GENERALIZED ABSOLUTE CONTINUITY OF
A FUNCTION OF WIENER’S CLASS

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In the present paper we give a criterion for a function of
Wiener’s class to belong to the class of generalized absolute
continuity, in terms of Fourier-Young coefficients \( \{c_k\} \). More
precisely, we prove the following theorem.

**Theorem.** Let \( A = (\lambda_{n,k}) \) be a normal almost periodic matrix of
real numbers such that \( \lambda_{n,k} \geq \lambda_{n,k+1} \) for all \( n \) and \( k \). Then
for any function \( f \) of Wiener’s class \( V_\nu \) \( (1 < \nu < 2) \) to be of
class of generalized absolute continuity \( A_p \) \( (1 < p < \infty) \) it is
necessary and sufficient that \( \left\{ |c_k|^2 \right\} \) is summable \( A \) to zero.

1. Introduction

Let \( f \) be a \( 2\pi \)-periodic function defined on \([0, 2\pi]\). We set

\[
V(f; a, b) = \sup \left\{ \sum_{t=1}^{n} |f(t_x) - f(t_{x-1})|^\nu \right\}^{1/\nu} \quad (1 \leq \nu < \infty),
\]

where the supremum has been taken with respect to all partitions
\( P : a = t_0 < t_1 < t_2 < \ldots < t_n = b \) of the segment \([a, b]\) contained in
\([0, 2\pi]\). We call \( V_\nu(f; a, b) \) the \( \nu \)th total variation of \( f \) on
\([a, b]\). If we denote the \( \nu \)th total variation of \( f \) on \([0, 2\pi]\) by

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253
$V_V(f)$, then we can define Wiener's class simply by

$$V_V = \{ f : V_V(f) < \infty \}.$$ 

It is clear that $V_1$ is the ordinary class of functions of bounded variation, introduced by Jordan. The class $V_V$ was first introduced by Wiener [7]. He [7] showed that functions of the class $V_V$ could only have simple discontinuities. We note [6] that

$$(1) \quad V_{V_1} \subset V_{V_2} \quad (1 \leq v_1 < v_2 < \infty)$$

is a strict inclusion. Hence for an arbitrary $1 \leq v < \infty$, Wiener's class $V_V$ is strictly larger than the class $V_1$. Wiener [7] also proved the following theorem.

**THEOREM A.** If $f \in V_v$ ($1 \leq v < \infty$) and $D[x_j] = f[x_j+0] - f[x_j-0]$ is the jump of $f$ at $x_j \in [0, 2\pi]$, then

$$V_{V_1}(f) = \sum_{j=0}^{\infty} |D[x_j]|_{V_1}$$

for all $v_1 > v$.

Recently we defined [6] the sequence of Fourier-Young coefficients by

$$C_{nk} = (2\pi)^{-1} \int_0^{2\pi} e^{ikt} \hat{f}(t) \, dt \quad (k = 0, \pm 1, \pm 2, \ldots)$$

which exists for every $f \in V_v$ ($1 \leq v < \infty$). Let $A = \{a_{nk}\}$ ($n, k = 0, 1, 2, \ldots$) be an infinite matrix of real numbers. A sequence

$\{C_{nk}\}$ is said to be summable $A$ if $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} C_{nk}$ exists; it is said to be summable $F_A$ if $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} C_{nk+v}$ exists uniformly in $v = 0, 1, 2, \ldots$. We also proved [5] the following theorem.

**THEOREM B.** Let $A = \{a_{nk}\}$ be an infinite matrix of real numbers such that $\lambda_{n,k} > \lambda_{n,k+1}$ for all $n$ and $k$. Then for every $f \in V_v$
A function of Wiener's class

(1 ≤ v < 2), the sequence \( \{ |C_k|^2 \} \) is summable to

\[
(hn)^{-1} \sum_{j=0}^{\infty} |D(x_j)|^2 \]

if and only if \( \Lambda \) is a normal almost periodic matrix.

2.

Love [2] first introduced pth power generalization of absolute continuity in the following way. For \( p > 1 \), \( A_p \) is the class of functions \( f \) which satisfy: given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
\left\{ E \mid |f(y_k) - f(x_k)|^p \right\}^{1/p} < \varepsilon
\]

for all finite sets of non-overlapping intervals \( \{(x_k, y_k)\} \) such that

\[
\sum_{i} (y_k - x_k)^p < \delta.
\]

In particular for \( p = 1 \), \( A_1 \) reduces to the class of absolutely continuous functions. It is known [2] that

\[
A_1 \subset A_p \subset C \quad (1 < p < \infty)
\]

are strict inclusions, where \( C \) denotes the class of continuous functions. Love [2] further proved the following theorem.

THEOREM C. If \( p > 1 \), a necessary and sufficient condition for \( f \) to be of \( A_p \) is that, given \( \varepsilon > 0 \), there is a subdivision

\( a = x_0 < x_1 < ... < x_n = b \) of \( [a, b] \) such that

\[
\sum_{i=1}^{n} \left( V_p(f; x_{i-1}, x_i) \right)^p < \varepsilon^p.
\]

3.

The main aim of this paper is to characterize the class \( A_p \) in terms of Fourier-Young coefficients of a function of Wiener's class \( V_v \). In other words we give a criterion for a function of Wiener's class to belong to the class \( A_p \). More precisely we prove the following theorem.
THEOREM 1. Let \( \Lambda = (\lambda_{n, k}) \) be a normal almost periodic matrix of real numbers such that \( \lambda_{n, k} > \lambda_{n, k+1} \) for all \( n \) and \( k \). Then for any function \( f \in V_\nu \) \((1 < \nu < 2)\) to be of the class \( A_p \) for every \( p > 1 \), it is necessary and sufficient that \( \left\{ |c_k|^2 \right\} \) is summable \( \Lambda \) to zero.

Proof. If \( f \in A_p \) for \( p > 1 \), it is clearly continuous and hence \( D[x_j] = 0 \) for all \( j = 1, 2, 3, \ldots \). It follows from Theorem B that \( \left\{ |c_k|^2 \right\} \) is summable \( \Lambda \) to zero.

Conversely suppose that \( \left\{ |c_k|^2 \right\} \) is summable \( \Lambda \) to zero, that is,

\begin{equation}
\lim_{n \to \infty} \sum_{k=1}^{\infty} \lambda_{n, k} |c_k|^2 = 0.
\end{equation}

But if \( f \in V_\nu \) \((1 < \nu < 2)\), then we easily obtain [5] by using Theorem B that

\begin{equation}
\lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n, k} |c_k|^2 = (1\pi)^{-1} \sum_{j=0}^{\infty} |D[x_j]|^2.
\end{equation}

From (3) and (4) we conclude that \( \sum_{j=0}^{\infty} |D[x_j]|^2 = 0 \). Since \( f \in V_\nu \) \((1 < \nu < 2)\), it follows from Theorem A that

\begin{equation}
\sum_{j=1}^{\infty} |D[x_j]|^2 = V_2(f),
\end{equation}

which is equal to zero. Now using Theorem C, we conclude that \( f \in A_p \) for every \( p > 1 \). This completes the proof of Theorem 1.

Applying Theorem 1 and Schwarz's inequality, we obtain the following theorem.

THEOREM 2. For \( f \in V_\nu \) \((1 < \nu < 2)\), the following statements are equivalent:

(1) \( f \in A_p \) for every \( p > 1 \);
A function of Wiener’s class

(2) $\{ |c_k|^2 \}$ is summable $\Lambda$ to zero by a normal almost periodic matrix such that $\lambda_{n,k} > \lambda_{n,k+1}$ for all $n$ and $k$.

(3) $|c_k|$ is summable $\Lambda$ to zero by a normal almost periodic matrix such that $\lambda_{n,k} > \lambda_{n,k+1}$ for all $n$ and $k$.

We can further reformulate Theorem 2 in the following:

**THEOREM 3.** For $f \in V_\nu$ ($1 < \nu < 2$) to be of the class $A_p$ for every $p > 1$, it is necessary that $\{ |c_k|^2 \}$ is summable $\Lambda$ to zero by each normal almost periodic matrix $\Lambda$ for which $\lambda_{n,k} > \lambda_{n,k+1}$ for all $n$ and $k$ and sufficient that $|c_k|$ is summable $\Lambda$ to zero by some normal almost periodic matrix for which $\lambda_{n,k} > \lambda_{n,k+1}$ for all $n$ and $k$.

Theorem 3 extends the various theorems on continuity to the generalized class of absolute continuity $A_p$ including those given by Wiener [7], Lozinski [3], Matveev [4] (cf. Bary [1], p. 256) and Siddiqi [5].

We also like to remark here that if $\nu \geq 2$, there is no necessary and sufficient condition for $f$ to be of the class $A_p$ in terms of the summability of the absolute value of its Fourier-Young coefficients. For we have the following two functions:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}; \quad g(x) = \sum_{k=1}^{\infty} \frac{\sin(kx+\ln k)}{k}.$$ 

It is easy to verify (cf. Zygmund [4], pp. 241-243) that both series in (5) converge for all $x$. We also note [4] that $f(x)$ is a discontinuous function belonging to $V_1$ and $g(x)$ belongs to $\text{Lip}_\infty$ and hence belongs to $V_2$. We can determine the values of the Fourier-Young coefficients $C_k(f) = C_k(g) = 1$ for $k = 1, 2, 3, \ldots$ and $C_0(f) = C_0(g) = 0$. In this way we obtain two functions $f$ and $g$ belonging to $V_\nu$ ($2 < \nu < \infty$); the first is discontinuous and hence does not belong to $A_p$ for $p > 1$. 

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and the second belongs to \( A_p \) for \( p > 1 \) such that \( C_k(f) = C_k(g) \) for 
\( k = 0, 1, 2, \ldots \). Hence Theorem 2 and Theorem 3 cannot be extended for \( v \geq 2 \) in terms of Fourier-Young coefficients.

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