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ERGODIC THEORY AND AVERAGING ITERATIONS

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1. Introduction. Suppose X is a Banach space and T a continuous linear operator on X. The significance of the asymptotic convergence of T for the approximate solution of the equation $(I - T)x = f$ by means of the Picard iterations was clearly shown in Browder's and Petryshyn's paper [**1**]. The results of [**1**] have stimulated further investigation of the Picard, and more generally, averaging iterations for the solution of linear and nonlinear functional equations [**2**; **3**; **4**; **8**; **9**]. Kwon and Redheffer [**8**] analyzed the Picard iteration under the mildest possible condition on T, namely that T be continuous and linear on a normed (not necessarily complete) space X. The results of [**8**] (still waiting to be extended for the averaging iterations) seem to give the most complete story of the Picard iterations for the linear case. Only when T is subject to some further restrictions, such as asymptotic A-boundedness and asymptotic A-regularity, one can agree with Dotson [**4**] that the iterative solution of linear functional equations is a special case of mean ergodic theory for affine operators. This thesis is rather convincingly demonstrated by results of De Figueiredo and Karlovitz [**2**], and Dotson [**3**], and most of all by Dotson's recent paper [**4**], in which the results of [**1**; **2**; **3**] are elegantly subsumed under the affine mean ergodic theorem of Eberlein-Dotson.

Our own investigation follows up the fact implicitly contained in the proof of Theorem 1 in [**1**] that X is the direct sum $N(I - T) \oplus R(I - T)^-$ if $\{T^n x\}$ converges in norm for all $x \in X$. (Here and everywhere in the paper, $N(A)$ and $R(A)$ denote the null space and the range of an operator A respectively, the bar denotes, as usual, closure in X.) A generalization of this decomposition for the ergodic subspace of a semigroup of linear operators ergodic in the sense of Eberlein is given in the subsequent section. Section 3 is devoted to the investigation of some properties of asymptotically A-bounded and asymptotically A-regular operators stemming largely from the decomposition theorem for the ergodic subspace of the semigroup $G = \{I, T, T^2, \ldots\}$, and gives some applications to the iterative solution of the equation $(I - T)x = f$.

The notation used in the paper is fairly standard. In addition to the symbols $N(A)$, $R(A)$ and $^-$ explained above, we use $B(X)$ to denote the Banach algebra of all continuous linear operators on X, $\rightarrow$ and $\rightharpoonup$ are employed to denote strong and weak convergence in X respectively. The action of a functional $w \in X^*$ on an element $x \in X$ is written as (x, w).

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2. Ergodic semigroups. Let X be a Banach space and G a semigroup of continuous linear operators on X. According to Eberlein [**5**], G is called *ergodic* if it possesses at least one system $\{A_\alpha | \alpha \in D\}$ of almost invariant integrals (D is a directed set). For the sake of completeness we list the conditions characterizing such a system:

I. $A_\alpha \in B(X)$ for each $\alpha \in D$.

II. For each $x \in X$ and all $\alpha \in D$, $A_\alpha x \in O(x) = \overline{\text{co}}\{Tx | T \in G\}$, where $\overline{\text{co}}(S)$ is the closed convex hull of S.

III. $||A_\alpha|| \leqq M$ for some $M > 0$ and all $\alpha \in D$.

IV. For each $T \in G$, $\lim_\alpha A_\alpha (I - T) = \lim_\alpha (I - T)A_\alpha = 0$.

$\{A_\alpha\}$ will be called a *weak* (*strong*) *system of almost invariant integrals* for G if IV is satisfied in the sense of the weak (strong) operator topology. Accordingly, G will be called weakly (strongly) ergodic. With each ergodic semigroup G we associate two subspaces N and R, where

(1) $$N = \bigcap_{T \in G} N(I - T), \qquad R = \text{sp}\{\bigcup_{T \in G} R(I - T)\},$$

with $\text{sp}(S)$ denoting the linear hull of S. In addition, the ergodic subspace E of an ergodic semigroup G is defined by

(2) $$E = \{x | x \in X, O(x) \cap N \neq \emptyset\}.$$

The mean ergodic theorem of Eberlein [**5**, Theorem 3.1] then states: Suppose G is strongly ergodic. Then:

(E1). $\{A_\alpha x\}$ is strongly convergent if and only if $\{A_\alpha x\}$ clusters weakly.

(E2). $\{A_\alpha x\}$ is strongly convergent if and only if $x \in E$.

(E3). If $x \in E$, the intersection $O(x) \cap N$ consists of a single point, namely the strong limit of $\{A_\alpha x\}$.

Analyzing the proof of Theorem 3.1 in [**5**], we observe that the strong convergence postulated in IV is only needed to establish the strong convergence of $\{A_\alpha x\}$ under the assumption that $\{A_\alpha x\}$ clusters weakly. Hence we have the following result for weakly ergodic semigroups: Suppose G is weakly ergodic. Then :

(E1)′. $\{A_\alpha x\}$ is weakly convergent if and only if $\{A_\alpha x\}$ clusters weakly.

(E2)′. $\{A_\alpha x\}$ is weakly convergent if and only if $x \in E$.

(E3)′. For each $x \in E$, $O(x) \cap N$ contains exactly the weak limit of $\{A_\alpha x\}$.

Given a strongly (weakly) ergodic semigroup G, we define an operator $Q : E \to X$ by

(3) $$Qx = O(x) \cap N.$$

(E2) (respectively (E2)′) implies that for any strong (respectively weak) system $\{A_\alpha\}$ of almost invariant integrals for G, $Qx = \lim_\alpha A_\alpha x$ ($x \in E$) in the corresponding topology. We show that an analogue of Theorem 4.1 in [**5**] is valid also for weakly ergodic semigroups.

LEMMA 1. *Suppose G is a weakly ergodic semigroup. The ergodic subspace E of G is a closed subspace of X invariant under each $T \in G$. The operator Q defined by* (3) *is a continuous linear mapping of E into itself such that $Q^2 = Q$ and $QT_E = TQ = Q$ for each $T \in G$.*

Proof. Let $\{A_\alpha\}$ be any weak system of almost invariant integrals for G. Since $A_\alpha x \rightharpoonup Qx$ for each $x \in E$, E is a subspace of X, and Q is linear. N is obviously contained in E, hence $Q : E \to E$ in view of (3). To establish the continuity of Q, we observe that for any $x \in X$, each w in the dual X^* of X and all $\alpha \in D$,

$$|(Qx, w)| \leqq |(A_\alpha x, w)| + |(Qx - A_\alpha x, w)| \leqq M||x|| \, ||w|| + |(Qx - A_\alpha x, w)|.$$

Passing to the limit as $\alpha \in D$, we obtain $|(Qx, w)| \leqq M||x||||w||$. Hence

$$||Qx|| = \sup_{||w||=1}|(Qx, w)| \leqq M||x||,$$

and $||Q|| \leqq M$. To prove the closure of E, suppose $x_n \to x$ for $x_n \in E$. It is easily verified that $\{Qx_n\}$ is a Cauchy sequence in the strong topology of X, so that $Qx_n \to y$ for some $y \in X$. For each $w \in X^*$, all $\alpha \in D$ and any positive integer n,

$$|(A_\alpha x - y, w)| \leqq M||x - x_n|| \, ||w|| + ||Qx_n - y|| \, ||w|| + |(A_\alpha x_n - Qx_n, w)|.$$

$A_\alpha x \rightharpoonup y$ is proved on choosing n sufficiently large and that α suitably. The invariance of E under T follows from IV, and the rest from Lemma 4.1 in [**5**].

We now describe the structure of the ergodic subspace in terms of the subspaces N and R defined in (1).

THEOREM 1 [**6**]. *If G is a weakly* (*strongly*) *ergodic semigroup with the ergodic subspace E, then*

$$E = N \oplus R^-.$$

The projection of E onto N associated with this direct sum is the operator Q defined by (3).

Proof. As shown in Lemma 1, Q is a continuous linear idempotent operator mapping E into itself. Hence $E = R(Q) \oplus N(Q)$ with $R(Q)$ closed. We establish that $R(Q) = N$ and $N(Q) = R^-$. For each $x \in E$, $Qx \in N$ by virtue of (3). If $y \in N$, $Qy = y$, and $y \in R(Q)$. Suppose $x \in N(Q)$. Then $Qx = 0 \in O(x)$. For each $\epsilon > 0$ there is $z = \sum_1^n a_i T_i x$ with $a_i \geqq 0$, $\sum_1^n a_i = 1$ and $T_i \in G$, such that $||z|| < \epsilon$. The vector $x - z = \sum_1^n a_i(I - T_i)x$ lies in R, and $||x - (x - z)|| = ||z|| < \epsilon$, which in turn means that $x \in R^-$. If, on the other hand, $y \in R$, $y = \sum_1^n (I - T_i)x_i$ for some $T_i \in G$ and some $x_i \in X$. By IV, each $(I - T_i)x_i$ lies in $N(Q)$, hence also $y \in N(Q)$. The inclusion $R^- \subset N(Q)$ then follows from $R \subset N(Q)$ as $N(Q)$ is closed.

Remark 1. Suppose G is a weakly ergodic semigroup with a weak system $\{A_\alpha\}$ of almost invariant integrals. In virtue of II, $\{A_\alpha x\}$ is bounded for each

$x \in X$. If $\{A_\alpha x\}$ clusters weakly for each $x \in X$ or if X is reflexive, $X = N \oplus R^-$.

Remark 2. A special case of Theorem 1 for the semigroup $G = \{I, T, T^2, \ldots\}$ with $A_n = n^{-1}(I + T + \ldots + T^{n-1})$ as a system of almost invariant integrals was proved by Yosida [**10**].

Let us consider the semigroup $G = \{I, T, T^2, \ldots\}$, where $T \in B(X)$. We show that in this case the subspaces N and R defined by (1) are given by the formulae

$$N = N(I - T), \quad R = R(I - T). \tag{4}$$

If $x \in N(I - T)$, $T^n x = x$ for each $n \in N$, and $x \in \bigcap_0^\infty N(I - T^n) = N$. The inclusion $N \subset N(I - T)$ is obvious. From the identity $I - T^n = (I - T)\sum_0^{n-1} T^k$ (with $\sum_0^{-1} = 0$) it follows that $R(I - T^n) \subset R(I - T)$ for all $n \geqq 0$, and $\mathrm{sp}\{\bigcup_0^\infty R(I - T^n)\} = R \subset R(I - T)$. The reverse inclusion is trivial.

3. Averaging iterations. T denotes a continuous linear operator on a Banach space X. A real infinite matrix $A = [a_{nj}]$ $(n, j \geqq 0)$ will be called *admissible* if A is nonnegative lower triangular with each row summing to 1. Following Dotson [**3**; **4**] we define the polynomials $a_n(t)$ and $b_n(t)$ $(n \geqq 0)$ by

$$a_n(t) = \sum_{j=0}^{n} a_{nj} t^j, \quad b_n(t) = (1 - a_n(t))/(1 - t).$$

Definition. Let A be an admissible matrix, and let $A_n = a_n(T)$ and $B_n = b_n(T)$ for each $n \geqq 0$.

(i) T is *asymptotically A-bounded* if $\|A_n\| \leqq M$ for some $M > 0$ and all $n \geqq 0$.

(ii) T is *weak* (*strong*) *asymptotically A-regular* if $\lim_n A_n(I - T) = 0$ in the weak (strong) operator topology.

(iii) T is *weak* (*strong*, *uniform*) *A-convergent* if T is weak asymptotically A-regular and $\{B_n\}$ converges in the weak (strong, uniform) operator topology.

(iv) T is *weakly* (*strongly*, *uniformly*) *asymptotically A-convergent* if T is weak asymptotically A-regular and $\{A_n\}$ converges in the weak (strong, uniform) operator topology.

It follows from the uniform boundedness principle that an asymptotically A-convergent operator T (in any of the three mentioned operator topologies) is also asymptotically A-bounded. In the case when A is the infinite unit matrix I, the preceding definition characterizes ordinary asymptotic boundedness, asymptotic regularity, convergence and asymptotic convergence. Note that the condition that T be weak asymptotically A-regular in parts (iii) and (iv) of the above definition is included only to guarantee $\lim_n A_n = Q = QT = TQ$. However, there is a class of admissible matrices A satisfying the equality automatically, namely the matrices with the property that, for each continuous linear operator T, $\lim_n TA_nx = \lim_n A_nx$ whenever $\{A_nx\}$ converges. It is not difficult to verify that the unit and Cesàro matrices belong to this class.

Suppose T is asymptotically A-bounded and weak (strong) asymptotically A-regular for some admissible A. It is immediately obvious that $\{A_n\}$ is a weak (strong) system of almost invariant integrals for $G = \{I, T, T^2, \ldots\}$. Let E be the ergodic subspace of G, and Q the operator defined by (3). According to Theorem 1 and the formula (4), $E = N(I - T) \oplus R(I - T)^-$, and $\{A_nx\}$ converges weakly (strongly) to Qx if and only if $x \in E$. Moreover, the operators A_n, B_n and Q satisfy the following conditions:

(A1) $(I - T)B_n = I - A_n$ for each $n \geqq 0$.

(A2) For each $x \in E$ and all $n \geqq 0$, $QA_nx = Qx$, and $QB_nx = \phi(n)Qx$, where $\phi(n)$ is a real valued function of n.

(A3) If $\lim_n a_{nj} = 0$ for each $j \geqq 0$ (in this case A will be called *Toeplitz*), $\lim_n \phi(n) = +\infty$.

This all can be easily deduced from Theorem 3 in [**3**].

In the sequel we consider the approximate solution of the equation $(I - T)x = f$ by means of the averaging iteration $x_n = A_nx_0 + B_nf$ [**3**; **4**] provided T is at least asymptotically A-bounded and weak asymptotically A-regular. This iteration can be viewed as a generalization of the Picard iteration $x_n = T^nx_0 + (\sum_0^{n-1} T^k)f$ which arises when $A = I$.

PROPOSITION 1. *Suppose T is asymptotically A-bounded and weak (strong) asymptotically A-regular for some admissible matrix A. If $f \in R(I - T)$, the sequence $\{x_n\} = \{A_nx_0 + B_nf\}$ converges weakly (strongly) to a solution x of the equation $(I - T)x = f$ if and only if $x_0 - y \in N(I - T) \oplus R(I - T)^-$ for some y with $(I - T)y = f$.*

Proof. Suppose $(I - T)y = f$ and put $y_n = A_ny + B_nf$. Then $y_n = A_ny + B_n(I - T)y = A_ny + (I - A_n)y = y$ in view of (A1). Furthermore, $x_n - y = x_n - y_n = A_n(x_0 - y)$, and $\{x_n\}$ converges weakly (strongly) if and only if $x_0 - y$ lies in the ergodic subspace $E = N(I - T) \oplus R(I - T)^-$ of $G = \{I, T, T^2, \ldots\}$. Let x be the limit of $\{x_n\}$. Since $x_n = y + A_n(x_0 - y)$, $x = y + Q(x_0 - y)$, and $(I - T)x = (I - T)y + (I - T)Q(x_0 - y) = f$.

Remark 3. A result related to the preceding proposition has been obtained by Kwon and Redheffer [**8**, Remark 1] for $A = I$, without the assumption of asymptotic boundedness and asymptotic regularity and with the subspace $\{x | \{T^nx\} \text{ converges strongly}\}$ in place of $N(I - T) \oplus R(I - T)^-$.

Next we consider the case when the equation $(I - T)x = f$ has a solution given by an averaging analogue of the Neumann series, namely a solution of the form $x = \lim_n B_nf$.

PROPOSITION 2. *Suppose T is asymptotically A-bounded and weak (strong)* *asymptotically A-regular for some admissible Toeplitz matrix A. The following are equivalent:*

(i) $\{B_n f\}$ *is weakly (strongly) convergent.*

(ii) $\{B_n f\}$ *has a weak cluster point.*

(iii) f *belongs to the image Y of $R(I - T)^-$ under $I - T$.*

Moreover, any cluster point of $\{B_n f\}$ is a solution of the equation $(I - T)x = f$ contained in $R(I - T)^-$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Suppose $B_n f \rightharpoonup x$ as $n = n_j \to \infty$. Then $(I - A_n)f = (I - T)B_n f \rightharpoonup (I - T)x$ as $n = n_j \to \infty$, $\{A_n f\}$ has a weak cluster point, and $f \in E = N(I - T) \oplus R(I - T)^-$ according to (E1)′ (respectively (E1)). Hence $QB_n f$ is defined and equal to $\phi(n)Qf$ by (A2). $\{\phi(n)Qf\}$ has a weak cluster point; since $\lim_n \phi(n) = +\infty$ in view of (A3), $Qf = 0$. Consequently, $(I - T)x = f - Qf = f$, and the cluster point x of $\{B_n f\} = \{(I - T)B_n x\}$ is contained in $R(I - T)^-$. This proves (iii) as well as the last statement of the present proposition.

(iii) $\Rightarrow$ (i). Suppose $(I - T)x = f$ for some $x \in R(I - T)^-$. Then $B_n f = B_n(I - T)x = x - A_n x$, and $\{B_n f\}$ is weakly (strongly) convergent. Let us remark that every solution of the equation $(I - T)x = f$ with $f \in Y$ lies in E as the coset $x + N(I - T)$ is contained in E whenever the particular solution x lies in $R(I - T)^-$.

Proposition 2 is related to Remarks 2, 4 and 5 of [**8**] in a similar way as described in our Remark 3.

Proposition 3. *Suppose T is weakly (strongly) A-convergent for some admissible Toeplitz matrix A. Then $\lim_n B_n = (I - T)^{-1}$ in the weak (strong) operator topology. Moreover, for each $f \in X$, the sequence $\{A_n x_0 + B_n f\}$ converges weakly (strongly) to the unique solution of the equation $(I - T)x = f$.*

Proof. A weakly (strongly) A-convergent operator T is also weak (strong) asymptotically A-convergent as follows from (A1). For each $x \in X$ we have $QB_n x = \phi(n)Qx$ in virtue of (A2). Since A is Toeplitz, (A3) holds, and $Q = 0$ on X, i.e., $N(I - T) = \{0\}$. Moreover, $X = R(I - T)^-$ in view of the decomposition theorem for the ergodic subspace X of $G = \{I, T, T^2, \ldots\}$, and $R(I - T)$ is closed by Proposition 2. Hence $(I - T)^{-1} \in B(X)$ by the Banach theorem. Since $\lim_n (I - T)B_n = \lim_n (I - A_n) = I$ in the weak (strong) operator topology, $\lim_n B_n = (I - T)^{-1}$. The rest of Proposition 3 follows immediately.

It is seen from the foregoing proof that $\lim_n A_n = 0$ and $R(I - T) = R(I - T)^-$ are necessary for T to be A-convergent. The next proposition shows that these conditions are also sufficient even if the matrix A is only admissible.

Proposition 4. *Let A be an admissible matrix. Suppose*

(a) $\lim_n A_n = 0$ *in the weak (strong) operator topology, and*
(b) $R(I - T)$ *is closed.*
Then T is weakly (strongly) A-convergent.

Proof. If the conditions (a) and (b) are fulfilled, T is asymptotically A-bounded and weak (strong) asymptotically A-regular, so that Theorem 1 applies. Moreover, (a) implies that $N(I - T) = \{0\}$, hence $X = R(I - T)^- = R(I - T)$ by (b), and $(I - T)^{-1} \in B(X)$. Then

$$\{B_n\} = \{(I - T)^{-1}(I - A_n)\}$$

converges weakly (strongly) to $(I - T)^{-1}$. Let us remark that (b) can be replaced by any of the following equivalent conditions: (b_1) $R(I - T) = X$, (b_2) $(I - T)^{-1}$ is bounded, (b_3) 1 does not belong to the continuous spectrum of T.

Kwon and Redheffer [**8**] gave an example of a shift operator T on a separable Hilbert space such that $\lim_n T^n = 0$ in the strong operator topology for which $(I - T)^{-1}$ is not continuous. This situation cannot occur if $A_n \to 0$ uniformly.

PROPOSITION 5. *Let A be an admissible matrix such that $||A_n|| = ||a_n(T)|| \to 0$ as $n \to \infty$. Then T is uniformly A-convergent, and $B_n = b_n(T) \to (I - T)^{-1}$ uniformly.*

Proof. T is clearly asymptotically A bounded and strong asymptotically A-regular. From $||A_n|| \to 0$ it follows that $||(I - T)B_n - I|| \to 0$ in virtue of (A1). Let N be a fixed positive integer with $||(I - T)B_N - I|| < \frac{1}{2}$. For each $x \in X$ and each $\epsilon > 0$ there is a positive integer n_0 such that

$$||(I - T)(B_n - B_m)B_N x|| < \tfrac{1}{2}\epsilon, \quad n, m > n_0.$$

Since

$$B_n x - B_m x = (I - (I - T)B_N)(B_n x - B_m x) + (I - T)(B_n - B_m)B_N x,$$

we get the inequality

$$||B_n x - B_m x|| < \tfrac{1}{2}||B_n x - B_m x|| + \tfrac{1}{2}\epsilon$$

valid for all $n, m > n_0$. Hence $||B_n x - B_m x|| < \epsilon$ for all $n, m > n_0$, and $\{B_n x\}$ converges in norm for each $x \in X$ as X is complete. For each $x \in X$, $x = \lim_n (I - T)B_n x = (I - T)(\lim_n B_n x)$ in norm, so that $X = R(I - T)$. $Q = 0$ on X, which proves $N(I - T) = \{0\}$. Therefore $(I - T)^{-1} \in B(X)$, and $||B_n - (I - T)^{-1}|| \leqq ||(I - T)^{-1}|| \, ||(I - T)B_n - I|| \to 0$ as $n \to \infty$.

As a consequence of Proposition 5 we obtain that $\sum_0^\infty T^n$ converges uniformly if and only if $||T^n|| \to 0$, or equivalently, if and only if $r(T) = \lim_n ||T^n||^{1/n} < 1$. If $\sum_0^\infty T^n$ converges weakly or strongly, $||T^n|| \leqq M$ for some $M > 0$ and all $n \geqq 0$, and $r(T) \leqq \lim_n M^{1/n} = 1$. However, even in the case when $\sum_0^\infty T^n$ converges strongly we can have $r(T) = 1$. To see this suppose X is a separable

Hilbert space with an orthonormal basis $\{e_k\}_1^\infty$. Following [**6**] define a linear diagonal operator T by

$$Te_k = (1 - k)k^{-1}e_k, \qquad k = 1, 2, \ldots .$$

T is selfadjoint, and $r(T) = ||T|| = \sup_k|(1 - k)k^{-1}| = 1$. For every k, $||T^n e_k|| \to 0$ as $n \to \infty$. Also any finite linear combination y of the basis vectors satisfies $||T^n y|| \to 0$ as $n \to \infty$. Each $x \in X$ can be approximated by such y, and the inequality $||T^n x|| \leqq ||x - y|| + ||T^n y||$ shows that also $||T^n x|| \to 0$ as $n \to \infty$. Given $f = \sum \alpha_k e_k \in X$, we define $x = \sum \lambda_k e_k$ with $\lambda_k = k(2k - 1)^{-1}\alpha_k$; $\sum |\lambda_k|^2$ converges as $|\lambda_k| \leqq |\alpha_k|$. It is easily verified that $f = (I - T)x$, hence $X = R(I - T)$. In view of Proposition 4, T is strongly convergent.

PROPOSITION 6. *Suppose T is weak (strong) asymptotically A-convergent for some admissible matrix A. Then $X = N(I - T) \oplus R(I - T)^-$. For each $f \in R(I - T)$ and any $x_0 \in X$, the sequence $\{A_n x_0 + B_n f\}$ converges weakly (strongly) to a solution x of the equation $(I - T)x = f$; x is of the form $x = Qx_0 + x^*$, where Qx_0 is the projection of x_0 into $N(I - T)$ in the direction of $R(I - T)^-$, and x^* is the unique solution of $(I - T)x = f$ in $R(I - T)^-$.*

Proof. The first two conclusions of the proposition follow from Theorem 1 and Proposition 1 respectively. Suppose $(I - T)y = f$ for some $y \in X$. Then $\{B_n f\} = \{(I - A_n)y\}$ converges weakly (strongly) to the element $x^* = (I - Q)y$ ($Q = \lim_n A_n$). Since $I - Q$ projects X onto $R(I - T)^-$, $x^* \in R(I - T)^-$, and $(I - T)x^* = (I - T)(I - Q)y = (I - T)y = f$. Hence x^* is a solution of $(I - T)x = f$ contained in $R(I - T)^-$; the uniqueness of such a solution is a consequence of the decomposition

$$X = N(I - T) \oplus R(I - T)^-.$$

The last statement in Proposition 6 then follows from the fact that $\lim_n(A_n x_0 + B_n f) = Qx_0 + x^*$ in the corresponding topology.

Remark 4. If we assume that T is strong asymptotically A-regular and that $\{A_n\}$ converges in the weak operator topology, (E1) supplies the result that $\{A_n x_0 + B_n f\}$ converges strongly for each $f \in R(I - T)$ and each $x_0 \in X$ as in Theorem 3 of [**4**]. Proposition 6 provides the additional insight pertaining to the decomposition of X and the form of a solution x of the equation $(I - T)x = f$.

Suppose T is asymptotically bounded and weak (strong) asymptotically regular. For any admissible matrix A, T is also asymptotically A-bounded. Indeed, if $||T^n|| \leqq M$ for some $M > 0$ and all $n \geqq 0$, then $||a_n(T)|| \leqq \sum_0^n a_{nj}||T^j|| \leqq M$. If A is also Toeplitz, T is weak (strong) asymptotically A-regular. Suppose $\lim_n T^n x = z$ for some $x \in X$. Then $\lim_n a_n(T)x = z$ [**9**]. If T is weak (strong) asymptotically regular, $\lim_n T^n(I - T)x = 0$ weakly (strongly) for each $x \in X$, and also $\lim_n a_n(T)(I - T)x = 0$ weakly (strongly) for each $x \in X$. Thus we are led to

PROPOSITION 7. *An operator T is weak (strong) asymptotically convergent if and only if:*

(a) *T is asymptotically bounded,*

(b) *T is weak (strong) asymptotically regular, and*

(c) *for some admissible Toeplitz matrix A, $\{A_n x\}$ clusters weakly for each $x \in X$.*

The proposition strengthens the Corollary to Theorem 5 in [**3**].

The conclusions of Proposition 6 are naturally valid with strong convergence throughout when T is uniform asymptotically A-convergent for some admissible A. In this case however the following stronger result can be obtained.

PROPOSITION 8. *Suppose T is uniform asymptotically A-convergent for some admissible matrix A. Then $R(I - T)$ is closed, and*

$$X = N(I - T) \oplus R(I - T).$$

Consequently, if $N(I - T) \neq \{0\}$, 1 is a simple pole of $(\lambda I - T)^{-1}$.

Proof. Let Q be the uniform limit of $\{A_n\} = \{a_n(T)\}$. Then $Q^2 = Q$ and $T^j Q = Q$ for each $j = 0, 1, \ldots,$ in view of Lemma 1. Hence $(T - Q)^j = T^j - Q$ for each $j = 0, 1, \ldots,$ and $a_n(T - Q) = \sum_{j=0}^{n} a_{nj}(T - Q)^j = \sum_{j=0}^{n} a_{nj}(T^j - Q) = a_n(T) - Q$. Then $\|a_n(T - Q)\| \to 0$ as $n \to \infty$. According to Proposition 5, $T - Q$ is uniformly A-convergent, and

$$(I - T + Q)^{-1} \in B(X).$$

In particular, $X = R(I - T + Q)$, and each $x \in X$ can be written in the form $x = (I - T)u + Qu$, where $Qu \in N(I - T)$. Suppose $x \in R(I - T)^-$. In view of the decomposition $X = N(I - T) \oplus R(I - T)^-$ which follows from Theorem 1, and the equality $x = Qu + (I - T)u$, Qu is necessarily 0, and $x = (I - T)u$. This proves $R(I - T)^- = R(I - T)$. The last statement in Proposition 8 is a direct consequence of the decomposition

$$X = N(I - T) \oplus R(I - T)$$

with $R(I - T)$ closed.

Proposition 8 is a generalization of the result obtained in [**7**] for a uniform asymptotically convergent operator T.

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