n)}$ as the maximal subobject in B^{n-1} whose inverse image is contained in $\overline{P}$.

In this manner we obtain for each sentence S (that is, no unquantified variables) a subobject of B^0, that is, an element of $(1, \Omega)$. We shall call such the *truth value* of the sentence, $t(S)$.

Given any set T of sentences, we say that an interpretation of L is a *model* of T if every sentence in T has truth value $1 \xrightarrow{t} \Omega$.

(Yes, that's why it's called "t".) Given sentences S_1, S_2 and a topos T we say that S_1 *implies* S_2 in T if every model of S_1 is a model of S_2, and denote same by $S_1 \models_T S_2$. We say that S_1 *strongly implies* S_2 in T if for every interpretation of L, $t(S_1) \le t(S_2)$, and denote same by $\models\!\!\models_T$. Note that $S_1 \models\!\!\models_T S_2$ is equivalent to $1 \models_T (S_1 \to S_2)$. Given any language L and sentences S_1, S_2 we say that S_1 *semantically implies* S_2 if for all topoi T, $S_1 \models_T S_2$. We denote same by $S_1 \models_* S_2$.

PROPOSITION 4.12. *If* $S_1 \models_* S_2$ *then for all* T, $S_1 \models\!\!\models_T S_2$.

Proof. Let $B \in T$ be an interpretation of L in T and suppose $t(S_1) \not\le t(S_2)$. Then in $T/t(S_1)$ we obtain a model of S_1 that is not a model of S_2. □

The definition of semantic implication reduces a host of assertions in intuitionistic logic to exercises in classical logic:

PROPOSITION 4.13. *For fixed* L, $\models_*$ *is recursively enumerable.* $\models_*$ *obeys Craig's interpolation theorem. If every finite subset of a given* T *has a model in some topos, then so does* T. □

For each monoid M, we obtain an intuitionistic logic $\models_{S^M}$. We suspect that the connection between such logics and classes of monoids will be a fruitful pursuit.

Problem: For every topos T is $\models_T$ the same partial ordering as $\models_{S^M}$ for some monoid M?

4.2. The boolean case

If we restrict our interpretation of a language to boolean topoi, we can replace $P \to Q$ with $\neg P \vee Q$ and $\forall$ with $\neg\exists\neg$. The advantage is that $\vee, \wedge, \neg$ and $\exists$ are all definable using only the predicates of

near-exactness. ($\overline{P} = \neg\overline{Q}$ iff $\overline{P} + \overline{Q} \to B^n$ is an isomorphism.)

More precisely, let

$$\mathsf{L} = \{\langle P_1, n\rangle, \ldots, \langle P_a, n_a\rangle, \langle f_1, m_1\rangle, \ldots, \langle f_b, m_b\rangle\} .$$

Let $J_{\mathcal{B}}$ be the set of interpretations of L in a boolean topos $\mathcal{B}$. The elements of $J_{\mathcal{B}}$ are of the form

$$\left\langle B, \overline{P}_1 \subset B^{n_1}, \ldots, \overline{P}_a \subset B^{n_a}, f_1 : B^{m_1} \to B, \ldots, f_b : B^{m_b} \to B\right\rangle .$$

PROPOSITION 4.21. *Given any elementary sentence S there exists a universally quantified Horn formula F in the predicates of near-exactness such that for each boolean $\mathcal{B}$ and $\langle B, \overline{P}_1, \ldots, \overline{f}_b\rangle \in J_{\mathcal{B}}$ it is the case that $F(B, \overline{P}_1, \ldots, \overline{f}_b)$ iff $\langle B, \overline{P}_1, \ldots, \overline{f}_b\rangle$ is a model of S. Also, there is an existentially quantified conjunction of near-exactness predicates $G(B, \overline{P}_1, \ldots, \overline{f}_b)$ with the same property.*

Proof. There is a tree. Its root is S, each branch a wff (well formed formula), each leaf a single variable, each branch point (we'll allow degenerate branch points) marked with either $P_1, \ldots, P_a, f_1, \ldots, f_b, \wedge, \neg, =$ or $\langle \exists, n\rangle$; and, if a branch point is marked P_i then n_i branches lead into it, they are all operator expressions $g_1, \ldots, g_{n_i}$ and the branch leading out is $P_i\left(g_1, \ldots, g_{n_i}\right)$; if the branch point is marked f_i then m_i branches lead in, they are all operator expressions $g_1, \ldots, g_{m_i}$ and the branch leading out is $f_i\left(g_1, \ldots, g_{m_i}\right)$; if the branch point is marked $\wedge$ then two branches lead in, both are predicates P, Q and the branch leading out is $P \wedge Q$; if the branch point is marked $\neg$ then one branch leads in, it is a predicate P and the branch out is $\neg P$; if it is marked $=$, two branches lead in, both operator expressions g, h, and the branch leading out is $g = h$; finally if the branch point is marked $\langle \exists, n\rangle$ then one branch leads in, it is a predicate $P(x_1, \ldots, x_m)$ and the branch

leading out is $\exists_{x_n} P(x_1, \ldots, x_m)$.

Let K be the number of variables in S . The quantified variables of F and G are defined as follows: for each variable in S , a map $B^K \xrightarrow{p_i} B$; for each branch other than a leaf, we introduce a variable $\bar{g} : B^K \to B$ if the branch is an operator expression, a variable $\bar{P} \to B^K$ if the branch is a predicate. To each leaf marked x_i we make correspond the new variable p_i .

For each branch point we define a near-exactness predicate as follows:

If a branch point is marked P_i then let $\bar{g}_1, \ldots, \bar{g}_n$ be the variables corresponding to the incoming branches; $\bar{P} \to B^K$ to the outgoing, and let A say that

$$\begin{array}{ccc} \bar{P} & \to & B^K \\ \downarrow & & \downarrow \langle \bar{g}_1, \ldots, \bar{g}_{n_i} \rangle \\ \bar{P}_i & \to & B^{n_i} \end{array}$$

is a pullback.

If a branch point is marked f_i , then let $\bar{g}_1, \ldots, \bar{g}_{n_i}$ be the variables corresponding to the incoming branches; $\bar{g} : B^K \to B$ to the outgoing and let A say that

$$\bar{g} = B^K \xrightarrow{\langle \bar{g}_1, \ldots, \bar{g}_{n_i} \rangle} B^{m_i} \xrightarrow{\bar{f}_i} B \ .$$

If a branch point is marked $\wedge$, then let $\bar{P} \to B^K$, $\bar{Q} \to B^K$ be the variables corresponding to the incoming branches: $\bar{R} \to B^K$ to the outgoing and let A say that $\bar{R} \to B^K$ is a pullback of $\begin{array}{ccc} & & \bar{P} \\ & & \downarrow \\ \bar{Q} & \to & B_K \end{array}$.

If a branch point is marked $\neg$ then let $\overline{P} \to B^K$ correspond to the incoming branch; $\overline{R} \to B^K$ is the outgoing, and let A say that $\begin{matrix} & \overline{P} \\ & \downarrow \\ \overline{R} \to & B_K \end{matrix}$ is a coproduct.

If a branch point is marked $=$, then let $\overline{g} : B^K \to B$, $\overline{h} : B^K \to B$ correspond to the incoming branches, $\overline{P} \to B^K$ to the outgoing, and let A say that $\overline{P} \to B^K$ is an equalizer of $\overline{g}$, $\overline{h}$.

If a branch point is marked $\langle \exists, n \rangle$, then let $\overline{P} \to B^K$ correspond to the incoming branch, $\overline{Q} \to B^K$ to the outgoing, and let A say that

$$\begin{array}{ccc} \overline{Q} & \longrightarrow & B^K \\ \downarrow & & \downarrow \langle p_1, \ldots, p_{n-1}, p_{n+1}, \ldots, p_k \rangle \\ \mathrm{Im}\left(\overline{P} \to B^K \to B^{K-1}\right) & \to & B^{K-1} \end{array}$$

is a pullback.

Finally, let $\overline{S} \to B^K$ be the variable assigned to the root. $F = \forall[\bigwedge A \to (\overline{S} \simeq B^K)]$. $G = \exists[\bigwedge A \wedge (\overline{S} \simeq B^K)]$. □

COROLLARY 4.22. *If a theory has a model in any non-degenerate boolean topos, it has a model in sets.*

Proof. By Theorem 3.23 there always exists a near-exact functor into sets, which functor must preserve G. □

COROLLARY 4.23. *For any boolean* $\mathcal{B}$ *if* $S_1 \vdash_S S_2$ *then* $S_1 \models_{\mathcal{B}} S_2$.

Proof. Suppose that $S_1 \vdash_S S_2$ but not $S_1 \models_{\mathcal{B}} S_2$. Then there exists an interpretation of L in $\mathcal{B}/t(S_1)$ that is a model for S_1 but not S_2. Let $t(S_2)$ be the truth value for S_2, and reflect the interpretation into Sh_j for j the closure operator such that $0 \to t(S_2)$ becomes an iso. We then have a model in Sh_j of $S_1 \wedge \neg S_2$. Then apply Corollary 4.22. □

Of course $\models_S$ is classical logic. In a later section we find necessary and sufficient conditions on B so that $\models_B$ coincides with $\models_S$.

5. Arithmetic in topoi

A pair $1 \xrightarrow{o} N$, $N \xrightarrow{s} N$ is a *natural numbers object*, or NNO for short, if for every $1 \xrightarrow{x} A \xrightarrow{t} A$ there exists unique $N \to A$ such that

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d] & N \arrow[d] \\ 1 \arrow[ur, "o"] \arrow[dr, "x"'] & & \\ & A \arrow[r, "t"'] & A \end{tikzcd} .$$

NNO's are clearly unique up to unique isomorphisms.

PROPOSITION 5.11. *If* $1 \xrightarrow{o} N \xrightarrow{s} N$ *is an* NNO *then* $\begin{pmatrix} o \\ s \end{pmatrix} : 1+N \to N$ *is an isomorphism.*

Proof. Let $u_1 : 1 \to 1+N$ and $u_2 : N \to 1+N$ be the coprojections. Define $s' : 1+N \to 1+N$ by $u_1 s' = ou_2$, $u_2 s' = su_2$. Since $1 \xrightarrow{o} N \xrightarrow{s} N$ is an NNO, there is a unique $g : N \to 1+N$ such that $og = u_1$ and $sg = gs'$. We claim that g is the inverse of $\begin{pmatrix} o \\ s \end{pmatrix} : 1+N \to N$.

That $g\begin{pmatrix} o \\ s \end{pmatrix} = 1$ follows from the uniqueness clause in the definition of NNO applied to the commutative diagram

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "g"] & N \arrow[d, "g"] \\ 1 \arrow[ur, "o"] \arrow[r, "u_1"] \arrow[dr, "o"'] & 1+N \arrow[r, "s'"] \arrow[d, "\begin{pmatrix} o \\ s \end{pmatrix}"] & 1+N \arrow[d, "\begin{pmatrix} o \\ s \end{pmatrix}"] \\ & N \arrow[r, "s"'] & N \end{tikzcd} .$$

That $\begin{pmatrix} o \\ s \end{pmatrix} g = 1$ is equivalent to $og = u_1$ and $sg = u_2$. The first we have, and the second comes by applying the uniqueness clause to the commutative diagrams

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "u_2"] & N \arrow[d, "u_2"] \\ 1 \arrow[ur, "o"] \arrow[r, "ou_2"'] & 1+N \arrow[r, "s'"'] & 1+N \end{tikzcd} , \quad \begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "sg"] & N \arrow[d, "sg"] \\ 1 \arrow[ur, "o"] \arrow[r, "ou_2"'] & 1+N \arrow[r, "s'"'] & 1+N \end{tikzcd} . \qquad \square$$

PROPOSITION 5.12. *If* $1 \xrightarrow{o} N \xrightarrow{s} N$ *is an* NNO *then* $N \to 1$ *is a coequalizer of* s *and* 1_N .

Proof. $N \to 1$ is epic (because there exists $1 \to N$) and it suffices to show that if $N \xrightarrow{s} N \xrightarrow{f} A = N \xrightarrow{f} A$ then there exists $1 \to A$ such that $N \xrightarrow{f} A = N \to 1 \to A$. Define $1 \xrightarrow{x} A = 1 \xrightarrow{o} N \xrightarrow{f} A$. Then both

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ 1 \arrow[ur, "o"] \arrow[r, "x"'] & A \arrow[r, "1"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d] & N \arrow[d] \\ 1 \arrow[ur, "o"] \arrow[dr, "x"'] & 1 \arrow[d, "x"] & 1 \arrow[d, "x"] \\ & A \arrow[r, "1"'] & A \end{tikzcd}$$

and the uniqueness condition on NNO's yields $N \xrightarrow{f} A = N \to 1 \xrightarrow{x} A$. $\square$

We shall show that the exactness conditions of the last two propositions characterize NNO's in topoi. In the last section we observed that near-exactness conditions are equivalent to stating that something is a model for a finite theory. Here we will see that the addition of a single coequalizer condition can yield a categorical (in the old sense) definition.

We consider first the category of sets:

PROPOSITION 5.13 for sets. *If* $A \xrightarrow{s} A$ *is monic and* $A \to 1$ *is a coequalizer of* s *and* 1_A *, then either* $A \simeq Z_n$ *,* $s(a) = a + 1$ *or* $A \simeq N$ *,* $s = s$ *.* □

Hence, if we add the requirement that s is *not* epic, we can characterize the natural numbers.

Let $\mathcal{T}$ be a well-pointed topos, $\mathcal{T} \xrightarrow{T} \mathcal{S}$ exact. If $1 \xrightarrow{0} N \xrightarrow{s} N$ is an NNO in $\mathcal{T}$, then $1 \to TN \xrightarrow{Ts} TN$ is the standard NNO in $\mathcal{S}$. (Any exact functor from a well-pointed topos to a non-degenerate topos is faithful.) Suppose then that in $\mathcal{T}$ there exists $1 \xrightarrow{x} N$ such that for no natural n does $1 \xrightarrow{0} N \xrightarrow{s} N \xrightarrow{s} \dots \xrightarrow{s} N = 1 \xrightarrow{x} N$. Then T just can't exist. We obtain such $\mathcal{T}$ simply by taking a non-principal ultra-power of $\mathcal{S}$.

In fact, we are using very little of the exactness of T . For well-pointed $\mathcal{T}$, non-degenerate $\mathcal{T}'$ and $T : \mathcal{T} \to \mathcal{T}'$ suppose $T(1) = 1$, $T(1+1) = 1 + 1$. Then T is faithful. ($1 + 1$ is a cogenerator for well-pointed topoi.) Because non-zero objects in well-pointed topoi are injective, T preserves monomorphisms if it is faithful. Hence if $1 \xrightarrow{0} N \xrightarrow{s} N$ is an NNO in $\mathcal{T}$ and $T(1) \simeq 1$, $T(1) + T(1) \simeq T(1+1)$, $\mathrm{Coeq}\left(T(1_N),\ T(s)\right) \simeq T\left(\mathrm{Coeq}(1_N,\ s)\right)$, then $1 \to TN \to TN$ is as described in Proposition 5.13. Thus for $\mathcal{T}$ a non-principal ultra-power of $\mathcal{S}$, there is no such $T : \mathcal{T} \to \mathcal{S}$. We can go one further step:

PROPOSITION 5.14 (the scarcity of right-exact functors). *If* $\mathcal{T}$ *is a non-principal ultra-power of* $\mathcal{S}$ *, then for every* $T : \mathcal{T} \to \mathcal{S}$ *such that* $T(1) + T(1) \simeq T(1+1)$ *and* $\mathrm{Coeq}\left(T(1_N),\ T(s)\right) \simeq T(\mathrm{Coeq}(1,\ s))$ *, (for example, right exact* T *), it is the case that* $T \equiv 0$ *.*

Proof. Suppose for some $A \in \mathcal{T}$, $TA \not\simeq 0$. Then because there exists $1 \to A$, $T(1) \not\simeq 0$. View T as a functor with values in $\mathcal{S}/T(1)$, choose $1 \to T(1)$ and define $T' = \mathcal{T} \to \mathcal{S}/T(1) \to \mathcal{S}/1$. T' may be alternatively

described by the pullbacks $\begin{matrix} T'(B) & \to & T(B) \\ \downarrow & & \downarrow \\ 1 & \to & T(1) \end{matrix}$. T' preserves at least the same colimits as T , and $T'(1) = 1$. Hence by our remarks above, T' can not exist. Nor can T . □

In particular, Theorem 3.24 can not be improved to make T exact. Moreover, we have laid to rest the idea that the exact embedding theorem for abelian categories has a nice generalization to "base" categories other than abelian. For:

COROLLARY 5.15. *No set of elementary conditions true for the category of sets implies exact (even right-exact) embeddability into the category of sets. (Even if you add countability.)*

Proof. We can take the complete elementary theory of S , and let T be an elementary submodel of a non-principal ultra-power of S and apply Proposition 5.14. □

The embedding theorem for abelian categories was motivated by the consequent metatheorem for the universal theory of exactness. The latter can be true without the former. But, alas, not for topoi. We need, first, a bit more about NNO's .

5.2. Primitive recursive functions in topoi

PROPOSITION 5.21 for topoi. *If* $1 \xrightarrow{o} N \xrightarrow{s} N$ *is an* NNO , *then for every* $A \xrightarrow{x} B$ *and* $B \xrightarrow{t} B$ *there exists unique* $A\times N \to B$ *such that*

$$\begin{tikzcd} & A\times N \arrow[r, "1\times s"] \arrow[d] & A\times N \arrow[d] \\ A \arrow[ur, "\langle 1,o\rangle"] \arrow[dr, "x"'] & & \\ & B \arrow[r, "t"'] & B \end{tikzcd}.$$

Proof. Transfer the problem to solving for

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d] & N \arrow[d] \\ 1 \arrow[ur, "o"] \arrow[dr] & & \\ & B^A \arrow[r, "t^A"'] & B^A \end{tikzcd} . \qquad \square$$

For the standard natural numbers in S we know that given $g : A \to B$, $h : A\times N\times B \to B$ there exists unique $f : A\times N \to B$ such that

$$f(a,\ 0) = g(a)\ ,$$

$$f(a,\ y+1) = h(a,\ y,\ f(a,\ y))\ .$$

(Usually A is a power of N , $B = N$.)

PROPOSITION 5.22 for topoi. *Let* $1 \xrightarrow{o} N \xrightarrow{s} N$ *be a* NNO , $g : A \to B$, $h : A\times N\times B \to B$ *given. There exists unique* $f : A\times N \to B$ *such that*

$$A \to A\times 1 \xrightarrow{1\times o} A\times N \xrightarrow{f} B = g\ ,$$

$$A\times N \xrightarrow{1\times s} A\times N \xrightarrow{f} B = A\times N \xrightarrow{p_1, p_2, f} A\times N\times B \xrightarrow{h} B\ .$$

Proof. (Notation: for any X , $X \xrightarrow{o} N$ means $X \to 1 \xrightarrow{o} N$.) Let k be such that

(5.231)
$$\begin{tikzcd} & A\times N \arrow[r, "1\times s"] \arrow[d, "k"] & A\times N \arrow[d, "k"] \\ A \arrow[ur, "\langle 1,o \rangle"] \arrow[r, "\langle 1,o,g \rangle"'] & A\times N\times B \arrow[r, "\langle p_1, p_2 s, h \rangle"'] & A\times N\times B \end{tikzcd} ,$$

as insured by Proposition 5.22. We will show that kp_3 works. First: $kp_1 = k\langle p_1,\ p_2 s,\ h\rangle p_1 = (1\times s)kp_1$ and $\langle 1,\ o\rangle kp_1 = 1$, hence

$$\begin{tikzcd} & A\times N \arrow[r, "1\times s"] \arrow[d, "kp_1"] & A\times N \arrow[d, "kp_1"] \\ A \arrow[ur, "\langle 1,o\rangle"] \arrow[r, "1"'] & A \arrow[r, "1"'] & A \end{tikzcd} .$$

p_1 works as well as kp_1 and Proposition 5.22 says, therefore, that $kp_1 = p_1$.

Second: $kp_2s = k\langle p_1, p_2s; h\rangle p_2 = (1\times s)kp_2$ and $\langle 1, o\rangle kp_2 = o$ hence

$$\begin{tikzcd} & A\times N \arrow[r, "1\times s"] \arrow[d, "kp_2"] & A\times N \arrow[d, "kp_2"] \\ A \arrow[ur, "\langle 1,o\rangle"] \arrow[r, "o"'] & N \arrow[r, "s"'] & N \end{tikzcd} ,$$

and p_2 works as well as kp_2 and again Proposition 5.22 says that $kp_2 = p_2$.

Finally, for the existence of f , define $f = kp_3$. Then $k = \langle p_1, p_2, f\rangle$ and

$$\langle 1, o\rangle f = \langle 1, o\rangle kp_3 = \langle 1, o, g\rangle p_3 = g ,$$

$$(1\times s)f = (1\times s)kp_3 = k\langle p_1, p_2s, h\rangle p_3 = \langle p_1, p_2, f\rangle h .$$

For the uniqueness, suppose f is as described in the proposition. Define $k = \langle p_1, p_2, f\rangle$, verify that (5.231) commutes and use Proposition 5.22. □

Note that h need not "depend" on A or N , or B . That is, given $h : A\times B \to B$ we could define $h' : A\times N\times B \to B = A\times N\times B \to A\times B \xrightarrow{h} B$ and apply the proposition.

Thus we can define on any NNO in a topos $a, m, e : N\times N \to N$ by

$$N \xrightarrow{\langle 1,o\rangle} N\times N \xrightarrow{a} N = 1_N \ , \quad N\times N \xrightarrow{1\times s} N\times N \xrightarrow{a} N = N\times N \xrightarrow{a} N \xrightarrow{s} N \ ,$$

$$N \xrightarrow{\langle 1,o\rangle} N\times N \xrightarrow{m} N = o \quad , \quad N\times N \xrightarrow{1\times s} N\times N \xrightarrow{m} N = N\times N \xrightarrow{\langle p_1, m\rangle} N\times N \xrightarrow{a} N \ ,$$

$$N \xrightarrow{\langle 1,o\rangle} N\times N \xrightarrow{e} N = os \quad , \quad N\times N \xrightarrow{1\times s} N\times N \xrightarrow{e} N = N\times N \xrightarrow{\langle p_1, e\rangle} N\times N \xrightarrow{m} N \ .$$

In the category of sets a, m, e are addition, multiplication and exponentiation; that is,

$$x + 0 = x \ , \quad x + (y+1) = (x+y) + 1 \ ,$$

$$x \cdot 0 = 0 \ , \quad x \cdot (y+1) = (x \cdot y) + x \ ,$$

$$x^0 = 1 \ , \qquad x^{(y+1)} = \left(x^y\right) \cdot x \ .$$

Take any elementary sentence S in the operators o, s, a, m, e. Add to it the six equations above which define a, m, e on N. We saw in Proposition 4.21 that there is a universal Horn exactness predicate that says that a given $\langle N, o, s, a, m, e\rangle$ is a model of S. Enlarge that formula to include $\begin{Bmatrix} o \\ s \end{Bmatrix} : 1 + N \simeq N$ and $\mathrm{Coeq}(1, s) = 1$. The universal quantification of that formula then says that the arithmetic of the NNO satisfies S. Hence,

THEOREM 5.23. *For any recursively enumerable set of elementary conditions* T *, true for the category of sets, there exists a model* T *of* T *and a universal Horn sentence in the predicates of exactness, true for sets but false for* T .

Proof. Add to T the axioms of a well pointed topos with an NNO. By Gödel's Incompleteness Theorem we know that there is an elementary sentence S true for standard arithmetic, whose translation, S', into a universal Horn sentence in the predicates of exactness is not a consequence of T (else number theory would be decidable).

Gödel's Completeness Theorem implies that there is a model of T $\cup \{\neg S'\}$. □

5.3. The exact characterization of the natural numbers

PROPOSITION 5.31 for topoi. *If a topos has* NNO *then*

$1 \xrightarrow{0} N \xrightarrow{s} N$ is an NNO *iff* $\begin{Bmatrix} 0 \\ s \end{Bmatrix} : 1 + N \to N$ *is iso and* $\mathrm{Coeq}(1, s) = 1$. □

Proof. Let $1 \xrightarrow{x} A \xrightarrow{t} A$ be such that $\begin{Bmatrix} x \\ t \end{Bmatrix}$ is iso and $\mathrm{Coeq}(1, t) = 1$. Let $1 \xrightarrow{0} N \xrightarrow{s} N$ be NNO and let

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ 1 \arrow[ur, "0"] \arrow[dr, "x"'] & & \\ & A \arrow[r, "t"'] & A \end{tikzcd}$$

commute. We wish to show f an isomorphism.

First, f is epi. We have a universal Horn sentence in exactness theory, namely that $\begin{Bmatrix} 0 \\ s \end{Bmatrix}$, $\begin{Bmatrix} x \\ t \end{Bmatrix}$ isomorphs and $\mathrm{Coeq}(1, s) = \mathrm{Coeq}(1, t) = 1$ imply f is epi. It suffices to prove it in well pointed topoi. Accordingly let $A' = \mathrm{Im}(f)$ and A'' the complement of A' . It suffices to show that $t(A'') \subset A''$, for such allows us to "split" t as $t|A' + t|A''$ and obtain a splitting $\mathrm{Coeq}(1, t) = \mathrm{Coeq}(1_{A'}, t|A') + \mathrm{Coeq}(1_{A''}, t|A'')$. If $A' \neq A$, then $A'' \neq 0$ and $\mathrm{Coeq}(1, t)$ is bigger than 1 . Hence it suffices to show $t(A'') \subset A''$.

Let $t' = t|A'$, $1 \xrightarrow{x'} A' = 1 \xrightarrow{0} N \to A'$. Then $\begin{Bmatrix} x' \\ t' \end{Bmatrix}$ is iso. The universal Horn sentence: "If $A' + A'' \simeq A$ and $\begin{Bmatrix} x' \\ t' \end{Bmatrix}$, $\begin{Bmatrix} x \\ t \end{Bmatrix}$ iso then $t(A'') \subset A''$ " is in the predicates of near-exactness and it suffices to prove it in $\mathcal{S}$, an easy matter.

Second, f is mono. Using just that $\begin{Bmatrix} 0 \\ s \end{Bmatrix}$ and $\begin{Bmatrix} x \\ t \end{Bmatrix}$ are isomorphs we can show that $\begin{matrix} 1 & \longrightarrow & 1 \\ 0\downarrow & & \downarrow \\ N & \xrightarrow[f]{} & A \end{matrix}$ and $\begin{matrix} N & \xrightarrow{f} & A \\ s\downarrow & & \downarrow t \\ N & \xrightarrow[f]{} & A \end{matrix}$ are pullbacks since such are sentences in the universal Horn theory of near-exactness and it suffices to prove them in $\mathcal{S}$.

Let $Q \rightarrowtail A$ be as described in Proposition 2.21, that is $\begin{tikzcd} Q \arrow[r, "1"] \arrow[d] & Q \arrow[d] \\ N \arrow[r] & A \end{tikzcd}$

is a pullback, and for all $X \to A$ such that $\begin{tikzcd} X \arrow[r, "1"] \arrow[d] & X \arrow[d] \\ N \arrow[r] & A \end{tikzcd}$ is a pullback,

there exists $X \to Q \to A = X \to A$. Then because $\begin{tikzcd} 1 \arrow[r] \arrow[d, "o"'] & 1 \arrow[d, "x"] \\ N \arrow[r] & A \end{tikzcd}$ is a pullback,

there exists $1 \to Q \to A = 1 \xrightarrow{x} A$. Because

$$\begin{tikzcd} Q \arrow[r, "1"] \arrow[d] & Q \arrow[d] \\ N \arrow[r] \arrow[d] & A \arrow[d] \\ N \arrow[r] & A \end{tikzcd}$$

is a pullback, there exists $Q \xrightarrow{g} Q$ such that

$Q \xrightarrow{g} Q \to A = Q \to A \xrightarrow{t} A$. Because $\begin{tikzcd} Q \arrow[r, "1"] \arrow[d] & Q \arrow[d] \\ N \arrow[r] & A \end{tikzcd}$ is a pullback we obtain

$\begin{tikzcd} & Q \arrow[r, "g"] \arrow[d] & Q \arrow[d] \\ 1 \arrow[ur] \arrow[dr] & & \\ & N \arrow[r] & N \end{tikzcd}$. Because $1 \xrightarrow{o} N \xrightarrow{s} N$ is a NNO we obtain $N \to Q$ such

that $N \to Q \to N = 1_N$. That is $Q \to N$ is epi and $Q \to A$ is an isomorph.

Hence so is f . □

Thus, exact functors between topoi with NNO's preserve NNO's . Hence,

THEOREM 5.32. *If* $T \to T'$ *is exact for* T, T' *well-pointed topoi with* NNO*'s , then the elementary arithmetics of* T, T' *coincide.*

Proof. Combine Propositions 4.21 and 5.31. □

We can push a bit further. By the existential second-order arithmetic of a topos, we mean the second-order sentences in arithmetic in which all second-order quantifiers are existential. Propositions 4.21 and 5.31 say that the truth of such a sentence in T is equivalent to an existential sentence in the exactness theory of T . Hence,

PROPOSITION 5.33. *The existential second-order arithmetic of a well-pointed topos is determined by its universal Horn theory of*

exactness. $\square$

5.4. Inferring the axiom of infinity

PROPOSITION 5.41 for topoi. *Given* $1 \xrightarrow{x} A \xrightarrow{t} A$ *there exists* $A' \rightarrowtail A$ *and*

$$\begin{tikzcd} & A' \arrow[r, "t'"] \arrow[d] & A' \arrow[d] \\ 1 \arrow[ur, "x'"] \arrow[dr, "x"'] & & \\ & A \arrow[r, "t"'] & A \end{tikzcd}$$

such that $\begin{Bmatrix} x' \\ t' \end{Bmatrix} : 1 + A' \to A'$ *is epi.*

Before proving Proposition 5.41 we show its consequences:

PROPOSITION 5.42 (the Peano property). *If* $\begin{Bmatrix} x \\ t \end{Bmatrix} : 1 + A \to A$ *is iso and* $\mathrm{Coeq}(x, t) = 1$ *then for every* $A' \subset A$ *such that* $\mathrm{Im}(x) \subset A'$, $t(A') \subset A'$ *it is the case that* $A' = A$.

Proof. The sentence is in the universal theory of exactness and it suffices to prove it in well-pointed topoi. By applying Proposition 5.41 to $1 \xrightarrow{x'} A' \xrightarrow{t'} A'$ we can assume that $\begin{Bmatrix} x' \\ t' \end{Bmatrix}$ is epi.

Let $A'' \subset A$ be the complement of A' . Using just that $\begin{Bmatrix} x \\ t \end{Bmatrix}$ is iso and $\begin{Bmatrix} x' \\ t' \end{Bmatrix}$ is epi we can show that $t(A'') \subset A''$ in S , hence everywhere. Thus t splits as $t' + t''$ $(t'' = t|A'')$ and $\mathrm{Coeq}(1, t) = \mathrm{Coeq}(1_{A'}, t') + \mathrm{Coeq}(1_{A''}, t'')$. If $A'' \neq 0$ then $\mathrm{Coeq}(1, t)$ is bigger than 1 . $\square$

THEOREM 5.43 for topoi. $1 \xrightarrow{0} N \xrightarrow{s} N$ *is an* NNO *iff* $\begin{Bmatrix} 0 \\ s \end{Bmatrix} : 1 + N \cong N$ *and* $\mathrm{Coeq}(1, s) = 1$.

Proof. The necessity of the exactness condition was Propositions 5.11, 5.12. For the sufficiency, note first that the Peano property above yields the uniqueness conditions, that is, if both $f, g : N \to A$ were such

that

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d] & N \arrow[d] \\ 1 \arrow[ur, "o"] \arrow[dr, "x"'] & & \\ & A \arrow[r, "t"'] & A \end{tikzcd} ,$$

then the equalizer of f, g would satisfy the hypotheses of Proposition 5.42 and hence would be all of N .

For the existential condition, let $1 \xrightarrow{y} B \xrightarrow{v} B$ be given, and apply Proposition 5.41 to $1 \xrightarrow{\langle o,y \rangle} N\times B \xrightarrow{s\times v} N\times B$ to obtain $1 \xrightarrow{x} A \xrightarrow{t} A$, $\begin{pmatrix} x \\ t \end{pmatrix}$ epi, maps $A \to N$, $A \to B$. It suffices to show that $A \to N$ is iso. Again, the Peano property says $A \to N$ is epi. We need:

LEMMA 5.431. *If* $\begin{pmatrix} x \\ t \end{pmatrix} : 1 + A \longrightarrow A$ *is epi,* $\begin{pmatrix} o \\ s \end{pmatrix} : 1 + N \to N$ *iso,* $\mathrm{Coeq}(1, s) = 1$ *, and*

$$\begin{tikzcd} & A \arrow[r, "t"] \arrow[d, "f"] & A \arrow[d, "f"] \\ 1 \arrow[ur, "x"] \arrow[dr, "o"'] & & \\ & N \arrow[r, "s"'] & N \end{tikzcd} ,$$

then $A \xrightarrow{f} N$ *is mono.*

Proof. The sentence is in the universal Horn theory of exactness and it suffices to prove it in well pointed topoi. Let $E \subset A\times A$ be the kernel-pair of f ; $E' = E - \Delta$, $N' = \mathrm{Im}(E' \to A \to N)$. We need $E = \Delta$, that is, $E' \simeq 0$, equivalently $N' \simeq 0$. Let N'' be the complement of N' . We need $N'' = N$.

Note that $1 \xrightarrow{g} N$ factors through N'' iff g has a unique lifting to A . We may verify that $1 \xrightarrow{o} N$ factors through N'' and that if $1 \xrightarrow{g} N$ does, then so does $1 \xrightarrow{gs} N$. That is, N'' satisfies the hypotheses of Proposition 5.42 and $N'' = N$. □

THEOREM 5.44 for topoi. *The following are equivalent:*

(a) there exists an NNO ;

(b) there exists a monomorphism $A \rightarrowtail A$ *and a map* $1 \to A$ *such that*
$\begin{array}{ccc} 0 & \to & 1 \\ \downarrow & & \downarrow \\ A & \to & A \end{array}$ *is a pullback;*

(c) there exists an isomorphism $1 + A \simeq A$.

Proof. Clearly *(a)* ⇒ *(c)* ⇒ *(b)*. For *(b)* ⇒ *(a)* apply Proposition 5.41 to obtain

$$\begin{tikzcd} & A' \arrow[r, "t'"] \arrow[d, tail] & A' \arrow[d, tail] \\ 1 \arrow[ur, "x'"] \arrow[r, "x"'] & A \arrow[r, "t"'] & A \end{tikzcd} ,$$

$\begin{Bmatrix} x' \\ t' \end{Bmatrix} : 1 + A' \to A'$ epic. It is clear that $\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow x' \\ A' & \xrightarrow[t']{} & A' \end{array}$ is a pullback,

hence $\begin{Bmatrix} x' \\ t' \end{Bmatrix}$ is monic, and an isomorphism.

Let $A' \to C$ be a coequalizer of $1_{A'}, t'$ and define

$1 \to C = 1 \xrightarrow{x'} A' \to C$. View $A' \to C$ as an object in T/C . Note that

$\begin{tikzcd}[column sep=small] A' \arrow[rr, "x'"] \arrow[dr] & & A' \arrow[dl] \\ & C & \end{tikzcd}$ and that $\begin{Bmatrix} x' \\ t' \end{Bmatrix} : 1 + A' \simeq A'$ remains true in T/C . Pullback

along $1 \to C$ to obtain $N \xrightarrow{s} N$ in $T/1$. We maintain the coproduct conditions and gain the coequalizer condition. □

THEOREM 5.45 for topoi. *If* $(1, \Omega) = 2$ *then either there exists an* NNO *or every mono-endo is auto and every epi-endo is auto.*

Proof. Suppose that $f : A \rightarrowtail A$ is not epi. Then $f^A : A^A \to A^A$ is

mono and if $\begin{array}{ccc} U & \to & 1 \\ \downarrow & & \downarrow \\ A^A & \to & A^A \end{array}$ is a pullback where $1 \to A^A$ corresponds to

$A \xrightarrow{1} A$, then $U \to 1$ can not be iso. Hence $U = 0$ and we can apply the last theorem.

If $g : A \twoheadrightarrow A$ is not mono, we can repeat the argument for $A^g : A^A \twoheadrightarrow A^A$. □

The proof of Theorem 5.41 is fairly easy in solvable topoi (see 2.7). We can there take A' as the smallest subobject such that $t(A') \subset A'$ and such that $\mathrm{Im}(x) \subset A'$.

Remarkable enough, even without solvability we can construct A' , not as an intersection but as a union.

Proof of Proposition 5.41. We define $B \subset \Omega^A$ to correspond to the family of subobjects of A such that $A' \subset \mathrm{Im}(x) \cup t(A')$ and show that $\cup B$ works. Define $B \subset \Omega^A$ as the equalizer of the identity map and

$$\Omega^A \xrightarrow{\langle 1, \ulcorner x \urcorner, \exists_t \rangle} \Omega^A \times \Omega^A \times \Omega^A \xrightarrow{1 \times \cup} \Omega^A \times \Omega^A \xrightarrow{\cap} \Omega^A$$

where $\ulcorner x \urcorner : \Omega^A \to \Omega^A = \Omega^A \to 1 \to \Omega^A$ and $1 \to \Omega^A$ corresponds to $A \to \Omega$ the characteristic map of $1 \xrightarrow{x} A$.

For any $A' \subset A$, the corresponding map $1 \to \Omega^A$ factors through $B \subset \Omega^A$ iff $A' \subset \mathrm{Im}(x) \cup t(A')$. Moreover:

such remains the case after application by any logical morphism.

Let

$$\begin{array}{ccc} C & \longrightarrow & B \times A \\ \Big| & & \downarrow \\ \Big| & & \Omega^A \times A \\ \downarrow & & \downarrow e \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

be a pullback.

For any $1 \to B$ let $\begin{array}{ccc} C' & \to & C \\ \downarrow & & \downarrow \\ 1 & \to & B \end{array}$ be a pullback. Then $C' \to C \to A$ is mono and $C' \subset \mathrm{Im}(x) \cup t(C')$

and such remains the case after application by any logical morphism.

The proof of this is obtained by noticing that $\begin{array}{ccc} C' & \to & 1\times A \\ \downarrow & & \downarrow \\ C & \to & B\times A \end{array}$ is a pullback, hence

$$\begin{tikzcd} C' \arrow[r] \arrow[dd] & 1\times A \arrow[d, "f\times 1"] \\ & \Omega^A\times A \arrow[d, "e"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

is a pullback, where $1 \xrightarrow{f} \Omega^A = 1 \to B \to \Omega^A$. But $(f\times 1)e = A \xrightarrow{g} \Omega$ the map which corresponds to f . $\begin{array}{ccc} C' & \to & A \\ \downarrow & & \downarrow g \\ 1 & \to & \Omega \end{array}$ is a pullback. Thus the characteristic map of C' corresponds to a map $1 \to \Omega^A$ which factors through B and $C' \subset \mathrm{Im}(x) \cup t(C')$.

We wish to show that $\mathrm{Im}(C \to A)$ works as A' as demanded by the proposition. The reversal of the above paragraph shows that given any $C' \rightarrowtail A$ such that $C' \subset \mathrm{Im}(x) \cup t(C')$ we can find $1 \to B$ such that $\begin{array}{ccc} C' & \to & C \\ \downarrow & & \downarrow \\ 1 & \to & B \end{array}$ is a pullback and $C' \to C \to A = C' \rightarrowtail A$. We wish to show, first, that $\mathrm{Im}(C \to A) \subset \mathrm{Im}\left(C \to A \xrightarrow{t} A\right)$, equivalently that in the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[dd] & 1+C \arrow[d] \\ & 1+A \arrow[d, "\binom{x}{t}"] \\ C \arrow[r] & A \end{tikzcd},$$

$P \to C$ is epic. (This last "equivalently" is a pair of sentences in the universal Horn theory of near-exactness and may be verified in sets.) Suppose $P \to C$ is not epic. We may transport the entire situation to T/C and obtain a map $1 \to TC$ such that in the pullback $\begin{array}{ccc} P_2 & \to & TP \\ \downarrow & & \downarrow \\ 1 & \to & TC \end{array}$, $P_2 \to 1$ is not epic. Because all our constructions are preserved by logical morphisms, we may drop the "T" and show that for any $1 \to C$ and pullback

$$\begin{tikzcd} P_2 \arrow[r] \arrow[dd] & 1+C \arrow[d] \\ & 1+A \arrow[d] \\ 1 \arrow[r] & C \rightarrow A \end{tikzcd},$$

$P_2 \to 1$ is epi.

Let $\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ 1 & \to C \to & B \end{array}$ be a pullback. Then

$$\begin{tikzcd} P_2 \arrow[r] \arrow[dd] & 1+C \arrow[d] \\ & 1+A \arrow[d] \\ 1 \arrow[r] & C' \rightarrow A \end{tikzcd}$$

is a pullback. Let $\begin{array}{ccc} P_3 & \to & 1+C' \\ \downarrow & & \downarrow \\ P_2 & \to & 1+C \end{array}$ be a pullback. It suffices to show that

$P_3 \to P_2 \to 1$ is epi.

Let $\begin{array}{ccc} P_4 & \to & 1+C' \\ \downarrow & & \downarrow \\ C' & \to & A \end{array}$ be a pullback. $P_4 \to C'$ *is* epic. Let $\begin{array}{ccc} P_5 & \to & P_4 \\ \downarrow & & \downarrow \\ 1 & \to & C' \end{array}$ be a

pullback. $P_5 \to 1$ is epic. But $\begin{array}{ccc} P_5 & \to & 1+C' \\ \downarrow & & \downarrow \\ 1 & \to & A \end{array}$ and $\begin{array}{ccc} P_3 & \to & 1+C' \\ \downarrow & & \downarrow \\ 1 & \to & A \end{array}$ are both

pullbacks and we have shown that $\mathrm{Im}(C \to A) \subset \mathrm{Im}(x) \cup \mathrm{Im}(C \to A \xrightarrow{t} A)$.

Call $A' = \mathrm{Im}(C \to A)$. We remarked above, that for any $C' \subset A$ such that $C' \subset \mathrm{Im}(x) \cup t(C')$ it is the case that $C' \to A = C' \to C \to A$ hence $C' \subset A'$. Thus A' is the maximal such subobject. But note that for $C' = A' \cup t(A')$ it is the case that $C \subset \mathrm{Im}(x) \cup t(C')$. Hence

$A \cup t(A') \subset A'$ and $t(A') \subset A'$, yielding $\begin{array}{ccc} A' & \xrightarrow{t'} & A' \\ \downarrow & & \downarrow \\ A & \xrightarrow[t]{} & A \end{array}$. For

$C' = \mathrm{Im}(x) \cup A'$ we have $C' \subset \mathrm{Im}(x) \cup t(C')$ and $\mathrm{Im}(x) \subset A'$, yielding

$$\begin{tikzcd} & A' \arrow[r, "t'"] \arrow[d] & A' \arrow[d] \\ 1 \arrow[ur, "x'"] \arrow[dr, "x"'] & & \\ & A \arrow[r, "t"'] & A \end{tikzcd} .$$

Finally, $A' \subset \mathrm{Im}(x) \cup t(A')$ directly yields $\begin{Bmatrix} x' \\ t' \end{Bmatrix} : 1 + A' \twoheadrightarrow A'$. □

5.5. One coequalizer for all

Given $f, g : A \to B$ let $R_{f,g} = \mathrm{Im}\left(A \xrightarrow{(f,g)} B\times B\right) \subset B\times B$ and $E_{f,g} \subset B\times B$ the kernel-pair of the coequalizer of f, g . By Corollary 3.32, $E_{f,g}$ is the smallest equivalence relation containing $R_{f,g}$.

Suppose $B \xrightarrow{h} C$ is epi. Let $E_h \subset B\times B$ be the equivalence relation induced by h , that is, $\begin{matrix} E_h & \to & B \\ \downarrow & & \downarrow \\ B & \to & C \end{matrix}$ a pullback. Then by Corollary 3.33, $B \xrightarrow{h} C$ is a coequalizer of f, g iff $E_{f,g} = E_h$. Note that $R_{f,g}$ and E_h are defined using only near-exactness.

$B \xrightarrow{h} C$ is a coequalizer of $f, g : A \to B$ iff h is epic and E_h is the smallest equivalence relation containing $R_{f,g}$.

In general, given $R \subset B\times B$ define $\equiv R$ to be the smallest equivalence relation containing R . Then a near-exact functor is exact iff it preserves the $\equiv$ operator on binary relations.

Given any $R \subset A\times C$ we can define a transformation $\mathrm{Rel}(-, A) \xrightarrow{\circ R} \mathrm{Rel}(-, C)$ as follows: for $Q \subset X\times A$ let $\begin{matrix} P & \to & Q \\ \downarrow & & \downarrow \\ R & \to & A \end{matrix}$ be a pullback and send Q to $\mathrm{Im}(P \to X\times C)$. $\circ R$ is natural and there exists $\circ R : \Omega^A \to \Omega^C$. (If R is the graph of a map f , then $\circ R = \exists_f$.) Given $R \subset B\times B$ define $\overline{R} \subset B\times B$ as $\Delta \cup R \cup \mathrm{Im}\left(R \to B\times B \xrightarrow{\tau} B\times B\right)$ where τ is the

twist map, and define $\Omega^{B\times B} \xrightarrow{\circ\overline{R}} \Omega^{B\times B}$ as above. In a topos with NNO let $k : N \to \Omega^{B\times B}$ be the map such that $1 \xrightarrow{o} N \xrightarrow{k} \Omega^{B\times B}$ corresponds to $B \xrightarrow{\Delta} B\times B$ and

$$\begin{tikzcd} N \arrow[r, "s"] \arrow[d, "k"'] & N \arrow[d] \\ \Omega^{B\times B} \arrow[r, "\circ\overline{R}"'] & \Omega^{B\times B} \end{tikzcd} .$$

We obtain a relation from N to $B\times B$, $Q \subset N\times B\times B$, $\mathrm{Im}(Q \to B\times B) = \,\overline{\equiv}R$. We may look at Q backwards as a map from $\overline{\equiv}R$ to Ω^N .

In well-pointed topoi, at least, we can define $(\Omega^N)^*$ to correspond to the non-empty subobjects of N and define $(\Omega^N)^* \to N$ to correspond to least elements, and obtain $\overline{\equiv}R \xrightarrow{f} N$. f has the following properties:

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & \overline{\equiv}R \arrow[d] \\ 1 \arrow[r, "o"'] & N \end{tikzcd}$$ *is a pullback and for* $$\begin{tikzcd} S \arrow[r] \arrow[d] & \overline{\equiv}R \arrow[d] \\ 1 \arrow[r, "x"'] & N \end{tikzcd} , \quad \begin{tikzcd} S' \arrow[r] \arrow[d] & \overline{\equiv}R \arrow[d] \\ 1 \arrow[r, "xs"'] & N \end{tikzcd}$$ *pullbacks,*

$S' = (S\circ\overline{R}) - S$.

These two properties, entirely in the language of near-exactness except for N itself, characterize $\overline{\equiv}R$. Hence:

PROPOSITION 5.51. *If* $T : \mathcal{T} \to \mathcal{T}'$ *is a near-exact functor of well-pointed topoi that preserves epimorphic families, then* T *is exact iff it preserves the coequalizer of* 1_N *and* s . □

Moreover given any diagram in a well-pointed topos, we can enlarge it and add $1 \xrightarrow{o} N \xrightarrow{s} N$ and translate any coequalizer condition into near-exactness conditions on the enlarged diagram. We can know the universal Horn theory of exactness of $\mathcal{T}$ if we know which two-sorted elementary theories have models in $\mathcal{T}$ in which one of the stipulated sorts is the natural numbers.

If $\mathcal{T}$ is a well-pointed topos with the *axiom of choice*, that is every object is projective, then by a Löwenheim-Skolem argument we can reduce the two-sorts to one and obtain the converse of Proposition 5.33:

THEOREM 5.52. *The universal Horn theory of exactness of a well-pointed topos with* NNO *and axiom of choice is determined by its existential second order arithmetic.* □

5.6. A standard recovery

PROPOSITION 5.61 **for topoi.** *If* T *is such that for every* $R \subset B\times B$, $\equiv R$ *is the union of the sequence* $\Delta, \overline{R}, \overline{R}^2, \ldots$, *through the standard natural numbers, then* T *may be exactly embedded in a power of* S.

Proof. By Theorem 3.24 we know that there is a collectively faithful family of non-exact functors into S, each of which preserves epimorphic families, hence unions. Hence the operation $R \mapsto \equiv R$ is preserved and by our remarks in the last section, such functors are exact. □

COROLLARY 5.62. *Countably complete topoi may be exactly embedded in a power of* S.

Proof. Whenever $\Delta, \overline{R}, \overline{R}^2, \ldots, \overline{R}^n, \ldots$ has a union it is $\equiv R$. □

From Corollary 5.15 we thus obtain:

PROPOSITION 5.63. *No set of elementary properties true for the category of sets implies exact (even right-exact) embeddability into a product of countably complete topoi.* □

By the standard maps from 1 to N we mean those of the form $1 \xrightarrow{0} N \xrightarrow{s} N \xrightarrow{s} N \xrightarrow{s} \cdots \xrightarrow{s} N$.

We say that N is of *standard generation* if the standard maps form a collectively epimorphic family. In a well-pointed topos, such is equivalent to $(1, N)$ having only standard maps.

THEOREM 5.64. *If* T *is a topos with* NNO *then it may be exactly embedded in a power of* S *if* N *is of standard generation.*

Proof. Theorem 5.23 says that there is a collectively faithful family of exact functors into well-pointed topoi each of which preserves epimorphic families, hence in each of which $(1, N)$ is standard. (Each has NNO by Proposition 5.42.)

By our remarks in the last section, $\equiv R$ is the union of the values of

a map $N \to \Omega^{B\times B}$, thus a union of a sequence over the standard natural numbers, and Proposition 5.61 applies. □

If T is solvable then N is well-pointed and standard generation is necessary.

COROLLARY 5.65. *If* T *is a solvable (for example, well-pointed) topos with* NNO *and every* $1 \to N$ *is standard then the existential second order arithmetic of* T *is standard.* □

It seems to me that Corollary 5.65 provides something of a semantics for existential second-order arithmetic.

6. Problems

Which categories can be exactly embedded in topoi?

Which well-pointed topoi can be exactly embedded in well-pointed topoi with NNO ?

Which of the latter can be exactly embedded in well-pointed topoi with the axiom of choice?

Which of the latter can be exactly embedded in well-pointed topoi with AC and an axiom of replacement?

Mitchell and Cole have independently shown that the latter are isomorphic to categories arising from models of Zermelo-Fraenkel. Hence we are asking for a metatheorem in which not *the* category of sets (what's that?) but *a* category of sets is the model.

The answers to the last two questions would tell us which existential second order theories of arithmetic are compatible with Z-F . Indeed a good question is whether each first-order arithmetic compatible with the axioms of well-pointed topoi is compatible with Z-F .

Using standard techniques transferred to well-pointed topoi we can see that each existential second order sentence in arithmetic is equivalent to one which asserts the existence of a single unary operator g that satisfies an equation involving g, $+$, $\times$, $-$. If we know that each such sentence implies (using the axioms of topoi) that g is bounded by a first-order definable operator (for example, recursive) then the existential second order theory is determined by the first order theory.

We should remark, in regard to the second question, that Theorem 5.23 remains true even if we replace the category of sets with the category of finite sets. The idea of the proof is to find a finite elementary theory such that the sentences true for all *finite* models are not recursively enumerable.

An example of such is the theory of ordered partial rings, with enough axioms to insure that the finite models of such are finite intervals of the integers. The theory of diophantine equations is thus a subset of the theory of its finite models.

Editor's note (1 May 1972). The Editor is very grateful to Mr T.G. Brook and especially also to Professor G.M. Kelly for reading the proofs with care and amending several mathematical errors. Some last-minute (and later) changes are due to the author.

Index of definitions

References

[1] J.C. Cole, "Categories of sets and models of set theory", Aarhus Universitet, Matematisk Institut, Preprint series, 52, 1971.

[2] Peter Freyd, "Several new concepts: lucid and concordant functors, pre-limits, pre-completeness, the continuous and concordant completions of categories", *Category theory, homology theory and their applications III* (Battelle Institute Conference, Seattle, Washington, 1968. Lecture Notes in Mathematics, 99, 196-241. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[3] John W. Gray, "The meeting of the Midwest Category Seminar in Zurich August 24-30, 1970, *Reports of the Midwest Category Seminar V* (Lecture Notes in Mathematics, 195, 248-255. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[4] F.W. Lawvere, "Quantifiers and sheaves", *Actes du Congrès International des Mathématiciens*, T.1 Nice, September 1970, 329-334. (Gauthier-Villars, Paris, 1971).

[5] F.W. Lawvere and M. Tierney, summarized in [3] above, *Reports of the Midwest Category Seminar V* (Lectures Notes in Mathematics, 195, 251-254. Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[6] J.-L. Verdier, "Séminaire de géometrie algébrique", fascicule 1, Mimeographed Notes *Inst. Hautes Études Sci.* (1963/64). To be published in *Théorie des topos et cohomologie étale des schémas* (Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York).

Department of Mathematics,
University of Pennsylvania,
Philadelphia,
Pennsylvania,
USA.