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EXTENSION OF THREE THEOREMS FOR FOURIER SERIES ON THE DISC TO THE TORUS

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We extend three well-known facts of Fourier series on the disc to Fourier series on the torus, a theorem of Riesz, a theorem of Szegö, and the fact that any function in H^1 can be factored as the product of two functions in H^2. Here the rôle of negative integers is played by the lattice points in the third quadrant. In earlier extensions of these theorems this rôle was played by half-planes.

1. Introduction

In the theory of functions of one complex variable it is well-known that a function f in the Hardy class H^1 can be factored in the form

$$f = gh \tag{1.1}$$

as the product of two functions g and h in H^2. The following question generalizing this fact to functions of two complex variables is raised in Helson and Lowdenslager ([2], p. 178): *Let R be a set of lattice points of the plane not containing the origin, which is closed under addition. Can every summable function f with Fourier series of the form*

$$f \sim a_{00} + \sum_{(m,n)\in R} a_{mn} e^{-i(m\theta+n\alpha)} \tag{1.2}$$

be factored as in (1.1), with the factors g and h being square summable functions with the same kind of Fourier series as f in (1.2)?

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Helson and Lowdenslager [2] gave a complete positive answer for some regions R, called *half-planes*, which have the following property:

$$(m,n) \in R \text{ if and only if } (-m,-n) \notin R \text{ , unless } m=n=0 .$$

The following interesting regions are typical half-planes:

$$S = \{(m,n) : m \leqslant -1, n \in Z\} \cup \{(0,n) : n \leqslant -1\} \tag{1.3}$$

$$T = \{(m,n) : m \in Z, n \leqslant -1\} \cup \{(m,0 : m \leqslant -1\} \tag{1.4}$$

The third quadrant

$$Q = \{(m,n) : m \leqslant 0, n \leqslant 0\} - \{(0,0)\} \tag{1.5}$$

because of it connection with the analytic functions of two variables (see ([2], p.178), [7]) as well as its connection with prediction theory of random fields (see [5] and [8]), seems the most important region to be considered. This is why the problem for the third quadrant has been explicitly raised by Helson and Lowdenslager ([2], p. 178).

The aim of this article is to give an answer to this factorisation problem by providing a set of sufficient conditions for that to happen. We should mention that the answer to the problem is in general in the negative (see Rudin [7], p. 67). We will also extend two further well-known facts from function theory on the unit disc to function theory on the torus: the *prediction error formula* due to Szegö and the theorem of F. and M. Reisz which proves that *every measure whose negative Fourier coefficients vanish is absolutely continuous.*

After setting up the necessary notation and terminology in section 2 we will prove these three theorems in sections 3, 4 and 5 respectively.

In order to prove our results we use techniques used by Helson and Lowdenslager in [2] together with some results concerning stationary fields.

We mention finally that, like the well-known strong connection between function theory on the unit disc and prediction theory of stationary random processes, there is quite a strong tie between function theory on the torus and prediction theory of stationary random fields. For more on this see Helson and Lowdenslager [2],[3], Korezlioglu and Loubaton [5], and Soltani [8].

2. Preliminaries

Let X_{mn} be an element of a Hilbert space H for all integers m and n. X_{mn} is called a *stationary field* on Z^2 if for all integers m,n,r,s the inner product of X_{mn} and X_{rs} depends *only* on $m-r$ and $n-s$, i.e., if we have

$$(X_{mn}, X_{rs}) = \rho(m-r, n-s) .$$

In this case $\rho(m,n) = (X_{mn}, X_{00})$ is a positive definite function on the group of lattice points Z^2. Thus there exists a nonnegative measure μ, called the *spectral measure* of the field X_{mn}, defined on the Borel sets of the torus

$$0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \alpha \leqslant 2\pi$$

such that

$$\rho(m,n) = \int e^{-i(m\theta+n\alpha)} d\mu , \quad \text{for all } m,n \in Z . \tag{2.1}$$

If μ is absolutely continuous with respect to the normalized Lebesgue measure $d\sigma = \frac{d\alpha d\theta}{4\pi^2}$, its Radon-Nikodym derivative w is called the *spectral density* of the field.

L^2_μ denotes the Hilbert space of all functions on the torus which are square summable with respect to the measure μ. From (2.1) it is clear that the operator

$$X_{mn} \to e^{-i(m\theta+n\alpha)}$$

extends to an *isomorphism* from

H_X = the closed subspace of H spanned by all X_{mn}'s ,

onto L^2_μ. This isomorphism is called the *Kolmogrov isomorphism* between the time domain and spectral domain.

For any subset M of Z^2 we define $H_X(M)$ (respectively $H_\mu(M)$) as the closed subspace spanned by all X_{mn}, (respectively $e^{-i(m\theta+n\alpha)}$), $(m,n) \in M$, in the Hilbert space H (respectively L^2_μ).

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Let $H_X^{m\infty}$, $H_X^{\infty n}$, and H_X^{mn} stand for $H_X(M)$ where M is the set $\{(r,s) : r \leqslant m, s \in Z\}$, $\{(r,s) : r \in Z, s \leqslant n\}$ and $\{(r,s) : r \leqslant m, s \leqslant n\}$ respectively. $H_\mu^{m\infty}$, $H_\mu^{\infty n}$ and H_μ^{mn} are defined similarly. For a spectral density w , by L_w^2 , $H_w^{m\infty}$, $H_w^{\infty n}$, and H_w^{mn} we will denote the corresponding spaces where μ is replaced by $w d\sigma$. Finally $P^{m\infty}$, $P^{\infty n}$, and P^{mn} stand for the orthogonal projections onto $H_X^{m\infty}$, $H_X^{\infty n}$ and H_X^{mn} respectively.

DEFINITION 2.1. A stationary field X_{mn} , $(m,n) \in Z^2$ is said to have a quarter-plane moving average representation if there is a white noise ν_{mn} and constants b_{mn} with $\sum_{(m,n)\in Z^2} |b_{mn}|^2 < \infty$ such that

$$X_{mn} = b_{00}\nu_{00} + \sum_{(p,q)\in Q} b_{pq}\, \nu_{m+p,n+q} , \tag{2.2}$$

$$H_X^{mn} = H_\nu^{mn} \quad \text{for all} \quad (m,n) \in Z^2 .$$

We need the following theorem proved by Soltani ([8], Theorem 4.3).

THEOREM 2.2. *Let* X_{mn} *be a stationary field with spectral measure* μ . *Then* X_{mn} *has a quarter-plane moving average representation if and only if it has a spectral density* w *satisfying the following conditions*

(i) $\log w \in L^1$,

(ii) *Fourier coefficients of* $\log w$ *vanish outside* $Q \cup (-Q) \cup \{(0,0)\}$,

(iii) $H_w^{00} = H_w^{0\infty} \cap H_w^{\infty 0}$.

We also need the following definitions.

DEFINITIONS 2.3.

(a) We say that the stationary random field X_{mn} has the commutative property if

$$P^{m\infty}\, P^{\infty n} = P^{mn} .$$

(b) A nonnegative measure μ is said to have the commutative property if its corresponding stationary field has the commutative property.

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The following theorem shows the connection of this commutative property with conditions (i), (ii), and (iii) of Theorem 2.2.

THEOREM 2.4. *The absolutely continuous nonnegative measure* μ *whose density* w *has the property* $\int \log w d\sigma > -\infty$ *has the commutative property if and only if it satisfies conditions (ii) and (iii) of theorem 2.2.*

Proof. The proof follows from Theorem 2.2 above and proposition 2.1.6 in Korezlioglu and Loubaton [5].

3. Factorization Theorem

In this section we will prove one of the main results of this article, namely a factorization theorem concerning factoring H^1 functions as a product of two H^2 functions (Theorem 3.1).

A summable nonnegative function w on the torus will be called *factorable* with respect to the half-plane S, defined by (1.3), if there exists a function ϕ of the form

$$\phi(\theta,\alpha) = c_{00} + \sum_{(m,n)\in S} c_{mn} e^{-i(m\theta+n\alpha)} \tag{3.1}$$

such that

$$w(\theta,\alpha) = |\phi(\theta,\alpha)|^2 . \tag{3.2}$$

Such a factor ϕ is called *optimal* if

$$|c_{00}|^2 = \exp(\int \log w d\sigma) \tag{3.3}$$

and the optimal factor is unique up to multiplication by a constant of modulus 1, [3]. Helson and Lowdenslager [2] have proved that a nonnegative summable function w has such an optimal factor with respect to S if and only if $\log w \in L^1$. In fact, to construct this factor they take the function H to be the projection of the function 1 on the subspace $H_w(S)$ and then show [2] that

$$e^{\lambda} = |1 + H|^2 w ,$$

where $\lambda = \int \log w d\sigma$, thus arriving at the factorization

$$w = \left|\frac{e^{\lambda/2}}{1 + H}\right|^2 = |\phi|^2 \tag{3.4}$$

with $\phi = \dfrac{e^{\lambda/2}}{1+H}$. It is then shown that the square summable function ϕ has the required series representation, namely

(3.5) $$\phi = \frac{e^{\lambda/2}}{1+H} = c_{00} + \sum_{(m,n)\in S} c_{mn} e^{-i(m\theta+n\alpha)} .$$

Now we can state and prove our theorem concerning the factorability of H^1 functions as the product of two H^2 functions.

THEOREM 3.1. *Let f be a summable function on the torus whose Fourier series is of the form*

(3.6) $$f \sim a_{00} + \sum_{(m,n)\in Q} a_{mn} e^{-i(m\theta+n\alpha)} ,$$

where Q is the third quadrant defined by (1.5). Suppose that

(i) $\log |f| \in L^1$,

(ii) Fourier coefficients of $\log |f|$ *vanish outside* $Q \cup (-Q) \cup \{(0,0)\}$,

(iii) $H^{0\infty}_{|f|} \cap H^{\infty 0}_{|f|} = H^{00}_{|f|}$.

Then there exist square summable functions g and h , with the same Foruier series as for f in (3.6), such that

$$f = gh .$$

Proof. Taking $w = |f|$ then w is a nonnegative function with $\log w \in L^1$ (by (i)) hence by what was proved above, w has the optimal factorization

(3.7) $$|f| = w = \left|\frac{e^{\lambda/2}}{1+H}\right|^2 .$$

Now working with the half-plane T of (1.4), instead of the half-plane S of (1.3), one can similarly factor w with respect to T as

(3.8) $$|f| = w = \left|\frac{e^{\lambda/2}}{1+K}\right|^2 ,$$

where K is the projection of 1 on $H_w(T)$. On the other hand, by Theorem 2.2, the stationary field X_{mn} corresponding to the density function w has a quarter-plane moving average representation, namely there exists a white noise ν_{mn} and constants b_{mn} with $\sum |b_{mn}|^2 < \infty$

such that

$$X_{mn} = b_{00}\nu_{00} + \sum_{(p,q)\in Q} b_{pq}\nu_{m+p,n+q}$$

$$H_X^{mn} = H_\nu^{mn} \quad \text{for all} \quad m,n \; .$$

Thus we see that $H_X(S) = H_\nu(S)$ and $H_X(T) = H_\nu(T)$. Using this fact one can see that the projections of X_{00} on $H_S(S)$ and $H_X(T)$ are the same and equal to the projection of X_{00} on $H_X(Q)$. In fact, these projections are simply

$$\sum_{(p,q)\in Q} b_{pq}\nu_{pq} \; .$$

Thus their Kolmogrov isomorphs are the same and belong to $H_\omega(Q)$. But their isomorphs are just H and K . Hence we have

(3.9) $$H = K \in H_\omega(Q) \; .$$

This means that there exists a sequence P_n of polynomials of the form (3.6) such that

$$P_n \to H \quad \text{in} \quad L_\omega^2 \; ,$$

or

$$1 + P_n \to 1 + H \quad \text{in} \quad L_\omega^2 \; ,$$

which implies that

$$1 + P_n \to 1 + H \quad \text{in} \quad L_\omega^1 \; ,$$

which means

$$(1 + P_n)\omega \to (1 + H)\omega \quad \text{in} \quad L_{d\sigma}^1 \; ,$$

hence

$$(1 + P_n)f \to (1 + H)f \quad \text{in} \quad L_{d\sigma}^1 \; .$$

This implies that $(1 + H)f$, and hence

$$h = e^{-\lambda/2}(1 + H)f$$

has the required Fourier series given in (3.5). Thus, taking the factor

g to be

$$g = \frac{e^{\lambda/2}}{1 + H}, \tag{3.10}$$

we have the factorization

$$f = gh .$$

We know that h, at least, has the required series representation. Now the function g given by (3.10) has a Fourier series of the form (3.5) and similarly we have

$$\frac{e^{\lambda/2}}{1 + K} = b_{00} + \sum_{(m,n)\in T} b_{mn} e^{-i(m\theta+n\alpha)}$$

Now since $H = K$, and hence

$$\frac{e^{\lambda/2}}{1 + H} = \frac{e^{\lambda/2}}{1 + K} = g ,$$

the function g has a Fourier series of the desired form:

$$g = d_{00} + \sum_{(m,n)\ \ Q} d_{mn} e^{-i(m\theta+n\alpha)} .$$

Finally we note that square summability of the factors follows from the fact that the factors g and h as given above have the following property

$$|g|^2 = |h|^2 = |f| . \tag{3.11}$$

This completes the proof of our theorem. □

As a corollary to Theorem 3.1, and particularly (3.11), one arrives at the following:

COROLLARY 3.2. *Any function on the unit sphere of the Hardy space* H^1 *of the torus can be factored as the product of two functions on the unit sphere of the Hardy space* H^2 *of the torus.*

To state the next theorem we need to give the following definition.

DEFINITION 3.3. Let h be a function in $L^p_{d\sigma}$ $(1 \leqslant p < \infty)$ with Fourier series of the form

$$h \sim a_{00} + \sum_{(m,n)\in Q} a_{mn} e^{-i(m\theta+n\alpha)}$$

then (a) the function h is called outer if

$$\int \log |h| d\sigma = \log \left| \int f d\sigma \right| = \log |a_{00}| > -\infty ,$$

(b) we call the function h to be strongly outer if its inverse h^{-1} lies in H_w^{00} with $w = |h|^2$.

REMARK 3.4. One can see that strongly outer functions are always outer and in the case of functions of one complex variable every outer function is strongly outer, too.

Now we can prove the following theorem.

THEOREM 3.5. *Let f be a summable function on the torus. Then f has a factorization*

$$f = gh$$

such that

(a) g and h are functions with

$$|f| = |g|^2 = |h|^2 ,$$

(b) h has Fourier series as in (3.6), and

(c) g is strongly outer

if and only if f has a Fourier series of the form (3.6) and (i), (ii), and (iii) of Theorem 3.1 hold.

Proof. If f is a summable function with a Fourier series of the form (3.6) which satisfies (i), (ii), and (iii), then the proof of Theorem 3.1 shows that the functions g and h employed there have the properties (a), (b), and (c).

Conversely, suppose that the summable function f can be factored as

$$f = gh$$

with g and h satisfying (a), (b), and (c). Then by (a) we have

$$\log |f| = 2 \log |g| .$$

From the fact that g , as an outer function, has the properties described for f in (i) and (ii), the corresponding results (i) and (ii) for f

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follow immediately. Now since g is strongly outer then $g^{-1} \in H^{00}_{|f|}$. Thus there exists a sequence P_n of trigonometric polynomials of the form (3.8) and a sequence of numbers a_n such that

$$a_n + P_n \to g^{-1} \quad \text{in} \quad L^2_{|f|} .$$

Hence

$$\int |a_n + P_n - g^{-1}|^2 \, |f| d\sigma \to 0 ,$$

which means

$$\int |(a_n + P_n)g - 1|^2 d\sigma \to 0 .$$

Thus 1 belongs to the closed subspace spanned by $ge^{i(m\theta+n\alpha)}$, $m \geq 0$, $n \geq 0$. Thus Theorem 2.18 of Soltani [8] implies that $H^{mn}_X = H^{m\infty}_X \cap H^{\infty n}_X$, and hence (iii) holds.

4. Szegö's Theorem

In this section we will give an extension of the following theorem of Szegö [9] which plays a key role in the prediction theory of stationary stochastic processes:

If μ is a finite nonnegative measure defined on the Borel set of the circle $0 \leq \theta < 2\pi$, whose absolutely continuous part is $w(e^{i\theta})d\theta/2\pi$ then we have

$$\exp\left(\int \log w \, d\sigma\right) = \operatorname*{Inf}_P \int |1 + P|^2 \, d\mu ,$$

where P ranges over the trigonometric polynomials of the form

$$P = a_1 e^{i\theta} + a_2 e^{2i\theta} + \ldots + a_n e^{ni\theta} .$$

The solution of the prediction problem for any region R of lattice points of Z^2 requires an appropriate generalization of Szegö's theorem for that region.

Helson and Lowdenslager [2] found the following generalisation of Szegö's theorem for the half-planes R which is important in the prediction of stationary fields with respect to the half-planes.

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THEOREM 4.1. *Let* μ *be a finite nonnegative measure on the torus whose absolutely continuous part is* $w(e^{i\theta}, e^{i\alpha})\, d\theta d\alpha/4\pi^2$. *Then*

$$\text{(4.1)} \qquad \operatorname*{Inf}_P \int |1 + P|^2 d\mu = \operatorname*{Inf}_P \int |1 + P|^2\, w d\sigma = \exp\left(\int \log w d\sigma\right) ,$$

where P *ranges over the trigonometric polynomials of the form*

$$\text{(4.2)} \qquad P = \sum_{(m,n)\in R} a_{mn} e^{-i(m\theta+n\alpha)} .$$

When the prediction of stationary fields with respect to a quarter-plane, say the third quadrant, is considered, we need an extension of Szegö's theorem for the third quadrant. In this section we give such an extension, however we need to assume that our measure μ has the commutative property. These kinds of conditions arise frequently whenever one is trying to extend a fact concerning functions of one complex variable to functions of two complex variables, with the set of nonnegative integers being now replaced by the third quadrant (compare Kallianpur and Mandrekar [4], Korezlioglu and Loubaton [5], and Soltani [8]).

THEOREM 4.2. *Let* μ *be a measure having the commutative property. Let* μ *and* w *be as in Theorem 4.1. Then*

$$\text{(4.3)} \qquad \operatorname*{Inf}_M \int |1 + M|^2\, d\mu = \exp\left(\int \log w d\sigma\right) ,$$

where M *ranges over the trigonometric polynomials of the form*

$$\text{(4.4)} \qquad M = \sum_{(m,n)\in Q} a_{mn} e^{-i(m\theta+n\alpha)} .$$

Proof. We first note that since the class of polynomials M in (4.4) is smaller than the class of polynomials P in (4.2), we have

$$\text{(4.5)} \qquad \operatorname*{Inf}_M \int |1 + M|^2\, d\mu \geqslant \exp\left(\int \log w d\sigma\right) .$$

Then we claim that the function H , namely the projection of 1 on $H_\mu(S)$, belongs to $H_w(Q)$. Hence there exists polynomials M_n of the form (4.4) such that

$$M_n \to H \quad \text{in} \quad L^2_\mu$$

or, equivalently,

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$$1 + M_n \to 1 + H \quad \text{in} \quad L^2_\mu ,$$

which implies

$$\int |1 + M_n|^2 d\mu \to \int |1 + H|^2 d\mu .$$

But this together with the fact that

$$\int |1 + H|^2 d\mu = \exp \left(\int \log w d\sigma\right) ,$$

(proven by Heldon and Lowdenslager [2]), shows that

$$(4.6) \qquad \int |1 + M_n|^2 d\mu \to \exp \left(\int \log w d\sigma\right) .$$

Now (4.5) and (4.6) imply the desired relation (4.3). Thus we just have to prove the above claim. To do this we first note that μ and hence its corresponding stationary field X_{mn} has the commutative property, and then we notice that since $X_{m,n-k} \in H_X^{\infty n}$, for all $k \geqslant 0$, we have $P^{\infty n} X_{m,n-k} = X_{m,n-k}$ and $P^{m-1\ \infty} X_{m,n-k} = P^{m-1\ \infty}\ P^{\infty n} X_{m,n-k} = P^{m-1\ n} X_{m,n-k}$, for all $k \geqslant 0$. Thus we get

$$P_{H_X(S)} X_{00} = P^{-1\infty} X_{00} + (X_{00} | H^{0,-1} - H_X^{-1,-1})$$

$$= P^{-1,0} X_{00} + P^{0,-1} X_{00} - P^{-1,-1} X_{00} .$$

Since each term in the right hand side belongs to $H_X(Q)$, the term on the left belongs to $H_X(Q)$ which means that its isomorph H must belong to $H_w(Q)$. This completes the proof of the claim and hence the theorem. □

This together with Theorem 2.4 gives the following corollary.

COROLLARY 4.3. *If w is a nonnegative summable function satisfying (i), (ii), and (iii) of Theorem 3.1, then we have*

$$\operatorname*{Inf}_M \int |1 + M|^2\, w d\sigma = \exp \left(\int \log w d\sigma\right) ,$$

where M ranges over all trigonometric polynomials of the form (4.4).

5. Riesz's Type Theorem

Continuing along the path of the last two sections, in this section we give an extension of the following result due to F. and M. Riesz [6] to the measures on the torus: *If* μ *is a bounded complex measure on the unit circle whose Fourier coefficients vanish for negative integers, then* μ *is absolutely continuous with respect to the Lebesgue measure.* Bochner [1] proved the following extension of this result for measures on the torus: *Suppose the complex measure* μ *on the torus has vanishing Fourier coefficients on a sector of plane with opening angle greater than* π *, then* μ *is absolutely continuous with respect to the Lebesgue measure on the torus.* Here, passing from the measures on the circle to measures on the torus, Bochner replaced the set of negative integers by a half-plane, but we are interested in replacing it by the third quadrant. We will use the prediction theoretical techniques of Helson and Lowdenslager [2] of their proof of the same theorem.

We start with the following lemma.

LEMMA 5.1. *Let* μ *be a complex measure whose total variation measure has the commutative property. If the Fourier coefficients of* μ_s *, the singular part of* μ *, vanish on* Q *, then its coefficient at (0,0) vanishes too.*

Proof. Let ν denote the total variation of μ . By theorem 4.2 and its proof the projection H of 1 on $H_\nu(Q)$ satisfies

$$\int |1 + H|^2 \, d\nu_s = 0 . \tag{5.1}$$

Since H is in $H_\nu(Q)$ there exists a sequence M_n of trigonometric polynomials of the form (4.4) such that

$$M_n \to H , \quad \text{in } L^2_\nu .$$

Hence

$$M_n \to H , \quad \text{in } L^2_{\nu_s} ,$$

which implies

$$1 + M_n \to 1 + H \quad \text{in} \quad L^2_{\nu_s} ,$$

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and hence

$$\int |1 + M_n|^2 \, d\nu_s \to \int |1 + H|^2 \, d\nu_s \,,$$

which, together with (5.1), implies

$$\int |1 + M_n|^2 \, d\nu_s \to 0 \,.$$

Now one can see that this implies

$$\int |1 + M_n| \, d\nu_s \to 0 \,, \tag{5.2}$$

and hence

$$\int (1 + M_n) \, d\mu_s \to 0 \,.$$

But, by our assumption, $\int M_n d\mu_s = 0$. Hence

$$\int d\mu_s = 0 \,,$$

and this completes the proof. □

Proofs of the following two results are similar to those of the corresponding results in Helson and Lowdenslager [2], and hence omitted.

COROLLARY 5.2. *Let* μ *be a complex measure whose total variation measure* ν *has the commutative property. If the Fourier coefficients of* μ_s *, the singular part of* μ *, vanish on*

$$Q^{rs} = \{(m,n) \; : \; m \leqslant r \; , \; n \leqslant s\} - \{(r,s)\} \tag{5.3}$$

then its Fourier coefficient at (r,s) *vanishes too.*

LEMMA 5.3. *Let* μ *be a complex measure on the torus whose total variation* ν *has the commutative property. If the Fourier coefficients of* μ *vanish on* Q^{rs} *, then the coefficients of its singular and absolutely continuous part vanish there separately.*

Now we can prove the following Riesz-Bochner type theorem, where the semigroup of negative integers of Z is now replaced by the semigroup of lattice points of the third quadrant Q of Z^2 .

THEOREM 5.4. *Let* T *be an open sector of the plane containing the third quadrant* Q . *If the Fourier coefficients of the complex measure* μ , *whose total variation has the commutative property, vanish on* T , *then* μ *is absolutely continuous.*

Proof. We can assume that this sector T is centred at the origin, since otherwise it will contain such a sector. Now since Q is contained in T , using Lemma 5.3 for $Q = Q^{00}$, we conclude that μ_s has no nonzero coefficient on Q , and by Lemma 5.1 even at the origin. On the other hand there exists a lattice point with second coordinate 1 in T . Calling this point $(m_0,1)$, then $Q^{m_0+1,1}$ clearly is contained in T . Thus applying Lemma 5.3, this time to $Q^{m_0+1,1}$, we conclude that the Fourier coefficients of μ_s on $Q^{m_0+1,1}$ are zero. Hence by corollary 5.2 its Fourier coefficient is zero at $(m_0+1,1)$ as well. Now one can see that if $m_0+2 \leqslant 0$ the Fourier coefficients of μ_s vanish on $Q^{m_0+2,1}$. In fact we have

$$Q^{m_0+2,1} \subset Q^{m_0+1,1} \cup Q \cup \{(m_0 + 1,1)\}$$

and we have already shown that the Fourier coefficients of μ_s vanish on $Q^{m_0+1,1}$, Q , and at $(m_0+1,1)$. Now using corollary 5.2 again we see that the Fourier coefficient of μ_s vanishes at $(m_0+2,1)$. If we continue in this fashion we see that the coefficients of μ_s on all lattice points of the form $(m,1)$ with $m \leqslant 0$ are zero. Now we can start an argument similar to that above with those lattice points in the left half-plane whose second coordinate is 2 instead of 1 . This will ensure us that all the corresponding Fourier coefficients vanish. Thus we can conclude that the coefficients of μ_s in the left half-plane are all zero. A similar argument shows that the Fourier coefficients of μ_s must vanish in the lower half-plane as well. Thus the Fourier coefficients of μ_s are zero in a sector with opening of $\frac{3\pi}{4}$ and hence the Bochner theorem implies the desired result.

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References

[1] S. Bochner, Boundary values of Analytic Functions in Several Variables and of almost Periodic Functions. *Ann. of Math.* 45 (1944), 708-722.

[2] H. Helson and D. Lowdenslager, Prediction Theory and Fourier Series in Several Variables. *Acta Math.* 99 (1959), 165-202.

[3] H. Helson and D. Lowdenslager, Prediction Theory and Fourier Series in Several Variables II. *Acta. Math.* 106 (1962), 175-213.

[4] G. Kallianpur and V. Mandrekar, Nondeterministic Random Fields and Wold and Halmos Decomposition for Commuting Isometries. Nov. 81. *Prediction Theory and Harmonic Analysis*, A Pesi Masani Volume, V. Mandrekar and H. Salehi, (Eds) North Holland (1983), 165-190.

[5] H. Korezlioglu and P. Loubaton, Spectral Factorization of wide sense stationary processes on Z^2, *J. Multivariate Anal.* (to appear).

[6] F. and M. Riesz, Über die Randwerte einer Analytischen Funktion. *Quaterième Congrès des Math. Scandinaves* (1916), 27-44.

[7] W. Rudin, *Function Theory in Polydiscs*. Benjamin (1969).

[8] A.R. Soltani, Extrapolation and Moving Average Representation for Stationary Random Fields and Beurling's Theorem, *Ann. Probab.*, 12 (1984), 120-132.

[9] G. Szegö, Beiträge zur Theorie der Toeplitzschen Formen (Erste Mitteilung). *Math. Z.* 6 (1920), 167-202.

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