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Remark on Integral Means of Derivatives of Blaschke Products

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Abstract. If B is the Blachke product with zeros $\{z_n\}$, then $|B'(z)| \le \Psi_B(z)$, where

$$\Psi_B(z) = \sum_n \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2}.$$

Moreover, it is a well-known fact that, for $0 < p < \infty$,

$$M_p(r,B') = \Big(\frac{1}{2\pi}\int_0^{2\pi} |B'(re^{i\theta})|^p\,d\theta\Big)^{1/p}, \quad 0 \le r < 1,$$

is bounded if and only if $M_p(r,\Psi_B)$ is bounded. We find a Blaschke product B_0 such that $M_p(r,B_0')$ and $M_p(r,\Psi_{B_0})$ are not comparable for any $\frac{1}{2} < p < \infty$. In addition, it is shown that, if $0 < p < \infty$, B is a Carleson–Newman Blaschke product and a weight ω satisfies a certain regularity condition, then

$$\int_{\mathbb{D}} |B'(z)|^p \omega(z)\,dA(z) \asymp \int_{\mathbb{D}} \Psi_B(z)^p \omega(z)\,dA(z),$$

where $dA(z)$ is the Lebesgue area measure on the unit disc.

1 Notation

Let $\mathcal{H}(\mathbb{D})$ be the collection of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. We say that $f \in \mathcal{H}(\mathbb{D})$ belongs to the Nevanlinna class N if

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta < \infty,$$

where $\log^+ 0 = 0$ and $\log^+ x = \max\{0, \log x\}$ for $0 < x < \infty$. The Hardy space H^p with $0 < p \le \infty$, which consists of $f \in \mathcal{H}(\mathbb{D})$ satisfying $\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty$, is a proper subspace of N [3]. Here

$$M_p(r,f) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\Big)^{1/p}, \quad 0 < p < \infty,$$

and $M_\infty(r,f) = \max_{|z|=r} |f(z)|$.

A function $\omega\colon \mathbb{D} \to [0,\infty)$ is called a *weight* if it is integrable over the unit disc $\mathbb{D}$. A weight ω is said to be *radial* if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω consists of $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) < \infty,$$

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where $dA(z)$ is the Lebesgue area measure on $\mathbb{D}$. In the classical case where $\omega(z) = (1-|z|)^\alpha$ for some $-1 < \alpha < \infty$, the notation A^p_α is used. The class $\widehat{\mathcal{D}}$ of doubling weights consists of radial weights ω such that $\widehat{\omega}(z) \lesssim \widehat{\omega}(\frac{1+|z|}{2})$, where $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\,ds$. A radial weight ω belongs to $\widehat{\mathcal{D}}$ if and only if

$$\sup_{0<r<1} \frac{(1-r)^p}{\widehat{\omega}(r)} \int_0^r \frac{\omega(s)}{(1-s)^p}\,ds < \infty \tag{1.1}$$

for some $0 < p < \infty$ [15]. If (1.1) holds for some fixed p, then we write $\omega \in \widehat{\mathcal{D}}_p$. For example, $\omega(z) = (1-|z|)^\alpha$ belongs to $\widehat{\mathcal{D}}_p$ if and only if $-1 < \alpha < p-1$.

The notation $a \lesssim b$ means that there exists a constant $C > 0$ such that $a \le Cb$, while $a \gtrsim b$ is understood in an analogous manner. If $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$.

2 Introduction and Main Results

The Blaschke product associated with $\{z_n\} \subset \mathbb{D} \setminus \{0\}$ satisfying $\sum_n (1-|z_n|) < \infty$ is defined by

$$B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z}_n z}, \quad z \in \mathbb{D}.$$

It is a bounded analytic function having a unimodular radial limit at almost every point on the boundary $\mathbb{T} = \{z \in \mathbb{D} : |z| = 1\}$; that is, B is an inner function [2, 3, 12]. A Blaschke product associated with a finite union of uniformly separated sequences is called a *Carleson–Newman Blaschke product*. Recall that a sequence $\{z_n\}$ in $\mathbb{D}$ is separated if

$$\inf_{n\neq k} \rho(z_n, z_k) > 0,$$

and uniformly separated if

$$\inf_k \prod_{n\neq k} \rho(z_n, z_k) > 0,$$

where $\rho(a,z) = |\varphi_a(z)|$ and $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$.

A Blaschke product B has a finite angular derivative at $e^{i\theta}$ provided that $B(e^{i\theta}) = \lim_{r\to 1^-} B(re^{i\theta})$ exists, $|B(e^{i\theta})| = 1$, and $B'(e^{i\theta}) = \lim_{r\to 1^-} B'(\mathrm{r}e^{i\theta})$ exists. If B does not have an angular derivative at $e^{i\theta}$, then we write $|B'(e^{i\theta})| = \infty$. By the Julia–Carathéodory theorem [21], we have

$$|B'(e^{i\theta})| = \liminf_{z\to e^{i\theta}} \frac{1-|B(z)|}{1-|z|}, \quad 0 \le \theta < 2\pi.$$

Using this fact, one can show that

$$|B'(e^{i\theta})| = \sum_n \frac{1-|z_n|^2}{|z_n - e^{i\theta}|^2}, \quad 0 \le \theta < 2\pi;$$

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see [1, Theorem 2] and [5, Théorème VI]. Hence, for $0 < p < \infty$, Hardy's convexity and the mean convergence theorem [3] yield

$$\begin{aligned}\|B'\|_{H^p}^p &= \lim_{s\to 1^-} M_p^p(s, B') = \frac{1}{2\pi}\int_0^{2\pi} |B'(e^{i\theta})|^p \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_n \frac{1-|z_n|^2}{|z_n - e^{i\theta}|^2}\Big)^p d\theta \\ &\gtrsim \int_0^{2\pi}\Big(\sum_n \frac{1-|z_n|^2}{|1-\overline{z}_n re^{i\theta}|^2}\Big)^p d\theta, \quad 0 \le r < 1.\end{aligned} \tag{2.1}$$

The asymptotic inequality in (2.1) is due to the fact that $|1 - se^{it}|^2 \asymp (1-s)^2 + t^2$ for $0 \le s < 1$ and $-\pi \le t \le \pi$.

If B is the Blaschke product associated with $\{z_n\}$, then

$$\frac{B'(z)}{B(z)} = \sum_n \frac{|z_n|^2 - 1}{(1-\overline{z}_n z)(z_n - z)} \quad \text{and} \quad |B(z)| \le \frac{|z_n - z|}{|1-\overline{z}_n z|}.$$

It follows that $|B'(z)| \le \Psi_B(z)$, where

$$\Psi_B(z) = \sum_n \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2}, \quad z \in \mathbb{D}.$$

Finally, using this inequality together with (2.1), we deduce that $B' \in H^p$ if and only if $M_p(r, \Psi_B)$ is bounded. The deduction above essentially originated in [1]. Next we give an example based on this result.

Example If $\frac{1}{2} < p < 1$ and B is the Blaschke product associated with a finite union of separated sequences $\{z_n\}$, then the following statements are equivalent:

(i) $B'' \in A^p_{p-1}$,
(ii) $B' \in H^p$,
(iii) $M_p(r, \Psi_B)$ is bounded for $0 \le r < 1$.

The equivalence between (ii) and (iii) is clear by the deduction above. If $B' \in H^p$, then [19, Theorem 3] yields

$$\sum_n (1-|z_n|)^{1-p} < \infty. \tag{2.2}$$

Moreover, by [11, Theorem 3.1], condition (2.2) implies $B'' \in A^p_{p-1}$. Hence, (ii) $\Rightarrow$ (i). For $0 < p \le 2$, we have $\{f : f' \in A^p_{p-1}\} \subset H^p$ by [22, Lemma 1.4], see also [4, Theorem 3]. If we choose $f = B'$, then the implication (i)$\Rightarrow$(ii) follows directly from this result.

It is possible that the derivative of a Blaschke product B does not belong to the Nevanlinna class N [5]; and consequently, $M_p(r, B')$ and $M_p(r, \Psi_B)$ are both unbounded for every $0 < p < \infty$. Therefore, it is natural to ask if it is true in general that $M_p(r, B') \asymp M_p(r, \Psi_B)$ as $r \to 1^-$. The following result implies the negative answer for $\frac{1}{2} < p < \infty$.

Theorem 2.1 *Let $\frac{1}{2} < p < \infty$ and $1 < \alpha < \infty$. Then there exists a Blaschke product B_α such that*

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(i) $M_p^p(r, B'_\alpha) \lesssim (1-r)^{1/2-p}$ *as* $r \to 1^-$,
(ii) $M_p^p(r, \Psi_{B_\alpha}) \gtrsim (1-r)^{1-2p}(\log \frac{e}{1-r})^{-\alpha p}$ *as* $r \to 1^-$.

In Section 3, Theorem 2.1 is proved by using a Blaschke product B_α with zeros on the positive real axis. Even in this special case, $M_p^p(r, \Psi_{B_\alpha})$ may grow much faster than $M_p^p(r, B'_\alpha)$. Nevertheless, if B is a Carleson–Newman Blaschke product, then $\|B'\|_{A^p_\omega}$ and $\|\Psi_B\|_{L^p_\omega}$ are comparable; see Theorem 2.2. It is worth noting that, in general, the sparsity of zeros of B does not force B' to H^p. For example, there exists a Blaschke product B with uniformly separated zeros $\{z_n\}$ such that B' does not belong to the Nevanlinna class N and $\sum_n (1-|z_n|)^\alpha < \infty$ for every $\alpha > \frac{1}{2}$ [14].

Theorem 2.2 *Let* $0 < p < \infty$ *and* $\omega \in \widehat{\mathcal{D}}_p$. *If* B *is a Carleson-Newman Blaschke product, then* $\|B'\|_{A^p_\omega} \asymp \|\Psi_B\|_{L^p_\omega}$.

The proof of Theorem 2.2 can be found in Section 4. Moreover, it is proved that, under an additional hypothesis for the weight ω, the statement of Theorem 2.2 for $\frac{1}{2} < p \le 1$ is valid also in the case where B is associated with a finite union of separated sequences. Furthermore, Section 4 contains an example showing that the statement of Theorem 2.2 is not true in general if B is an arbitrary Blaschke product.

3 Proof and Modification of Theorem 2.1

The main purpose of this section is to prove Theorem 2.1. We begin with Propositions 3.1 and 3.2, which play key roles in the proof. It is worth noting that the proof of Proposition 3.1 uses some ideas from [7].

Proposition 3.1 *Let* $0 < p < \infty$ *and* $k \in \mathbb{N}$, *and let* $g : [k, \infty) \to (0,1)$ *be a continuous and strictly decreasing function satisfying* $\int_k^\infty g(x)\,dx < \infty$. *In addition, let* N_t *be the unique real number on* $[k+1, \infty)$ *such that* $g(N_t) = t$ *for* $0 < t \le g(k+1)$. *If* B *is the Blaschke product associated with the sequence* $\{z_n\}$, *where*

$$z_n = 1 - g(n), \quad n = k, k+1, \dots,$$

then

$$M_p^p(r, \Psi_B) \gtrsim \int_{1-r}^{g(k+1)} \Big(\int_k^{N_t} \frac{dx}{g(x)} \Big)^p dt, \quad r \to 1^-.$$

Proof We begin by noting that the monotonicity and continuity of g imply the existence of the unique real number N_t on $[k+1, \infty)$ such that $g(N_t) = t$ for $0 < t \le g(k+1)$.

Let $z = re^{i\theta}$ for $-\pi \le \theta \le \pi$ and $0 \le r < 1$. Then, by [7, Lemma 3], we find $0 < R < 1$ such that

$$\frac{1-|z_n|^2}{|1-\overline{z}_n z|^2} = \frac{1-|z_n|^2}{|1-|z_n|z|^2} \asymp \frac{1-|z_n|}{((1-r)+|\theta|+(1-|z_n|))^2}, \quad R < r < 1,$$

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where the comparison constants depend on R and B. Hence, for $R < r < 1$ and $-\pi \le \theta \le \pi$, we have $\Psi_B(re^{i\theta}) \asymp f_B((1-r)+|\theta|)$, where

$$f_B(t) = \sum_{n=k}^{\infty} \frac{1-|z_n|}{(t+(1-|z_n|))^2} = \sum_{n=k}^{\infty} \frac{g(n)^{-1}}{(tg(n)^{-1}+1)^2}, \quad 0 < t < \infty.$$

It follows that

$$f_B(t) \ge \frac{1}{4}\sum_{n=k}^{N_t} g(n)^{-1}, \quad 0 < t \le g(k+1).$$

Now, by the estimates above, we have

$$\begin{aligned} M_p^p(r,\Psi_B) &\asymp \int_0^{\pi} f_B((1-r)+\theta)^p \, d\theta = \int_{1-r}^{\pi+(1-r)} f_B(t)^p \, dt \\ &\ge \int_{1-r}^{g(k+1)} f_B(t)^p \, dt \ge 4^{-p} \int_{1-r}^{g(k+1)} \Big(\sum_{n=k}^{N_t} g(n)^{-1} \Big)^p dt \\ &\asymp \int_{1-r}^{g(k+1)} \Big(\int_k^{N_t} \frac{dx}{g(x)} \Big)^p dt, \quad r \to 1^-. \end{aligned}$$

This completes the proof. ∎

For $\xi \in \mathbb{T}$ and $1 < \eta < \infty$, the approaching region

$$\Omega_\eta(\xi) = \{z \in \mathbb{D} : |1-\overline{\xi}z| \le \eta(1-|z|)\}$$

is known as a *Stolz domain*. Denote the family of all Blaschke products whose zeros lie in some Stolz domain by $\mathfrak{B}$.

Proposition 3.2 *Let $0 < p < \infty$ and $B \in \mathfrak{B}$. Then there exists $R = R(B) \in [0,1)$ such that*

(i)

$$M_p^p(r,\Psi_B) \lesssim \begin{cases} 1, & p < \frac{1}{2}, \\ \log(\frac{e}{1-r}), & p = \frac{1}{2}, \\ (1-r)^{1-2p}, & p > \frac{1}{2}, \end{cases}$$

for $R < r < 1$;

(ii)

$$M_p^p(r,B') \lesssim \int_0^{2\pi} \Big(\frac{1-|B(re^{i\theta})|}{1-r} \Big)^p d\theta \lesssim I(r) = \begin{cases} 1, & p < \frac{1}{2}, \\ \log(\frac{e}{1-r}), & p = \frac{1}{2}, \\ (1-r)^{1/2-p}, & p > \frac{1}{2}, \end{cases}$$

for $R < r < 1$.

Note that the asymptotic inequality $M_p^p(r,B') \lesssim I(r)$ was earlier proved in [20]. Some special cases of Proposition 3.2(ii) are also possible to verify using results in [8, 9], as mentioned in [20]. The proof here uses a straightforward method that is based on the inequalities $|B'(z)| \le \Psi_B(z)$ and $|B'(z)| \lesssim (1-|z|)^{-1}$ for all $z \in \mathbb{D}$. Due to this, the upper bound is also valid for the integral mean M_p of $\min\{\Psi_B(re^{i\theta}), (1-r)^{-1}\}$.

Proof Assume, without loss of generality, that the zero-sequence $\{z_n\}$ of B is contained in $\Omega_\eta(1)$ for some $1<\eta<\infty$, and set

$$f_B(t)=\sum_n \frac{1-|z_n|}{(t+(1-|z_n|))^2},\quad 0<t<\infty.$$

By [7, Lemmas 3 and 4], we find $R=R(\eta)\in(0,1)$ such that $\Psi(re^{i\theta})\asymp f_B((1-r)+|\theta|)$ for $R<r<1$ and $-\pi\le\theta\le\pi$. Hence,

$$M_p^p(r,\Psi_B)\asymp\int_{1-r}^{\pi+(1-r)} f_B(t)^p\,dt\lesssim\int_{1-r}^{\pi+(1-r)}\frac{dt}{t^{2p}},\quad R<r<1;$$

and consequently, assertion (i) follows.

By the Schwarz–Pick lemma and [12, Theorem 3.5], we obtain

$$\begin{aligned}|B'(re^{i\theta})|&\le\frac{1-|B(re^{i\theta})|^2}{1-r^2}=\sum_n|B_n(re^{i\theta})|^2\frac{1-|z_n|^2}{|1-\overline{z}_nre^{i\theta}|^2}\le\Psi(re^{i\theta})\\&\asymp f_B((1-r)+|\theta|)\lesssim\theta^{-2},\quad R<r<1,\quad -\pi\le\theta\le\pi,\end{aligned}$$

where

$$B_1(z)=1\quad\text{and}\quad B_k(z)=\prod_{j=1}^{k-1}\frac{z_j-z}{1-\overline{z}_jz},\quad k\in\mathbb{N}\setminus\{1\},\quad z\in\mathbb{D}.$$

Hence, for $\frac12<p<\infty$, we have

$$\begin{aligned}M_p^p(r,B')&\lesssim\int_{-\pi}^{\pi}\Big(\frac{1-|B(re^{i\theta})|}{1-r}\Big)^p d\theta\lesssim(1-r)^{-p}\int_0^{\sqrt{1-r}}d\theta+\int_{\sqrt{1-r}}^{\pi}\frac{d\theta}{\theta^{2p}}\\&\asymp(1-r)^{1/2-p},\quad R<r<1.\end{aligned}$$

This gives assertion (ii) for $\frac12<p<\infty$. The remaining cases are direct consequences of assertion (i). ∎

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 Let $\frac12<p<\infty$ and $1<\alpha<\infty$, and set

$$g(x)=\frac{1}{x(\log x)^\alpha},\quad x\ge2.$$

Let B_α be the Blaschke product associated with the sequence $\{z_n\}$, where $z_n=1-g(n)$ for $n\in\mathbb{N}\setminus\{1\}$. Then B_α satisfies assertion (i) by Proposition 3.2(ii).

We prove assertion (ii) using Proposition 3.1. Let $h(t)=(t(\log\frac1t)^\alpha)^{-1}$ for $0<t<\frac1e$. Since g is decreasing and

$$g\big(h(t)\big)=\frac{t(\log\frac1t)^\alpha}{\Big(\log\Big(\frac{1}{t(\log\frac1t)^\alpha}\Big)\Big)^\alpha}\ge t=g(N_t),\quad 0<t<\frac1e,$$

we have $h(t)\le N_t$. Hence, Proposition 3.1 yields

$$\begin{aligned}M_p^p(r,\Psi_B)&\gtrsim\int_{1-r}^{t_0}\Big(\int_2^{h(t)}\frac{dx}{g(x)}\Big)^p dt\asymp\int_{1-r}^{t_0}h(t)^{2p}\big(\log h(t)\big)^{\alpha p}dt\\&\asymp\int_{1-r}^{t_0}\frac{dt}{t^{2p}(\log\frac1t)^{\alpha p}}\asymp(1-r)^{1-2p}\Big(\log\frac{e}{1-r}\Big)^{-\alpha p},\quad r\to1^-,\end{aligned}$$

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for some sufficiently small $t_0 = t_0(B_\alpha) \in (0,1)$. This completes the proof. ■

There exists a Blaschke product B_α such that the statement of Theorem 2.1 is also valid if $M_p^p(r, B'_\alpha)$ is replaced by

$$\Phi_{B_\alpha,p}(r) = \int_0^{2\pi} \Big(\sum_n |B_n(re^{i\theta})|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n re^{i\theta}|^2} \Big)^p \, d\theta,$$

where $\{z_n\}$ is the zero-sequence of B_α and

$$B_1(z) = 1 \quad \text{and} \quad B_k(z) = \prod_{j=1}^{k-1} \frac{z_j - z}{1-\overline{z}_j z}, \quad k \in \mathbb{N} \setminus \{1\}, \quad z \in \mathbb{D}.$$

This is due to the proofs of Theorem 2.1 and Proposition 3.2. Hence, by Theorem 2.1, the term $|B_n(re^{i\theta})|^2$ may affect essentially to the size of $\Phi_{B_\alpha,p}$.

The proof of Theorem 2.1 gives an example in which $M_p(r, B'_\alpha)$ and $M_p(r, \Psi_{B_\alpha})$ are not comparable for any $\frac{1}{2} < p < \infty$. If we allow the choice of α to be dependent on p, then the following result offers an alternative counter example.

Theorem 3.3 *Let $\frac{1}{2} < p < \infty$ and $1 < \alpha < \min\{2, 2p\}$. Let B_α be the Blaschke product associated with the sequence $\{z_n\}$, where $z_n = 1 - (\frac{1}{n})^\alpha$ for $n \in \mathbb{N}$. Then*

(i) $M_p^p(r, B'_\alpha) \lesssim (1-r)^{1/2-p}$ *as* $r \to 1^-$,

(ii) $M_p^p(r, \Psi_{B_\alpha}) \gtrsim (1-r)^{1-p-p/\alpha}$ *as* $r \to 1^-$.

Proof Assertion (i) is a direct consequence of Proposition 3.2. We prove the assertion (ii) using Proposition 3.1. Since $g(x) = x^{-\alpha}$ for $x \geq 1$ and $N_t = t^{-1/\alpha}$ for $0 < t < \frac{1}{4} < (\frac{1}{2})^\alpha$, we obtain

$$M_p^p(r, \Psi_{B_\alpha}) \gtrsim \int_{1-r}^{1/4} \Big(\int_1^{t^{-1/\alpha}} x^\alpha \, dx \Big)^p dt \asymp \int_{1-r}^{1/4} t^{-p-p/\alpha} \, dt \asymp (1-r)^{1-p-p/\alpha}$$

as $r \to 1^-$. This completes the proof. ■

It is worth noting that, by using a Blaschke product B in $\mathfrak{B}$, we cannot find a counter example in the case $0 < p < \frac{1}{2}$. This is due to the fact that $B' \in \bigcap_{0<p<\frac{1}{2}} H^p$; see Proposition 3.2. For $p = \frac{1}{2}$, it might be possible because, for example, the derivative of the Blaschke product with zeros

$$z_n = 1 - \frac{1}{n(\log n)^2}, \quad n \in \mathbb{N} \setminus \{1\},$$

does not belong to $H^{1/2}$ [7]. Nevertheless, Proposition 3.2 does not offer a sharp enough upper bound for $M_{1/2}(r, B')$. Hence a sharper upper bound depending on B is needed if one intend to prove a counterpart of Theorem 2.1 for $p = \frac{1}{2}$ using $B \in \mathfrak{B}$.

As mentioned in Section 2, there exist Blaschke products B such that $B' \notin N \supset \bigcup_{0<p<\infty} H^p$. This means that it might be possible to find a Blaschke product B such that $M_p(r, B')$ and $M_p(r, \Psi_B)$ are not comparable for $p < \frac{1}{2}$. Nevertheless, if the zeros of B behave wild, it might be laborious to verify sufficiently sharp bounds for $M_p(r, B')$ and $M_p(r, \Psi_B)$.

4 Proof and Related Results of Theorem 2.2

We begin with the proof of Theorem 2.2.

Proof of Theorem 2.2 By [6, Chpt. VI, Lemma 3.3] and [13, Lemma 21], the Blaschke product B with zeros $\{z_n\}$ is a Carleson–Newman Blaschke product if and only if

$$\sup_{a\in\mathbb{D}} \sum_n (1-|\varphi_a(z_n)|^2) < \infty. \tag{4.1}$$

Assume that B is a Carleson-Newman Blaschke product. Using (4.1) together with the inequality $1-r^2 \le -2\log r$ for $0<r\le 1$ and the fact that $(1-e^{-x})/x$ is decreasing on $(0,\infty)$, one can show that

$$1-|B(z)| \gtrsim \sum_n (1-|\varphi_{z_n}(z)|^2) = (1-|z|^2)\Psi_B(z), \quad z\in\mathbb{D};$$

see details, for example, in the proof of [17, Theorem 4.1]. Moreover, for $0<p<\infty$ and $\omega\in\widehat{\mathcal{D}}_p$, [16, Theorem 1] together with the Schwarz–Pick lemma yields

$$\|\Theta'\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} \Big(\frac{1-|\Theta(z)|}{1-|z|}\Big)^p \omega(z)\,dA(z) \tag{4.2}$$

for any inner function Θ. In particular, (4.2) is valid if Θ is the Blaschke product B. By combining these facts, we obtain $\|B'\|_{A^p_\omega} \gtrsim \|\Psi_B\|_{L^p_\omega}$. Since $|B'(z)|\le\Psi_B(z)$ for all $z\in\mathbb{D}$, the assertion is proved. ∎

A radial weight ω belongs to $\mathcal{D}$ if there exist $C=C(\omega)\ge 1$, $\alpha=\alpha(\omega)>0$ and $\beta=\beta(\omega)\ge\alpha$ such that

$$C^{-1}\Big(\frac{1-r}{1-t}\Big)^\alpha \widehat{\omega}(t) \le \widehat{\omega}(r) \le C\Big(\frac{1-r}{1-t}\Big)^\beta \widehat{\omega}(t), \quad 0\le r\le t<1. \tag{4.3}$$

It is worth noting that the right-hand side inequality of (4.3) is valid for some β if and only if $\omega\in\widehat{\mathcal{D}}$ [15]. Using this notation, we state and prove the following counterpart of Theorem 2.2 for $\frac12 < p \le 1$.

Theorem 4.1 *Let $\frac12<p\le 1$ and $\omega\in\mathcal{D}\cap\widehat{\mathcal{D}}_{2p-1}$. If B is a Blaschke product associated with a finite union of separated sequences, then $\|B'\|_{A^p_\omega}\asymp\|\Psi_B\|_{L^p_\omega}$.*

Proof Under the given hypotheses, [18, Theorem 1] gives

$$\|B'\|^p_{A^p_\omega} \asymp \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}, \tag{4.4}$$

where $\{z_n\}$ is the zero-sequence of B. In addition, by the proof of [18, Proposition 4], we obtain

$$\|B'\|^p_{A^p_\omega} \le \|\Psi_B\|^p_{L^p_\omega} \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}. \tag{4.5}$$

More precisely, the asymptotic inequality in (4.5) can be proved by using the fact that x^p is sub-additive for $0<p\le 1$ together with the Forelli–Rudin estimate [10, Theorem 1.7], then dividing the radial integral into two parts, from zero to $|z_n|$ and the rest,

and finally estimating in a natural manner. The assertion follows by combining (4.4) and (4.5). ■

We close this note with the following example, which shows that the statements of Theorems 2.2 and 4.1 are not true in general if B is an arbitrary Blaschke product.

Example Let $\frac{1}{2} < p < \infty$ and $\omega(z) = (1-|z|)^\alpha$, where $\alpha = \alpha(p) = p - \frac{3}{2} + x_p$ and $x_p = \min\{\frac{1}{4}, \frac{p}{2} - \frac{1}{4}\}$. Since $-1 < \alpha < \min\{p-1, 2p-2\}$, it is clear that $\omega \in \mathcal{D} \cap \widehat{\mathcal{D}}_q$, where $q = q(p) = \min\{p, 2p-1\}$. Let B be the Blaschke product associated with $\{z_n\}$, where

$$z_n = 1 - \frac{1}{n(\log n)^2}, \quad n \in \mathbb{N} \setminus \{1\}.$$

By Proposition 3.2 and the proof of Theorem 2.1, we have $B' \in A^p_\alpha$, while

$$\begin{aligned}\|\Psi_B\|^p_{L^p_\alpha} &\gtrsim \int_R^1 (1-r)^{1+\alpha-2p}\Big(\log\frac{e}{1-r}\Big)^{-2p} dr \\ &\geq \int_R^1 (1-r)^{-p/2-3/4}\Big(\log\frac{e}{1-r}\Big)^{-2p} dr = \infty\end{aligned}$$

for some $R = R(p) \in [0,1)$.

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