Bull. Aust. Math. Soc. **85** (2012), 380–394
doi:10.1017/S0004972711003479

ON SOME REFINEMENTS AND CONVERSES OF MULTIDIMENSIONAL HILBERT-TYPE INEQUALITIES

MARIO KRNIĆ

(Received 3 June 2011)

Abstract

The main objective of this paper is a study of some new refinements and converses of multidimensional Hilbert-type inequalities with nonconjugate exponents. Such extensions are deduced with the help of some remarkable improvements of the well-known Hölder inequality. First, we obtain refinements and converses of the general multidimensional Hilbert-type inequality in both quotient and difference form. We then apply the results to homogeneous kernels with negative degree of homogeneity. Finally, we consider some particular settings with homogeneous kernels and weighted functions, and compare our results with those in the literature.

2010 *Mathematics subject classification*: primary 26D15.

Keywords and phrases: Hilbert-type inequality, nonconjugate exponents, Hölder's inequality, refinement, converse.

1. Introduction

Hilbert's inequality is one of the most significant weighted inequalities in mathematical analysis and its applications. Through the years, Hilbert-type inequalities have been discussed by numerous authors, who either reproved them using various techniques, or applied and generalised them in many different ways.

Although classical, Hilbert's inequality is still of interest to many mathematicians. Some recent results concerning Hilbert's inequality concern various choices of kernels, weighted functions and sets of integration, as well as an extension to the multidimensional case and investigations of the best possible constant factors involved in the inequalities (see, for instance, [1, 3, 4, 6, 7]). All these results are equipped with conjugate exponents p_i, that is, such that

$$\sum_{i=1}^{n} 1/p_i = 1, \quad p_i > 1, n \geq 2.$$

In this paper we make reference to the paper by Brnetić *et al.* [5] which provides a unified treatment of multidimensional Hilbert-type inequalities in the setting of

This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grant 036–1170889–1054.

nonconjugate exponents. Before we state our corresponding result, we recall the definition of nonconjugate parameters.

Let p_i be real parameters satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1, \quad p_i > 1, i = 1, 2, \ldots, n. \tag{1.1}$$

The parameters p_i' are defined as their associated conjugates, that is,

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Since $p_i > 1$, it follows that $p_i' > 1$, $i = 1, 2, \ldots, n$. In addition, we define

$$\lambda = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'}. \tag{1.3}$$

Clearly, (1.1) and (1.2) imply that $0 < \lambda < 1$. Finally, we introduce parameters q_i defined by

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, 2, \ldots, n, \tag{1.4}$$

assuming that $q_i > 0$, $i = 1, 2, \ldots, n$. The above conditions (1.1)–(1.4) establish the n-tuple of nonconjugate exponents and were given by Bonsall [2] more than half a century ago. The above conditions also imply the relations $\lambda = \sum_{i=1}^{n} 1/q_i$ and $1/q_i + 1 - \lambda = 1/p_i$, $i = 1, 2, \ldots, n$. Of course, if $\lambda = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, which represents the setting with conjugate parameters.

We now state the general multidimensional Hilbert-type inequality from [5], in this setting. Suppose that $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces and

$$K : \prod_{i=1}^{n} \Omega_i \to \mathbb{R}, \quad \phi_{ij} : \Omega_j \to \mathbb{R}, \quad f_i : \Omega_i \to \mathbb{R}, \quad i, j = 1, 2, \ldots, n,$$

are nonnegative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then

$$\int_{\mathbf{\Omega}} K^{\lambda}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mu(\mathbf{x}) \le \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.5}$$

where

$$\omega_i(x_i) = \bigg(\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j)\, d\hat{\mu}^i(\mathbf{x}) \bigg)^{1/q_i} \tag{1.6}$$

and

$$\begin{gathered} \mathbf{\Omega} = \prod_{i=1}^{n} \Omega_i, \quad \hat{\mathbf{\Omega}}^i = \prod_{j=1, j\neq i}^{n} \Omega_j, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n), \\ d\mu(\mathbf{x}) = \prod_{i=1}^{n} d\mu_i(x_i), \quad d\hat{\mu}^i(\mathbf{x}) = \prod_{j=1, j\neq i}^{n} d\mu_j(x_j). \end{gathered} \tag{1.7}$$

Abbreviations as in (1.7) will be valid throughout this paper. Also, note that $\|\cdot\|_{p_i}$ denotes the usual norm in $L^{p_i}(\Omega_i)$, that is,

$$\|\phi_{ii}\omega_i f_i\|_{p_i} = \left(\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\right)^{1/p_i}, \quad i = 1, 2, \ldots, n.$$

In addition, we suppose that all integrals converge and omit such conditions in what follows.

It is important to emphasise that the multidimensional inequality (1.5) provides a unified treatment of Hilbert-type inequalities and extends results from [1, 3, 4, 6] to the case of nonconjugate exponents. For more details, the reader is referred to [5].

The main purpose of this paper is to establish some general refinements and converses of the multidimensional Hilbert-type inequality (1.5). More precisely, such extensions can be established with the help of some remarkable refinements and converses of the well-known Hölder inequality, which will be presented in the next section.

The paper is organised in the following way. In Section 2 we establish two pairs of refinements and converses of Hölder's inequality, one in the difference form, and the other one in the quotient form. With the help of such improvements, in Section 3 we establish our main results, that is, refinements and converses of the Hilbert-type inequality (1.5). Further, in Section 4 we apply our main results to homogeneous kernels with negative degree of homogeneity. Finally, in Section 5 we consider our results equipped with some particular kernels and weight functions, and compare our results with those previously known in the literature.

The techniques that will be used in the proofs are mainly based on classical real analysis, especially on Fubini's theorem and improvements of Hölder's inequality which will be presented in due course.

2. Auxiliary results

The starting point in obtaining Hilbert's inequality is the well-known Hölder inequality, that is,

$$\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\, d\mu(x) \le \prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}, \tag{2.1}$$

where $F_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are nonnegative measurable functions on σ-finite measure space (Ω, Σ, μ) and α_i are positive real numbers such that $\sum_{i=1}^n \alpha_i = 1$.

Our first result in this section yields an improvement of the above Hölder inequality in the quotient form.

Lemma 2.1. *Let (Ω, Σ, μ) be a σ-finite measure space and let $F_i : \Omega \to \mathbb{R}$ be nonnegative measurable functions, $i = 1, 2, \ldots, n$. If $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$, then the*

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

following inequalities hold:

$$\left(\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\right)^{n\max_{1\le i\le n}\{\alpha_i\}} \le \frac{\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}} \le \left(\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\right)^{n\min_{1\le i\le n}\{\alpha_i\}}. \tag{2.2}$$

Proof. The left-hand side of Hölder's inequality (2.1) can be rewritten as

$$\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x) = \int_\Omega \left(\prod_{i=1}^n F_i^{\beta_i}(x)\right)^{1-nm} \cdot \left(\prod_{i=1}^n F_i^{1/n}(x)\right)^{nm} d\mu(x),$$

where $m = \min_{1\le i\le n}\{\alpha_i\}$ and $\beta_i = (\alpha_i - m)/(1 - nm)$, $i = 1, 2, \ldots, n$.

Since $1 - nm \ge 0$, the application of Hölder's inequality to the previous relation yields the inequality

$$\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x) \le \left(\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)\,d\mu(x)\right)^{1-nm} \cdot \left(\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)\right)^{nm}. \tag{2.3}$$

On the other hand, the right-hand side of Hölder's inequality (2.1) can be rewritten as

$$\prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i} = \left(\prod_{i=1}^n \|F_i^{\beta_i}\|_{1/\beta_i}\right)^{1-nm} \cdot \left(\prod_{i=1}^n \|F_i^{1/n}\|_n\right)^{nm}. \tag{2.4}$$

Now (2.3) and (2.4) imply that

$$\frac{\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}} \le \left(\frac{\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{\beta_i}\|_{1/\beta_i}}\right)^{1-nm} \cdot \left(\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\right)^{nm}. \tag{2.5}$$

Note that $\sum_{i=1}^n \beta_i = 1, \beta_i \ge 0$, so yet another application of Hölder's inequality implies

$$\frac{\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{\beta_i}\|_{1/\beta_i}} \le 1,$$

that is, from (2.5) we get the right inequality in (2.2).

The left inequality in (2.2) is proved in a similar way. Namely, we use the decomposition

$$\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x) = \int_\Omega \left(\prod_{i=1}^n F_i^{\alpha_i}(x)\right)^{1/nM} \cdot \left(\prod_{i=1}^n F_i^{\gamma_i}(x)\right)^{1-1/nM} d\mu(x),$$

where $M = \max_{1\le i\le n}\{\alpha_i\}$, $\gamma_i = (M - \alpha_i)/(nM - 1)$, $i = 1, 2, \ldots, n$, and apply Hölder's inequality as in the first part of the proof. □

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

Clearly, the quotient of the left- and right-hand sides of Hölder's inequality (2.1) is bounded via the quotient of the same type involving equal exponents. Moreover, since

$$\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x) \le \prod_{i=1}^n \|F_i^{1/n}\|_n,$$

the right inequality in (2.2) yields a refinement, while the left one yields the converse of Hölder's inequality (2.1). The interpolating inequalities (2.2) will be referred to as the refinement and converse of Hölder's inequality in the quotient form.

On the other hand, our next result provides the refinement and the converse of Hölder's inequality in the difference form.

Lemma 2.2. *Let* (Ω, Σ, μ) *be a* σ*-finite measure space and let* $F_i : \Omega \to \mathbb{R}$ *be nonnegative measurable functions,* $i = 1, 2, \ldots, n$. *If* $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$, *then the following inequalities hold:*

$$\begin{aligned} n \min_{1\le i\le n}\{\alpha_i\} \prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}\Big(1 - \frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\Big) \\ \le \prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i} - \int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x) \\ \le n \max_{1\le i\le n}\{\alpha_i\} \prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}\Big(1 - \frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\Big). \end{aligned} \tag{2.6}$$

Proof. Inequalities (2.6) can be derived through the refinement and the converse of the classical arithmetic–geometric mean inequality. Namely, the difference between the weighted arithmetic and geometric means can be rewritten as

$$\sum_{i=1}^n \alpha_i t_i - \prod_{i=1}^n t_i^{\alpha_i} = \sum_{i=1}^n (\alpha_i - m)t_i + m\sum_{i=1}^n t_i - \Big(\prod_{i=1}^n t_i^{\beta_i}\Big)^{1-nm} \cdot \Big(\prod_{i=1}^n t_i^{1/n}\Big)^{nm}, \tag{2.7}$$

where $m = \min_{1\le i\le n}\{\alpha_i\}$, $\beta_i = (\alpha_i - m)/(1 - nm)$, $t_i > 0$, $i = 1, 2, \ldots, n$. In addition, the arithmetic–geometric mean inequality yields

$$\Big(\prod_{i=1}^n t_i^{\beta_i}\Big)^{1-nm} \cdot \Big(\prod_{i=1}^n t_i^{1/n}\Big)^{nm} \le (1 - nm)\prod_{i=1}^n t_i^{\beta_i} + nm\prod_{i=1}^n t_i^{1/n}. \tag{2.8}$$

Thus, from (2.7) and (2.8),

$$\sum_{i=1}^n \alpha_i t_i - \prod_{i=1}^n t_i^{\alpha_i} \ge (1 - nm)\Big(\sum_{i=1}^n \beta_i t_i - \prod_{i=1}^n t_i^{\beta_i}\Big) + nm\Big(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{1/n}\Big),$$

that is,

$$\sum_{i=1}^n \alpha_i t_i - \prod_{i=1}^n t_i^{\alpha_i} \ge nm\Big(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{1/n}\Big), \tag{2.9}$$

since $\sum_{i=1}^n \beta_i = 1$ and $\sum_{i=1}^n \beta_i t_i - \prod_{i=1}^n t_i^{\beta_i} \ge 0$.

Now, if we replace t_i by $F_i(x)/\int_\Omega F_i(x)\,d\mu(x)$ and taking into account that

$$\int_\Omega F_i(x)\,d\mu(x) = \|F_i^{\alpha_i}\|_{1/\alpha_i}^{1/\alpha_i} = \|F_i^{1/n}\|_n^n,$$

(2.9) takes the form

$$\sum_{i=1}^n \frac{\alpha_i f_i(x)}{\int_\Omega F_i(x)\,d\mu(x)} - \frac{\prod_{i=1}^n F_i^{\alpha_i}(x)}{\prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}} \geq nm\Big(\frac{f_i(x)}{n\int_\Omega F_i(x)\,d\mu(x)} - \frac{\prod_{i=1}^n F_i^{1/n}(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\Big),$$

that is,

$$1 - \frac{\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{\alpha_i}\|_{1/\alpha_i}} \geq nm\Big(1 - \frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x)}{\prod_{i=1}^n \|F_i^{1/n}\|_n}\Big),$$

after integrating over Ω with respect to the measure μ.

To prove the right inequality in (2.6) we start with the relation

$$\begin{aligned} &nM\Big(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{1/n}\Big) \\ &\quad= \sum_{i=1}^n (M-\alpha_i)t_i + \sum_{i=1}^n \alpha_i t_i - nM\Big(\prod_{i=1}^n t_i^{\alpha_i}\Big)^{1/nM} \cdot \Big(\prod_{i=1}^n t_i^{\gamma_i}\Big)^{1-1/nM}, \end{aligned} \tag{2.10}$$

where $M = \max_{1\leq i\leq n}\{\alpha_i\}$ and $\gamma_i = (M-\alpha_i)/(nM-1)$, $i = 1, 2, \ldots, n$. Further, the arithmetic–geometric mean inequality yields

$$nM\Big(\prod_{i=1}^n t_i^{\alpha_i}\Big)^{1/nM} \cdot \Big(\prod_{i=1}^n t_i^{\gamma_i}\Big)^{1-1/nM} \leq \prod_{i=1}^n t_i^{\alpha_i} + (nM-1)\Big(\prod_{i=1}^n t_i^{\gamma_i}\Big). \tag{2.11}$$

Therefore, (2.10) and (2.11) imply that

$$nM\Big(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{1/n}\Big) \geq \sum_{i=1}^n \alpha_i t_i - \prod_{i=1}^n t_i^{\alpha_i} + (nM-1)\Big(\sum_{i=1}^n \gamma_i t_i - \prod_{i=1}^n t_i^{\gamma_i}\Big),$$

that is,

$$nM\Big(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{1/n}\Big) \geq \sum_{i=1}^n \alpha_i t_i - \prod_{i=1}^n t_i^{\alpha_i},$$

since $\sum_{i=1}^n \gamma_i t_i \geq \prod_{i=1}^n t_i^{\gamma_i}$, $\sum_{i=1}^n \gamma_i = 1$. The rest of the proof follows along the same lines as the proof of the left inequality in (2.6). □

Obviously, since

$$\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)\,d\mu(x) \leq \prod_{i=1}^n \|F_i^{1/n}\|_n,$$

the left inequality in (2.6) yields the refinement, while the right one provides the converse of Hölder's inequality. The interpolating series of inequalities (2.6) will be referred to as the refinement and the converse of Hölder's inequality in the difference form.

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

3. Main results

In this section we establish our main results, that is, refinements and converses of the multidimensional Hilbert-type inequality (1.5) in both quotient and difference form. Such results will be derived with the help of the interpolating series of (2.2) and (2.6), presented in the previous section.

As we have already mentioned, the starting point in obtaining the Hilbert-type inequality is Hölder's inequality (2.1). However, Hölder's inequality includes conjugate parameters and we shall use Bonsall's idea [2] about reduction of nonconjugate exponents to the setting which includes conjugate exponents. Regarding definitions (1.1)–(1.4) of nonconjugate exponents, the essence of the above idea is the apparently trivial observation that $\sum_{i=1}^{n} 1/q_i + (1-\lambda) = 1$ and the application of Hölder's inequality to conjugate exponents q_i, $i = 1, 2, \ldots, n$, and $1/(1-\lambda)$. In this way, by using refinements and converses of Hölder's inequality from Section 2, we obtain refinements and converses of the multidimensional Hilbert-type inequality (1.5) as well.

Our first result provides the refinement and the converse of the Hilbert-type inequality (1.5) in the quotient form.

THEOREM 3.1. *Suppose that p_i, p_i', q_i, $i = 1, 2, \ldots, n$, and λ are real parameters satisfying (1.1)–(1.4). Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces, and let $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, be nonnegative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then*

$$\frac{\left(\int_{\mathbf{\Omega}} (K^n(\mathbf{x}) \prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j))^{1/(n+1)}\, d\mu(\mathbf{x})\right)^{(n+1)M}}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{2Mp_i}}$$
$$\leq \frac{\int_{\mathbf{\Omega}} K^{\lambda}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}}$$
$$\leq \frac{\left(\int_{\mathbf{\Omega}} (K^n(\mathbf{x}) \prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j))^{1/(n+1)}\, d\mu(\mathbf{x})\right)^{(n+1)m}}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{2mp_i}}, \tag{3.1}$$

where $\omega_i : \Omega_i \to \mathbb{R}$ is defined by (1.6) and $m = \min\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$.

PROOF. The left-hand side of the Hilbert-type inequality (1.5) can be rewritten in the form

$$\int_{\mathbf{\Omega}} K^{\lambda}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x}) \cdot F_{n+1}^{1-\lambda}(\mathbf{x})\, d\mu(\mathbf{x}),$$

where

$$F_i(\mathbf{x}) = K(\mathbf{x}) \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j), \quad i = 1, 2, \ldots, n, \tag{3.2}$$

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

and

$$F_{n+1}(\mathbf{x}) = \prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{p_i}(x_i). \tag{3.3}$$

Now we consider the interpolating inequalities of (2.2) with $n+1$ instead of n and the parameters $\alpha_i = 1/q_i$, $i = 1, 2, \ldots, n$, and $\alpha_{n+1} = 1-\lambda$. Clearly, due to the definitions of nonconjugate exponents, $\sum_{i=1}^{n+1} \alpha_i = 1$. Moreover, by Fubini's theorem,

$$\begin{aligned}
\|F_i^{1/q_i}\|_{q_i} &= \bigg(\int_{\mathbf{\Omega}} K(\mathbf{x}) \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j)\, d\mu(\mathbf{x})\bigg)^{1/q_i} \\
&= \bigg(\int_{\Omega_i} \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \bigg(\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j)\, d\hat{\mu}^i(\mathbf{x})\bigg) d\mu_i(x_i)\bigg)^{1/q_i} \\
&= \bigg(\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\bigg)^{1/q_i} = \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/q_i}, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

and

$$\begin{aligned}
\|F_{n+1}^{1-\lambda}\|_{1/(1-\lambda)} &= \bigg(\int_{\mathbf{\Omega}} \prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu(\mathbf{x})\bigg)^{1-\lambda} \\
&= \prod_{i=1}^{n} \bigg(\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\bigg)^{1-\lambda} = \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i(1-\lambda)},
\end{aligned}$$

which yields

$$\prod_{i=1}^{n} \|F_i^{1/q_i}\|_{q_i} \cdot \|F_{n+1}^{1-\lambda}\|_{1/(1-\lambda)} = \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i(1/q_i+1-\lambda)} = \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}. \tag{3.4}$$

Similarly,

$$\|F_i^{1/(n+1)}\|_{n+1} = \bigg(\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\bigg)^{1/(n+1)} = \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/(n+1)}, \quad i = 1, 2, \ldots, n,$$

and

$$\|F_{n+1}^{1/(n+1)}\|_{n+1} = \prod_{i=1}^{n} \bigg(\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\bigg)^{1/(n+1)} = \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/(n+1)},$$

that is,

$$\prod_{i=1}^{n+1} \|F_i^{1/(n+1)}\|_{n+1} = \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{2p_i/(n+1)}. \tag{3.5}$$

We now need to compute the product of the functions F_i, $i = 1, 2, \ldots, n+1$:

$$\prod_{i=1}^{n+1} F_i(\mathbf{x}) = K^n(\mathbf{x}) \prod_{i=1}^{n} \frac{(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{i,j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j). \tag{3.6}$$

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

Finally, if we substitute (3.4)–(3.6) into (2.2) with n replaced by $n+1$, we get (3.1) and the proof is complete. □

REMARK 3.2. According to inequalities (2.2), we conclude that the left inequality in (3.1) yields the converse, while the right one yields the refinement of the Hilbert-type inequality (1.5) in the quotient form.

On the other hand, regarding inequalities (2.6), we also obtain the refinement and the converse of (1.5) in the difference form.

THEOREM 3.3. *Let p_i, p_i', q_i, $i = 1, 2, \ldots, n$, and λ be real parameters satisfying (1.1)–(1.4), and let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2, \ldots, n$, be σ-finite measure spaces. If $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, are nonnegative measurable functions with $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, then*

$$
\begin{aligned}
&(n+1)m \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} \\
&\quad \times \left(1 - \frac{\int_{\mathbf{\Omega}} (K^n(\mathbf{x}) \prod_{i=1}^n (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^n \phi_{ij}^{q_i}(x_j))^{1/(n+1)}\, d\mu(\mathbf{x})}{\prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i}^{2p_i/(n+1)}}\right) \\
&\quad \le \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} - \int_{\mathbf{\Omega}} K^\lambda(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mu(\mathbf{x}) \\
&\quad \le (n+1)M \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} \\
&\qquad \times \left(1 - \frac{\int_{\mathbf{\Omega}} (K^n(\mathbf{x}) \prod_{i=1}^n (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^n \phi_{ij}^{q_i}(x_j))^{1/(n+1)}\, d\mu(\mathbf{x})}{\prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i}^{2p_i/(n+1)}}\right),
\end{aligned}
\tag{3.7}
$$

where $\omega_i : \Omega_i \to \mathbb{R}$ is defined by (1.6) and $m = \min\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$.

PROOF. The proof is very similar to the proof of Theorem 3.1. Namely, we use the same decomposition of the left-hand side of the multidimensional Hilbert-type inequality (1.5), involving functions F_i, $i = 1, 2, \ldots, n+1$, defined by (3.2) and (3.3). Now the result follows after substituting (3.4)–(3.6) into inequalities (2.6) with $n+1$ instead of n, and the parameters $\alpha_i = 1/q_i$, $i = 1, 2, \ldots, n$, and $\alpha_{n+1} = 1 - \lambda$. □

REMARK 3.4. Considering inequalities (2.6), we conclude that the left inequality in (3.7) yields the refinement, while the right one provides the converse of the Hilbert-type inequality (1.5) in the difference form.

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

4. Applications to homogeneous functions

In this section, we apply our general results to homogeneous functions with negative degree of homogeneity. Further, regarding the notation from the previous section, we assume that $\Omega_i = R_+$, equipped with the nonnegative Lebesgue measure $d\mu_i(x_i) = dx_i$, $i = 1, 2, \ldots, n$. In addition, $\mathbf{\Omega} = \mathbb{R}_+^n$ and $d\mathbf{x} = dx_1 dx_2 \ldots dx_n$.

We introduce real parameters A_{ij}, $i, j = 1, 2, \ldots, n$, provided that $\sum_{i=1}^n A_{ij} = 0$, and we write $\alpha_i = \sum_{j=1}^n A_{ij}$. Next, we consider the set of power functions $\phi_{ij} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\phi_{ij}(x_j) = x_j^{A_{ij}}. \tag{4.1}$$

Clearly, this set of power functions satisfies the condition

$$\prod_{i,j=1}^n \phi_{ij}(x_j) = \prod_{j=1}^n \prod_{i=1}^n x_j^{A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n A_{ij}} = 1,$$

since $\sum_{i=1}^n A_{ij} = 0$. Therefore, the functions ϕ_{ij}, $i, j = 1, 2, \ldots, n$, satisfy the conditions of Theorems 3.1 and 3.3.

Recall that the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s} K(\mathbf{x})$ for all $t > 0$. Furthermore, for $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^n u_j^{a_j} \, \hat{d}^i\mathbf{u}, \quad i = 1, 2, \ldots, n, \tag{4.2}$$

where $\hat{\mathbf{u}}^i = (u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n)$, $\hat{d}^i\mathbf{u} = du_1 \ldots du_{i-1}\, du_{i+1} \ldots du_n$, provided that the above integral converges.

Therefore, in this setting we can find an explicit formula for the weighted function (1.6). More precisely, we use the substitution $x_j = u_j x_i$, $j \neq i$, that is, $\hat{d}^i\mathbf{x} = x_i^{n-1}\hat{d}^i\mathbf{u}$, while the homogeneity of the kernel K yields the relation $K(\mathbf{x}) = x_i^{-s} K(\hat{\mathbf{u}}^i)$. Moreover, regarding definition (4.2),

$$\begin{aligned} \omega_i(x_i) &= \bigg(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{j=1, j\neq i}^n x_j^{q_i A_{ij}} \, \hat{d}^i\mathbf{x}\bigg)^{1/q_i} \\ &= \bigg(x_i^{n-1-s+\sum_{j=1, j\neq i}^n q_i A_{ij}} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^n u_j^{q_i A_{ij}} \, \hat{d}^i\mathbf{u}\bigg)^{1/q_i} \\ &= x_i^{(n-1-s)/q_i + \alpha_i - A_{ii}} k_i^{1/q_i}(q_i \mathbf{A_i}), \end{aligned} \tag{4.3}$$

where $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $i = 1, 2, \ldots, n$.

In the following, we give variants of Theorems 3.1 and 3.3 which include homogeneous kernels in this setting. First, we give the interpolating series of inequalities in the quotient form.

Corollary 4.1. *Let p_i, p_i', q_i, $i = 1, 2, \ldots, n$, and λ be as in (1.1)–(1.4), and let A_{ij}, $i, j = 1, 2, \ldots, n$, be real parameters such that $\sum_{i=1}^n A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a*

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

nonnegative measurable homogeneous function of degree $-s$, $s>0$, *and* $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\ldots,n$, *are nonnegative measurable functions, then*

$$\frac{(\int_{\mathbb{R}_+^n}(K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}}f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x})^{(n+1)M}}{(\prod_{i=1}^n k_i(q_i\mathbf{A_i}))^M\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}^{2Mp_i}}$$
$$\leq\frac{\int_{\mathbb{R}_+^n}K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)\,d\mathbf{x}}{\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}}$$
$$\leq\frac{(\int_{\mathbb{R}_+^n}(K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}}f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x})^{(n+1)m}}{(\prod_{i=1}^n k_i(q_i\mathbf{A_i}))^m\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}^{2mp_i}},\tag{4.4}$$

where $\alpha_i=\sum_{j=1}^n A_{ij}$, $i=1,2,\ldots,n$, $m=\min\{1/q_1,1/q_2,\ldots,1/q_n,1-\lambda\}$, $M=\max\{1/q_1,1/q_2,\ldots,1/q_n,1-\lambda\}$, *and* $k_i(\cdot)$, $i=1,2,\ldots,n$, *are defined by* (4.2).

Proof. The proof is a direct consequence of Theorem 3.1. Namely, if we substitute the functions ϕ_{ij} and ω_i, $i,j=1,2,\ldots,n$, defined respectively by (4.1) and (4.3), into (3.1), we get (4.4) after straightforward computation. □

The next result yields the interpolating series of inequalities in the difference form.

Corollary 4.2. *Suppose that* p_i, p_i', q_i, $i=1,2,\ldots,n$, *and* λ *are as in* (1.1)–(1.4), *and* A_{ij}, $i,j=1,2,\ldots,n$, *are real parameters satisfying* $\sum_{i=1}^n A_{ij}=0$. *If* $K:\mathbb{R}_+^n\to\mathbb{R}$ *is a nonnegative measurable homogeneous function of degree* $-s$, $s>0$, *and* $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\ldots,n$, *are nonnegative measurable functions, then*

$$(n+1)m\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}$$
$$\times\Bigg(1-\frac{\int_{\mathbb{R}_+^n}(K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}}f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x}}{(\prod_{i=1}^n k_i(q_i\mathbf{A_i}))^{1/(n+1)}\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}^{2p_i/(n+1)}}\Bigg)$$
$$\leq\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}-\int_{\mathbb{R}_+^n}K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)\,d\mathbf{x}$$
$$\leq(n+1)M\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}$$
$$\times\Bigg(1-\frac{\int_{\mathbb{R}_+^n}(K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}}f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x}}{(\prod_{i=1}^n k_i(q_i\mathbf{A_i}))^{1/(n+1)}\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}^{2p_i/(n+1)}}\Bigg),\tag{4.5}$$

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1, 2, \dots, n$, $m = \min\{1/q_1, 1/q_2, \dots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \dots, 1/q_n, 1-\lambda\}$, *and* $k_i(\cdot)$, $i = 1, 2, \dots, n$, *are defined by* (4.2).

Proof. We use Theorem 3.3. More precisely, if we insert the functions ϕ_{ij} and ω_i, $i, j = 1, 2, \dots, n$, defined respectively by (4.1) and (4.3), into (3.7), we get (4.5) after straightforward computation. □

5. Some examples

In this section, we discuss the results from the previous section in some particular settings. More precisely, we consider a homogeneous function $K : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$K(x) = \Big(\sum_{i=1}^n x_i\Big)^{-s}, \qquad s > 0. \tag{5.1}$$

Clearly, K is a homogeneous function of degree $-s$. In this setting, the constant (4.2) can be expressed in terms of the usual gamma function Γ. For that purpose, we use the well-known formula

$$\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{(1+\sum_{i=1}^{n-1} u_i)^{\sum_{i=1}^n a_i}}\, \hat{d}^n\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i)},$$

which holds for $a_i > 0$, $i = 1, 2, \dots n$ (see, for instance [1]). In this way, the constant factors $k_i(q_i\mathbf{A_i})$, $i = 1, 2, \dots, n$, involved in inequalities (4.4) and (4.5) become

$$k_i(q_i\mathbf{A_i}) = \frac{\Gamma(s-n+1-q_i\alpha_i+q_iA_{ii})}{\Gamma(s)} \prod_{j=1, j\neq i}^n \Gamma(1+q_iA_{ij}), \qquad i = 1, 2, \dots, n,$$

provided that $A_{ij} > -1/q_i$, $i \neq j$, and $A_{ii} - \alpha_i > (n-s-1)/q_i$.

In what follows we consider some special choices of the parameters A_{ij}, $i, j = 1, 2, \dots, n$, which will give us known results in the literature.

Example 5.1. Let $A_{ii} = (n-s)(\lambda q_i - 1)/q_i^2$ and $A_{ij} = (s-n)/(q_iq_j)$, $i, j = 1, 2, \dots, n$, $i \neq j$. Then one easily verifies that $\sum_{i=1}^n A_{ij} = \sum_{j=1}^n A_{ij} = 0$, so the interpolating series of inequalities (4.4) reduces to

$$\begin{aligned}
&\frac{\big(\int_{\mathbb{R}_+^n}((\sum_{i=1}^n x_i)^{-ns}\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(n-s)(\lambda q_i-n)/q_i} f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x}\big)^{(n+1)M}}{\mathcal{K}_M \prod_{i=1}^n \|x_i^{(n-1-s)/q_i} f_i\|_{p_i}^{2Mp_i}}\\
&\leq \frac{\int_{\mathbb{R}_+^n}(\sum_{i=1}^n x_i)^{-\lambda s}\prod_{i=1}^n f_i(x_i)\,d\mathbf{x}}{\mathcal{K}\prod_{i=1}^n \|x_i^{(n-1-s)/q_i} f_i\|_{p_i}}\\
&\leq \frac{\big(\int_{\mathbb{R}_+^n}((\sum_{i=1}^n x_i)^{-ns}\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(n-s)(\lambda q_i-n)/q_i} f_i^{2p_i}(x_i))^{1/(n+1)}\,d\mathbf{x}\big)^{(n+1)m}}{\mathcal{K}_m \prod_{i=1}^n \|x_i^{(n-1-s)/q_i} f_i\|_{p_i}^{2mp_i}},
\end{aligned} \tag{5.2}$$

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

where

$$\mathcal{K}_M = \frac{1}{\Gamma(s)^M} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + s - n}{p_i}\Big)^M \prod_{i=1}^{n} \Gamma\Big(\frac{q_i + s - n}{q_i}\Big)^{(n-1)M},$$

$$\mathcal{K} = \frac{1}{\Gamma(s)^\lambda} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + s - n}{p_i}\Big)^{1/q_i} \prod_{i=1}^{n} \Gamma\Big(\frac{q_i + s - n}{q_i}\Big)^{\lambda-(1/q_i)},$$

$$\mathcal{K}_m = \frac{1}{\Gamma(s)^m} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + s - n}{p_i}\Big)^m \prod_{i=1}^{n} \Gamma\Big(\frac{q_i + s - n}{q_i}\Big)^{(n-1)m},$$

provided that $s > n - \min_{1\le i\le n}\{p_i, q_i\}$.

Remark 5.2. If p_i are conjugate exponents, that is, $p_i = q_i$, $i = 1, 2, \ldots, n$, and $\lambda = 1$, then the constant $\mathcal{K}$ from (5.2) reduces to $\mathcal{K} = 1/(\Gamma(s)) \prod_{i=1}^{n} \Gamma((p_i + s - n)/p_i)$, that is, the middle term in (5.2) represents the quotient between the left- and right-hand sides of the corresponding multidimensional inequality in [1]. Therefore, the interpolating series of inequalities (5.2) can be regarded as the nonconjugate extension of the corresponding result from [1].

Example 5.3. If $A_{ii} = (\lambda q_i - 1)/(\lambda q_i^2)$ and $A_{ij} = -1/(\lambda q_i q_j)$, $i, j = 1, 2, \ldots, n$, $i \ne j$, then it follows that $\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = 0$. Now, if the degree of homogeneity of kernel (5.1) is $1 - n$, so that $s = n - 1$, the interpolating series of inequalities (4.4) becomes

$$\begin{aligned}
&\frac{(\int_{\mathbb{R}_+^n} ((\sum_{i=1}^{n} x_i)^{-n(n-1)} \prod_{i=1}^{n} x_i^{1-n/(\lambda q_i)} f_i^{2p_i}(x_i))^{1/(n+1)}\, d\mathbf{x})^{(n+1)M}}{\mathcal{L}_M \prod_{i=1}^{n} \|f_i\|_{p_i}^{2Mp_i}}\\
&\quad\le \frac{\int_{\mathbb{R}_+^n} (\sum_{i=1}^{n} x_i)^{-\lambda(n-1)} \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x}}{\mathcal{L} \prod_{i=1}^{n} \|f_i\|_{p_i}}\\
&\quad\le \frac{(\int_{\mathbb{R}_+^n} ((\sum_{i=1}^{n} x_i)^{-n(n-1)} \prod_{i=1}^{n} x_i^{1-n/(\lambda q_i)} f_i^{2p_i}(x_i))^{1/(n+1)}\, d\mathbf{x})^{(n+1)m}}{\mathcal{L}_m \prod_{i=1}^{n} \|f_i\|_{p_i}^{2mp_i}},
\end{aligned} \tag{5.3}$$

where

$$\mathcal{L}_M = \frac{1}{[(n-2)!]^M} \Big(\prod_{i=1}^{n} \Gamma\Big(\frac{1}{\lambda p_i'}\Big)\Big)^M,$$

$$\mathcal{L} = \frac{1}{[(n-2)!]^\lambda} \Big(\prod_{i=1}^{n} \Gamma\Big(\frac{1}{\lambda p_i'}\Big)\Big)^\lambda,$$

$$\mathcal{L}_m = \frac{1}{[(n-2)!]^m} \Big(\prod_{i=1}^{n} \Gamma\Big(\frac{1}{\lambda p_i'}\Big)\Big)^m.$$

Remark 5.4. The middle term in (5.3) represents the quotient between the left- and right-hand sides of the nonconjugate Hilbert-type inequality which was proved by Bonsall [2], for $n = 3$.

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

Note that the parameters A_{ij}, $i, j = 1, 2, \ldots, n$, were symmetric in the previous two examples. We conclude this paper with an example where the above parameters are not symmetric.

EXAMPLE 5.5. Suppose that A_i, $i = 1, 2, \ldots, n$, are real parameters satisfying relations $(n-s-1)/q_{i-1} < A_i < 1/q_{i-1}$, provided that $s > n-2$. We use the convention that $q_0 = q_n$. We define parameters A_{ij}, $i, j = 1, 2, \ldots, n$, by

$$A_{ij} = \begin{cases} A_i & j = i, \\ -A_{i+1} & j = i+1, \\ 0 & \text{otherwise}, \end{cases}$$

where the indices are taken modulo n from the set $\{1, 2, \ldots, n\}$. Now, in this setting equipped with the homogeneous kernel (5.1), the series of inequalities in (4.4) reads

$$\begin{aligned} &\frac{(\int_{\mathbb{R}^n_+}((\sum_{i=1}^n x_i)^{-ns} \prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_{i-1})A_i-(2p_i-q_i)A_{i+1}} f_i^{2p_i}(x_i))^{1/(n+1)}\, d\mathbf{x})^{(n+1)M}}{\mathcal{R}_M \prod_{i=1}^n \|x_i^{(n-1-s)/q_i+A_i-A_{i+1}} f_i\|_{p_i}^{2Mp_i}} \\ &\quad \le \frac{\int_{\mathbb{R}^n_+}(\sum_{i=1}^n x_i)^{-\lambda s} \prod_{i=1}^n f_i(x_i)\, d\mathbf{x}}{\mathcal{R} \prod_{i=1}^n \|x_i^{(n-1-s)/q_i+A_i-A_{i+1}} f_i\|_{p_i}} \\ &\quad \le \frac{(\int_{\mathbb{R}^n_+}((\sum_{i=1}^n x_i)^{-ns} \prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_{i-1})A_i-(2p_i-q_i)A_{i+1}} f_i^{2p_i}(x_i))^{1/(n+1)}\, d\mathbf{x})^{(n+1)m}}{\mathcal{R}_m \prod_{i=1}^n \|x_i^{(n-1-s)/q_i+A_i-A_{i+1}} f_i\|_{p_i}^{2mp_i}}, \end{aligned} \tag{5.4}$$

where

$$\mathcal{R}_M = \frac{1}{\Gamma(s)^M}\bigg(\prod_{i=1}^n \Gamma(s-n+1+q_iA_{i+1})\Gamma(1-q_iA_{i+1})\bigg)^M,$$

$$\mathcal{R} = \frac{1}{\Gamma(s)^\lambda} \prod_{i=1}^n \Gamma(s-n+1+q_iA_{i+1})^{1/q_i}\Gamma(1-q_iA_{i+1})^{1/q_i},$$

$$\mathcal{R}_m = \frac{1}{\Gamma(s)^m}\bigg(\prod_{i=1}^n \Gamma(s-n+1+q_iA_{i+1})\Gamma(1-q_iA_{i+1})\bigg)^m.$$

REMARK 5.6. Note that the middle term in (5.4) represents the corresponding multidimensional inequality in [3] in the conjugate setting.

REMARK 5.7. It is important to emphasise that the multidimensional inequalities (in the nonconjugate setting) represented by the middle terms in (5.2)–(5.4) were also derived in [5]. Therefore, our relations (5.2)–(5.4) represent refinements and converses of appropriate results from [5].

In conclusion, let us mention that we can also derive variants of (5.2)–(5.4) in the difference form. They are omitted here.

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press

References

[1] Y. Bicheng and T. M. Rassias, 'On the way of weight coefficients and research for the Hilbert-type inequalities', *Math. Inequal. Appl.* **6** (2003), 625–658.
[2] F. F. Bonsall, 'Inequalities with nonconjugate parameters', *Quart. J. Math. Oxford Ser.* **2**(2) (1951), 135–150.
[3] I. Brnetić and J. Pečarić, 'Generalization of Hilbert's integral inequality', *Math. Inequal. Appl.* **7** (2004), 199–205.
[4] I. Brnetić and J. Pečarić, 'Generalization of inequalities of Hardy-Hilbert's type', *Math. Inequal. Appl.* **7** (2004), 217–225.
[5] I. Brnetić, M. Krnić and J. Pečarić, 'Multiple Hilbert and Hardy-Hilbert inequalities with nonconjugate parameters', *Bull. Aust. Math. Soc.* **71** (2005), 447–457.
[6] M. Krnić and J. Pečarić, 'Hilbert's inequalities and their reverses', *Publ. Math. Debrecen* **67** (2005), 315–331.
[7] Y. Li and B. He, 'On inequalities of Hilbert's type', *Bull. Aust. Math. Soc.* **76** (2007), 1–13.

MARIO KRNIĆ, Faculty of Electrical Engineering and Computing, Unska 3,
10000 Zagreb, Croatia
e-mail: mario.krnic@fer.hr

https://doi.org/10.1017/S0004972711003479 Published online by Cambridge University Press