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On Closed Ideals in a Certain Class of Algebras of Holomorphic Functions

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Abstract. We recently introduced a weighted Banach algebra $\mathcal{A}_G^n$ of functions that are holomorphic on the unit disc $\mathbb{D}$, continuous up to the boundary, and of the class $C^{(n)}$ at all points where the function G does not vanish. Here, G refers to a function of the disc algebra without zeros on $\mathbb{D}$. Then we proved that all closed ideals in $\mathcal{A}_G^n$ with at most countable hull are standard. In this paper, on the assumption that G is an outer function in $C^{(n)}(\overline{\mathbb{D}})$ having infinite roots in $\mathcal{A}_G^n$ and countable zero set $h_0(G)$, we show that all the closed ideals I with hull containing $h_0(G)$ are standard.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$, and let $\mathcal{A}$ be the disc algebra formed by all functions holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let G be in $\mathcal{A}$ such that $G(z) \neq 0$ for all $z \in \mathbb{D}$. For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, define $\mathcal{A}_G^n$ as the completion of the space of polynomials with respect to the norm

$$\|f\|_{G,n} = \sum_{k=0}^{n} \sup_{z\in\mathbb{D}} |G^k f^{(k)}(z)|.$$

These algebras were introduced in [5] as generalizations of the algebras studied in [1] and [2], which in our notation correspond to the case of $G(z) = z^2 - 1$.

The algebras $\mathcal{A}_{z^2-1}^n$ are isomorphic copies of Banach algebras, which appear to be natural ranges of the Gelfand transform of certain convolution Sobolev algebras of functions on the real half-line $\mathbb{R}_+$.

Using results from [6], we prove in [5] that for $G \in \mathcal{A}$ which is differentiable up to the boundary $\mathbb{T}$ of $\mathbb{D}$, the closed ideals $I \subset \mathcal{A}_G^n$ with at most countable hull

$$h_0(I) := \{z \in \mathbb{T} : f(z) = 0, f \in I\}$$

are standard. (See the definition in Section 2.) This partially extends the results of [2].

This paper concerns the case of an outer function G, which, moreover, is of class $C^{(n)}$ on $\overline{\mathbb{D}}$ and vanishes on an at most countable set. Hence the function G is an element of $\mathcal{A}_G^n$. The main result provides conditions assuring that all closed ideals contained in the closed ideal generated in $\mathcal{A}_G^n$ by G are standard. Accordingly to Proposition 3.1, the class of above ideals coincides with the class of ideals which satisfy $\{z \in \mathbb{T} : G(z) = 0\} \subset h_0(I)$.

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2 Standard Ideals in the Algebras $\mathcal{A}^n$ and $\mathcal{A}^n_G$

Let $\mathcal{A}^n$ be the algebra of functions holomorphic on the disc $\mathbb{D}$ and of the class $C^{(n)}$ up to the boundary, provided with the norm $\|f\|_n = \sum_{k=0}^n \sup_{z\in\mathbb{D}} |f^{(k)}(z)|$. This norm is visibly stronger than the norm $\|\cdot\|_{G,n}$ restricted to $\mathcal{A}^n$. Taking into account that the space of complex polynomials is dense in $(\mathcal{A}^n, \|\cdot\|_n)$, we conclude that $\mathcal{A}^n$ is continuously and densely contained in $\mathcal{A}^n_G$.

With a closed ideal J in $\mathcal{A}^n$ we associate the function U_J, which is the greatest common inner divisor of all non-zero elements of J. Let

$$E_j(J) = \{z \in \mathbb{T} : f(z) = f'(z) = \cdots = f^{(j)}(z) = 0 \ f \in J\}, \quad 0 \le j \le n.$$

The sets $E_j(J)$ are compact and form a descending family. Let $J(U_J; E_0(J), \dots, E_n(J))$ denote the closed ideal of functions $f \in \mathcal{A}^n$ such that U_J divides f (denoted by $U_J|f$) and $f^{(j)}|_{E_j(J)} \equiv 0$, $0 \le j \le n$. The above closed ideals are called standard.

B. I. Korenblyum proved that all closed ideals of $\mathcal{A}^n$ are standard.

Theorem 2.1 ([4]) *Let J be a closed nontrivial ideal in $\mathcal{A}^n$. Then*

$$J = J\big(U_J; E_0(J), \dots, E_n(J)\big).$$

The convergence of a sequence of functions with respect to the norm $\|\cdot\|_{G,n}$ implies the uniform convergence on $\overline{\mathbb{D}}$ and the almost uniform convergence of the derivatives of the functions up to the order n on the set $\mathbb{T} \setminus h_0(G)$. It follows that every element of $\mathcal{A}^n_G$ is a function holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and of the class $C^{(n)}$ on $\overline{\mathbb{D}} \setminus h_0(G)$, since the space of polynomials is dense in $\mathcal{A}^n_G$ by definition.

The following result gives a connection between Korenblyum's theorem and the problem of the structure of ideals in the algebra $\mathcal{A}^n_G$.

For $G \in \mathcal{A}^n$ and $f \in \mathcal{A}^n_G$ define $\gamma(f) = G^n f$.

Proposition 2.2 *If $G \in \mathcal{A}^n$ and $f \in \mathcal{A}^n_G$, then $\gamma(f) \in \mathcal{A}^n$ and $\gamma: \mathcal{A}^n_G \to \mathcal{A}^n$ is an injective bounded linear operator.*

Proof For $0 \le k \le n$ we can write $(G^n)^{(k)} = G^{n-k} F_k$, where $F_k \in \mathcal{A}$ depends polynomially on G and on the derivatives of G up to the order k. For $f \in \mathcal{A}^n_G$ and $0 \le i \le n$, we have

$$(G^n f)^{(i)} = \sum_{m=0}^{i} \binom{i}{m} f^{(m)} (G^n)^{(i-m)} = \sum_{m=0}^{n} \binom{i}{m} f^{(m)} G^m G^{n-i} F_{i-m}. \tag{2.1}$$

For $i \le n$ this function has a continuous extension on the closure $\overline{\mathbb{D}}$, hence $\gamma(f) = G^n f \in \mathcal{A}^n$.

Equation (2.1) implies

$$\sup_{z\in\mathbb{D}} |(G^n f)^{(i)}(z)| \le C \sum_{m=0}^{n} \sup_{z\in\mathbb{D}} |f^{(m)} G^m(z)|,$$

where the constant C depends only on G and i. It follows that γ is a continuous operator. It is obvious that γ is injective. ∎

Let I be an ideal in $\mathcal{A}^n_G$. Set $h_0(I) = \{z \in \mathbb{T} : f(z) = 0, \ f \in I\}$, and for $1 \le j \le n$ let

$$h_j(I) = \{z \in \mathbb{T} \setminus h_0(G) : f(z) = f'(z) = \cdots = f^{(j)}(z) = 0\}.$$

The set $h_0(I)$ is compact, while the sets $h_j(I)$, $1 \le j \le n$, are relatively closed in $\mathbb{T} \setminus h_0(I)$. An ideal I in $\mathcal{A}^n_G$ is said to be standard if it is of the form

$$\begin{aligned} I &= I(U_I; h_0(I), h_1(I), \ldots, h_n(I)) \\ &:= \{f \in \mathcal{A}^n_G : U_I | f, \ f^{(k)}|_{h_k(I)} \equiv 0, \ 0 \le k \le n\}. \end{aligned}$$

The following theorem was proved in [5].

Theorem 2.3 *Let $G \in \mathcal{A}^1$ be a function nowhere vanishing in $\mathbb{D}$. Every closed ideal I in $\mathcal{A}^n_G$ such that $h_0(I)$ is at most countable is standard.*

3 Main Results

A function f holomorphic on $\mathbb{D}$ is called an outer function if the unique inner function dividing f is constant. For $F \in \mathcal{A}^n_G$ let I_F denote the ideal algebraically generated in $\mathcal{A}^n_G$ by F and let $\overline{I}_F$ be its closure in $\mathcal{A}^n_G$.

Proposition 3.1 *Let $G \in \mathcal{A}^n$ be an outer function such that the hull $h_0(G)$ is at most countable. Then for arbitrary $k \in \mathbb{N}$,*

$$\overline{I}_{G^k} = I(1; h_0(G), \varnothing, \ldots, \varnothing).$$

For $k \ge n$, we have $I_{G^k} \subset \mathcal{A}^n$.

Proof The first assertion follows by Theorem 2.3. For $k \ge n$ and $f = G^k h$ with $h \in \mathcal{A}^n_G$, we obtain $f = \gamma(G^{k-n} h) \in \mathcal{A}^n$ by Proposition 2.2. ■

Theorem 3.2 *Let $G \in \mathcal{A}^n$ be an outer function such that the hull $h_0(G)$ is at most countable. Suppose that there exists in $\overline{I}_G$ an approximate unit, that is, a sequence (ϕ_m) such that $\phi_m f \to f$ for every $f \in \overline{I}_G$. Then for every closed ideal I in $\mathcal{A}^n_G$ contained in $\overline{I}_G$, the $\mathcal{A}^n$-ideal $J(U_I; h_0(I), h_1(I) \cup h_0(G), \ldots, h_n(I) \cup h_0(G))$ is dense in I.*

Proof By Proposition 3.1 the assumption that the $\mathcal{A}^n_G$-ideal I is contained in $\overline{I}_G$ is equivalent to the condition $h_0(G) \subset h_0(I)$. The sets $h_j(I)$ are relatively closed in $\mathbb{T} \setminus h_0(G)$, therefore the sets $h_j(I) \cup h_0(G)$ are compact.

Once again by Proposition 3.1 the ideal $I_{G^{2n}}$ is dense in $\overline{I}_G$. The space $\mathcal{A}^n$ is dense in $\mathcal{A}^n_G$. Consequently, for every $m \in \mathbb{N}$, there is $h_m \in \mathcal{A}^n_G$ such that for $\psi_m = G^{2n} h_m$ it holds $\|\phi_m - \psi_m\|_{G,n} \le \frac{1}{m}$. Evidently, the sequence (ψ_m) is also an approximate unit in $\overline{I}_{G^k}$. In particular, for every $f \in I$ we have that $\lim_{m \to \infty} G^{2n} h_m f = f$. The elements $h_m G^n f$ belong to $I \cap \mathcal{A}^n$, as well as the elements $h_m G^{2n} f$, which, moreover, vanish on $h_0(G)$ with all derivatives of order $j \le n$. This proves that

$$\psi_m f \in J\big(U_I; h_0(I), h_1(I) \cup h_0(G), \ldots, h_n(I) \cup h_0(G)\big)$$

and concludes the proof. ■

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Theorem 3.3 *Under the assumptions of Theorem 3.2 the closed ideal I is standard.*

Proof Denote $\widetilde{I} = I(U_I; h_0(I), \dots, h_n(I))$. Then $\widetilde{I}$ is an ideal in $\mathcal{A}^n_G$ that contains I and is contained in $\overline{I}_G$. By Theorem 3.2 the set

$$\widetilde{J} := J(U_{\widetilde{I}}; h_0(\widetilde{I}), h_1(\widetilde{I}) \cup h_0(G), \dots, h_n(\widetilde{I}) \cup h_0(G))$$

is dense in $\widetilde{I}$. On the other hand, $U_{\widetilde{I}} = U_I$, $h_j(\widetilde{I}) = h_j(I)$, $0 \le j \le n$, hence $\widetilde{J} = J(U_I; h_0(I), h_1(I) \cup h_0(G), \dots, h_n(I) \cup h_0(G))$. The latter set is dense in I by Theorem 3.2. We obtain $I = \widetilde{I}$. ■

4 Constructions of Approximate Units

There is a natural candidate for a (bounded) approximate identity in the ideal $\overline{I}_G$. Every outer function is of the form $G = e^F$ for some function F holomorphic on $\mathbb{D}$. The functions $\phi_m = G^{\frac{1}{m}} = e^{\frac{1}{m}F}$ are well defined on $\overline{\mathbb{D}}$ and, as proved subsequently, form a bounded sequence with respect to the norm $\|\cdot\|_{G,n}$. For every $f \in \overline{I}_G$ the convergence $\phi_m f \to f$ does hold. However, it is not obvious that ϕ_m belongs to $\mathcal{A}^n_G$. In this section we give some sufficient condition for $\phi_m \in \mathcal{A}^n_G$ to hold.

The following result was proved in [5].

Lemma 4.1 *Let X be a compact topological space. Suppose that $C(X) \ni g_m \to g$ uniformly on X. Suppose that g vanishes on a closed set $S \subset X$. Let ψ_m be a bounded sequence in $C(X)$ such that $\psi_m \to 1$ almost uniformly on $X \setminus S$. Then $g_m\psi_m \to g$ uniformly.*

Denote by $\mathfrak{A}^n_G$ the subspace of functions g in the disc algebra such that for every $0 \le j \le n$ the function $G^j g^{(j)}$ extends continuously to $\overline{\mathbb{D}}$ and vanishes on $h_0(G)$. The function $\|\cdot\|_{G,n}$ is a norm on $\mathfrak{A}^n_G$. Notice that $\mathcal{A}^n_G$ is a closed subspace of $\mathfrak{A}^n_G$. In fact, for every $z_0 \in h_0(G)$ and $0 \le j \le n$ the functional "$g \to G^j g^{(j)}(z_0)$" is continuous on $\mathcal{A}^n_G$ and vanishes on polynomial functions, hence it vanishes on $\mathcal{A}^n_G$.

Theorem 4.2 *Suppose that for some $0 < \delta < 1$ the function G satisfies the condition*

$$M = \sup_{(z,t)\in\mathbb{D}\times[\delta,1)} \frac{|G(z)|}{|G(tz)|} < \infty.$$

Then $\mathfrak{A}^n_G = \mathcal{A}^n_G$.

Proof For $f \in \mathfrak{A}^n_G$, $0 < t \le 1$, denote $f_t(z) = f(tz)$. Since $f_t \in \mathcal{A}^n$, it suffices to prove that $\lim_{t\nearrow 1} f_t = f$ with respect to the norm $\|\cdot\|_{G,n}$.

To see this, note that $\lim_{t\to 1}(G^j f^{(j)})_t = G^j f^{(j)}$ uniformly on $\overline{\mathbb{D}}$ for $0 \le j \le n$.

Now,

$$G^j(f_t)^{(j)} = G^j t^j (f^{(j)})_t = t^j\Big(\frac{G}{G_t}\Big)^j (G^j f^{(j)})_t.$$

Take t_m such that $0 < t_m < 1$ and $t_m \to 1$ as $m \to \infty$. Then the functions $g_m = (G^j f^{(j)})_{t_m}$ tend uniformly to $g = G^j f^{(j)}$ as $m \to \infty$ and the latter function vanishes on $h_0(G)$.

Moreover, the functions $\psi_m = (G/G_{t_m})^j$ converge to 1 almost uniformly on $\mathbb{D} \setminus h_0(G)$. By Lemma 4.1 the proof follows. ■

The next result follows by simple calculation.

Proposition 4.3 *Let $G \in \mathcal{A}^n$ be an outer function. The sequence $\phi_m = G^{\frac{1}{m}}$ belongs to the space $\mathfrak{A}_G^n$ and is bounded therein.*

We need also the following result whose proof is straightforward by induction.

Lemma 4.4 *Let $p > 0$, $k \in \mathbb{N}$. Then*

$$(G^p)^{(k)} = pG^{p-1}G^{(k)} + p(p-1) \sum_{\mathbf{r}=(r_0,\dots,r_{k-1})} a_{\mathbf{r}} G^{p-1+r_0}(G')^{r_1} \cdots (G^{(k-1)})^{r_{k-1}},$$

where $r_0 \geq -k$, $r_j \in \{0, 1, \dots, k\}$ for $j = 1, \dots, k-1$ and the coefficients $\alpha_{\mathbf{r}}$ are polynomials in p whose degree and coefficients depend only on k.

Proposition 4.5 *Let $G \in \mathcal{A}^n$ be an outer function. Let $\phi_m = G^{\frac{1}{m}}$. Then $\phi_m G \to G$ in the space $\mathfrak{A}_G^n$.*

Proof Without loss of generality we can suppose that $\|G\|_\infty = 1$.
By Lemma 4.4 we obtain for $p = 1 + \frac{1}{m}$:

$$G^k\Big(\phi_m G\Big)^{(k)} = \Big(1 + \frac{1}{m}\Big)\phi_m G^k G^{(k)} + \\ + \Big(1 + \frac{1}{m}\Big)\frac{1}{m} \sum_{\mathbf{r}=(r_0,\dots,r_{k-1})} a_{\mathbf{r}} \phi_m G^{r_0+k}(G')^{r_1} \cdots (G^{(k-1)})^{r_{k-1}}.$$

The first term on the right-hand side tends to $G^k G^{(k)}$ uniformly by Proposition 4.3 and Lemma 4.1. Other terms of the finite sum on the right-hand contain non-negative powers of G and of the derivatives of G of order less or equal to n, hence they are bounded with respect to $\|\cdot\|_\infty$ norm. Thanks to the coefficient $\frac{1}{m}$ these terms tend to zero uniformly. ■

Theorem 4.6 *Let $G \in \mathcal{A}^n$ be an outer function such that $h_0(G)$ is at most countable. Suppose that G satisfies one of the following conditions:*

(i) *there exists $0 < \delta < 1$ such that $\sup_{(z,t)\in\mathbb{D}\times[\delta,1)} \frac{|G(z)|}{|G(tz)|} < \infty$;*

(ii) *$G \in \mathcal{A}^\infty(\mathbb{D})$ and all zeros of G are of infinite multiplicity.*

Then $\phi_m = G^{\frac{1}{m}}$ is an approximate unit in the closed ideal $\overline{I}_G$ generated in $\mathcal{A}_G^n$.
In consequence, every closed ideal in $\mathcal{A}_G^n$ contained in $\overline{I}_G$ is standard.

Proof If condition (i) is satisfied, then $\mathcal{A}_G^n = \mathfrak{A}_G^n$ by Theorem 4.2. The sequence ϕ_m belongs $\mathcal{A}_G^n$ and is bounded in this space. Every element of the sequence vanishes on $h_0(G)$ and, according to Proposition 3.1, belongs to $\overline{I}_G$.

Then, since the sequence $(\phi_m)_m$ is bounded in $\mathcal{A}_G^n$, the convergence $\phi_m G \to G$ implies the convergence $\phi_m f \to f$ for every $f \in \overline{I}_G$. On the other hand, if G is an outer function with all zeros of infinite multiplicity, then $G^{\frac{1}{m}} \in \mathcal{A}^n(\mathbb{D}) \subset \mathcal{A}^n_G(\mathbb{D})$ and the proof follows by the same arguments as above.

Theorem 3.3 proves the second assertion. ∎

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