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GENERALIZED ABSOLUTE CONTINUITY OF A FUNCTION OF WIENER'S CLASS

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In the present paper we give a criterion for a function of Wiener's class to belong to the class of generalized absolute continuity, in terms of Fourier-Young coefficients $\{C_k\}$. More precisely, we prove the following theorem.

THEOREM. *Let* $\Lambda = (\lambda_{n,k})$ *be a normal almost periodic matrix of real numbers such that* $\lambda_{n,k} \geq \lambda_{n,k+1}$ *for all* n *and* k . *Then for any function* f *of Wiener's class* V_ν $(1 < \nu < 2)$ *to be of class of generalized absolute continuity* A_p $(1 < p < \infty)$ *it is necessary and sufficient that* $\left\{|C_k|^2\right\}$ *is summable* Λ *to zero.*

1. Introduction

Let f be a 2π-periodic function defined on $[0, 2\pi]$. We set

$$V\ (f;\ a,\ b) = \sup\left\{\sum_{i=1}^{n} |f(t_i)-f(t_{i-1})|^\nu\right\}^{1/\nu} \quad (1 \leq \nu < \infty) ,$$

where the supremum has been taken with respect to all partitions $P : a = t_0 < t_1 < t_2 < \ldots < t_n = b$ of the segment $[a, b]$ contained in $[0, 2\pi]$. We call $V_\nu(f;\ a,\ b)$ the νth total variation of f on $[a, b]$. If we denote the νth total variation of f on $[0, 2\pi]$ by

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$V_\nu(f)$, then we can define Wiener's class simply by

$$V_\nu = \{f : V_\nu(f) < \infty\} .$$

It is clear that V_1 is the ordinary class of functions of bounded variation, introduced by Jordan. The class V_ν was first introduced by Wiener [7]. He [7] showed that functions of the class V_ν could only have simple discontinuities. We note [6] that

$$(1) \qquad V_{\nu_1} \subset V_{\nu_2} \quad (1 \le \nu_1 < \nu_2 < \infty)$$

is a strict inclusion. Hence for an arbitrary $1 \le \nu < \infty$, Wiener's class V_ν is strictly larger than the class V_1 . Wiener [7] also proved the following theorem.

THEOREM A. *If* $f \in V_\nu$ $(1 \le \nu < \infty)$ *and* $D(x_j) = f(x_j+0) - f(x_j-0)$ *is the jump of* f *at* $x_j \in [0, 2\pi]$ *, then*

$$V_{\nu_1}(f) = \sum_{j=0}^{\infty} |D(x_j)|^{\nu_1}$$

for all $\nu_1 > \nu$.

Recently we defined [6] the sequence of Fourier-Young coefficients by

$$C_k = (2\pi)^{-1} \int_0^{2\pi} e^{ikt} df(t) \quad (k = 0, \pm 1, \pm 2, \ldots)$$

which exists for every $f \in V_\nu$ $(1 \le \nu < \infty)$. Let $\Lambda = (\lambda_{n,k})$ $(n, k = 0, 1, 2, \ldots)$ be an infinite matrix of real numbers. A sequence $\{C_k\}$ is said to be summable Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} C_k$ exists; it is said to be summable F_Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} C_{k+\nu}$ exists uniformly in $\nu = 0, 1, 2, \ldots$. We also proved [5] the following theorem.

THEOREM B. *Let* $\Lambda = (\lambda_{n,k})$ *be an infinite matrix of real numbers such that* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k *. Then for every* $f \in V_\nu$

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$(1 \le \nu < 2)$, *the sequence* $\left\{|C_k|^2\right\}$ *is summable* Λ *to*

$(4\pi)^{-1} \sum_{j=0}^{\infty} |D(x_j)|^2$ *if and only if* Λ *is a normal almost periodic matrix.*

2.

Love [2] first introduced pth power generalization of absolute continuity in the following way. For $p > 1$, A_p is the class of functions f which satisfy: given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left\{\Sigma \mid f(y_k)-f(x_k) \mid^p\right\}^{1/p} < \varepsilon$$

for all finite sets of non overlapping intervals $\{(x_k, y_k)\}$ such that

$$\left\{\Sigma(y_k-x_k)^p\right\}^{1/p} < \delta .$$

In particular for $p = 1$, A_1 reduces to the class of absolutely continuous functions. It is known [2] that

(2) $$A_1 \subset A_p \subset C \quad (1 < p < \infty)$$

are strict inclusions, where C denotes the class of continuous functions. Love [2] further proved the following theorem.

THEOREM C. *If* $p > 1$, *a necessary and sufficient condition for* f *to be of* A_p *is that, given* $\varepsilon > 0$, *there is a subdivision* $a = x_0 < x_1 < \dots < x_n = b$ *of* $[a, b]$ *such that*

$$\sum_{i=1}^{n} \left(V_p(f; x_{i-1}, x_i)\right)^p < \varepsilon^p .$$

3.

The main aim of this paper is to characterize the class A_p in terms of Fourier-Young coefficients of a function of Wiener's class V_ν . In other words we give a criterion for a function of Wiener's class to belong to the class A_p . More precisely we prove the following theorem.

THEOREM 1. *Let* $\Lambda = (\lambda_{n,k})$ *be a normal almost periodic matrix of real numbers such that* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k . *Then for any function* $f \in V_\nu$ $(1 < \nu < 2)$ *to be of the class* A_p *for every* $p > 1$, *it is necessary and sufficient that* $\{|C_k|^2\}$ *is summable* Λ *to zero.*

Proof. If $f \in A_p$ for $p > 1$, it is clearly continuous and hence $D(x_j) = 0$ for all $j = 1, 2, 3, \ldots$. It follows from Theorem B that $\{|C_k|^2\}$ is summable Λ to zero.

Conversely suppose that $\{|C_k|^2\}$ is summable Λ to zero, that is,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_{n,k}|C_k|^2 = 0 . \tag{3}$$

But if $f \in V_\nu$ $(1 < \nu < 2)$, then we easily obtain [5] by using Theorem B that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}|C_k|^2 = (4\pi)^{-1} \sum_{j=0}^{\infty} |D(x_j)|^2 . \tag{4}$$

From (3) and (4) we conclude that $\sum_{j=0}^{\infty} |D(x_j)|^2 = 0$. Since $f \in V_\nu$ $(1 < \nu < 2)$, it follows from Theorem A that

$$\sum_{j=1}^{\infty} |D(x_j)|^2 = V_2(f) ,$$

which is equal to zero. Now using Theorem C, we conclude that $f \in A_p$ for every $p > 1$. This completes the proof of Theorem 1.

Applying Theorem 1 and Schwarz's inequality, we obtain the following theorem.

THEOREM 2. *For* $f \in V_\nu$ $(1 < \nu < 2)$, *the following statements are equivalent:*

(1) $f \in A_p$ *for every* $p > 1$;

(2) $\left\{|C_k|^2\right\}$ *is summable* Λ *to zero by a normal almost periodic matrix such that* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k ;

(3) $|C_k|$ *is summable* Λ *to zero by a normal almost periodic matrix such that* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k .

We can further reformulate Theorem 2 in the following:

THEOREM 3. *For* $f \in V_\nu$ $(1 < \nu < 2)$ *to be of the class* A_p *for every* $p > 1$ *, it is necessary that* $\left\{|C_k|^2\right\}$ *is summable* F_Λ *to zero by each normal almost periodic matrix* Λ *for which* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k *and sufficient that* $\left\{|C_k|^2\right\}$ *is summable* Λ *to zero by some normal almost periodic matrix for which* $\lambda_{n,k} > \lambda_{n,k+1}$ *for all* n *and* k .

Theorem 3 extends the various theorems on continuity to the generalized class of absolute continuity A_p including those given by Wiener [7], Lozinskiĭ [3], Matveev [4] (*cf.* Bary [1], p. 256) and Siddiqi [5].

We also like to remark here that if $\nu \geq 2$, there is no necessary and sufficient condition for f to be of the class A_p in terms of the summability of the absolute value of its Fourier-Young coefficients. For we have the following two functions:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \; ; \quad g(x) = \sum_{k=1}^{\infty} \frac{\sin k(x+\ln k)}{k} . \tag{5}$$

It is easy to verify (*cf.* Zygmund [8], pp. 241-243) that both series in (5) converge for all x . We also note [8] that $f(x)$ is a discontinuous function belonging to V_1 and $g(x)$ belongs to $\mathrm{Lip}_{\frac{1}{2}}$ and hence belongs to V_2 . We can determine the values of the Fourier-Young coefficients $C_k(f) = C_k(g) = 1$ for $k = 1, 2, 3, \ldots$ and $C_0(f) = C_0(g) = 0$. In this way we obtain two functions f and g belonging to V_ν $(2 < \nu < \infty)$; the first is discontinuous and hence does not belong to A_p for $p > 1$

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and the second belongs to A_p for $p > 1$ such that $c_k(f) = c_k(g)$ for $k = 0, 1, 2, \ldots$. Hence Theorem 2 and Theorem 3 cannot be extended for $\nu \geq 2$ in terms of Fourier-Young coefficients.

References

[1] N.K. Bary, *A treatise on trigonometric series*, Volume I (translated by Margaret F. Mullins. Pergamon Press, London, New York, Paris, 1964).

[2] E.R. Love, "A generalization of absolute continuity", *J. London Math. Soc.* 26 (1951), 1-13.

[3] S. Lozinski, "On a theorem of N. Wiener", *C.R. (Doklady) Acad. Sci. URSS (N.S.)* 49 (1945), 542-545.

[4] Б.В. Матвеев [V.A. Matveev], "О теоремах Винера и Лозинского" [On the theorems of Wiener and Lozinskii], *Исследования по современным проблемам конструктивной теории функций* [*Studies of contemporary problems in the constructive theory of functions*], 460-468 (Proc. Second All-Union Conf., Baku, 1962. Izdat. Akad. Nauk Azerbaidzan. SSR, Baku, 1965).

[5] Rafat N. Siddiqi, "Determination of jump of a function of V_p by means of its Fourier-Stieltjes coefficients", *Rend. Circ. Mat. Palermo* (2) 23 (1974), 135-144.

[6] Rafat N. Siddiqi, "Fourier-Young coefficients of a function of Wiener's class V_p ", *Canad. J. Math.* 28 (1976), 753-759.

[7] N. Wiener, "The quadratic variation of a function and its Fourier coefficients", *Massachusetts J. Math.* 3 (1924), 72-94.

[8] A. Zygmund, *Trigonometric series*, Volume I (Cambridge University Press, Cambridge, 1959).

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