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ON k-CYCLED REFINEMENTS OF CERTAIN GRAPHS

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ABSTRACT. A graph is k-cycled if all its cycles are integral multiples of an integer $k \geq 2$. We determine the structure of refinements of K_n and $K_{n,m}$ which are k-cycled.

1. **Introduction.** All graphs considered in this paper are finite, undirected, without loops and multiple edges. Notions not defined here can be found in [1]. Let $e = uv$ be an edge of a graph G. The edge will be called *subdivided* if it is replaced by a vertex w, called a *refinement vertex* and by the edges uw and wv. A graph is called a *subdivision* of G, if it is obtained from G by a subdivision of an edge of G. A *refinement* $\hat{G}$ of G, is a graph isomorphic to a graph obtained from G by a finite sequence of subdivisions. Note that end vertices of edges of G may be refinement vertices. A graph all of whose cycles are integral multiples of an integer $k \geq 2$ will be called a *k-cycled graph.* A refinement $\hat{G}$ of G which is k-cycled, is called a *k-cycled refinement* of G. Let $\hat{G}$ be a k-cycled refinement of G. An edge of G with $l(\text{mod}\, k)$ refinement vertices in $\hat{G}$, is called an *edge of order* $l+1$ (with respect to $\hat{G}$), or simply an edge of order $l+1$. A refinement $\hat{G}$ of G, in which all edges of G are of order k is an example of a k-cycled refinement of G. A refinement $\hat{G}$ of G such that all edges of G are of order k or $k/2$ if k is even (order k if k is odd) is called a *$(k, k/2)$-refinement* of G.

For a graph G, let $V(G)$ and $E(G)$ denote the vertex and edge set of G, respectively. As usual K_n, $K_{n,m}$ and Q^n denote the complete graph on n vertices, the complete bipartite graph on n and m vertices and the n-dimensional cube, respectively. For $V \subset V(G)$, the *induced subgraph* $\langle V \rangle$, is the maximal subgraph of G with vertex set V.

In [2], the first author examined refinements of K_n with a minimal number of edges, which are subgraphs of the m-cube. Such refinements are in particular 2-cycles and naturally the problem of characterizing 2-cycled refinements of K_n arose. In this article, we determine the structure of k-cycled refinements of K_n and $K_{n,m}$. These results are used to calculate $m(G, k)$, defined as the minimal

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number of edges of a k-cycled refinement of G, for the corresponding two graphs.

2. **Main results.** First we prove the following:

LEMMA 1. *If G is a 3-edge connected graph and $\hat{G}$ is a k-cycled refinement of G, then $\hat{G}$ is a $(k, k/2)$-refinement.*

Proof. Let $e = uv \in E(G)$. From the 3-edge connectivity of G and Menger's Theorem (see [1]), it follows that there are at least two edge disjoint paths in G, between u and v, not containing e. Therefore there are two cycles in G with a single common edge e. Since $\hat{G}$ is k-cycled, let the corresponding cycles in $\hat{G}$ be of length l_1k and l_2k. If the order of e is denoted by m, then $l_1k + l_2k - 2m \equiv 0 \pmod k$, as this is a cycle of $\hat{G}$. Hence $2m \equiv 0 \pmod k$, which completes the proof of the lemma. □

In the case of 2-edge connected graphs, Lemma 1 is trivially true for $k = 2$. If however $k > 2$ and G is 2-edged connected, then k-cycled refinements of G are not necessarily $(k, k/2)$-refinements.

If k is odd, the structure of all k-cycled refinements of a 3-edge connected graph is determined by Lemma 1.

COROLLARY 1. *For k odd, $\hat{G}$ is a k-cycled refinement of a 3-edge connected graph G, if and only if all edges of G are of order k.*

We now restrict our attention to k even.

Let $K_{l,n-l}^{(l)}$ denote the following class of $(k, k/2)$-refinements of K_n. Partition $V(K_n)$ into two disjoint subsets V_1, V_2, such that $|V_1| = l$, $|V_2| = n - l$ $(0 \le l \le n)$. In the subgraph $K_{l,n-l}$, of K_n, generated by V_1 and V_2, each edge is of order $k/2$. In $\langle V_1 \rangle$ and $\langle V_2 \rangle$ each edge is of order k.

A refinement $\hat{G}$ of G will be called *k-triangular* (*k-squared*), if all triangles (squares) of G become cycles which are integral multiples of $k \ge 2$ in $\hat{G}$.

We now prove our main result for complete graphs:

THEOREM 1. *For a refinement $\hat{K}_n$ of K_n, $(n \ge 4)$ and even k the following assertions are equivalent.*

(1) *$\hat{K}_n$ is k-triangular and it is a $(k, k/2)$-refinement of K_n.*

(2) *$\hat{K}_n$ is isomorphic to $K_{l,n-l}^{(k)}$ for some $0 \le l \le n$.*

(3) *$\hat{K}_n$ is k-cycled.*

Proof. Let l be the maximal integer such that there exists a set $V_1 \subseteq V(K_n)$, $|V_1| = l$ and all edges of $\langle V_1 \rangle$ are of order k. Obviously $l \ge 2$ since $\hat{K}_n$ must contain edges of order k.

Let $V_2 = V(K_n) - V_1$. If $V_2 = \phi$, $\hat{K}_n$ is isomorphic to $K_{0,n}^{(k)}$. Assume therefore $v \in V_2$ and let $vw \in E(K_n)$ where $w \in V_1$. If vw is of order k then by (1) any

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edge incident with v and a vertex of V_1 must be of order k, contradicting the maximality of l. Therefore all edges from V_1 to V_2 are of order $k/2$. But then due to (1) all edges of $\langle V_2\rangle$ (if $|V_2|\geq 2$), must be of order k, proving (2) from (1). To show (3) from (2), note that any cycle in $\hat{K}_n$ all whose vertices are in V_i ($i=1,2$) is a multiple of k. Cycles containing vertices of V_1 and V_2 contain an even number of edges of K_n of order $k/2$, proving (3).

If we assume (3) $\hat{K}_n$ is in particular k-triangular and by Lemma 1 it is a $(k, k/2)$-refinement of K_n, proving (1). □

Let k be even and $K_{n,m}$ a complete bipartite graph with vertex sets N and M. A $(k, k/2)$-refinement of $K_{n,m}$ is called *proper* if there exist partitions $N=N_1\cup N_2$, $M=M_1\cup M_2$, $N_1\cap N_2=\phi$, $M_1\cap M_2=\phi$, such that

(1) edges joining a vertex in N_i to a vertex in M_i ($i=1,2$) are of order k and

(2) edges joining a vertex in M_1 (N_1) to a vertex in N_2 (M_2) are of order $k/2$.

The following is an equivalent definition.

For a fixed $v\in M$, any edge vw ($w\in N$) is either of order k or of order $k/2$.

Suppose that $x\in M$ ($x\neq v$) then, either

(1′) for each $y\in N$ the edge xy is of order $k/2$ if and only if vy is of order $k/2$; or

(2′) for each $y\in N$ the edge xy is of order $k/2$ if and only if vy is of order k.

THEOREM 2. *For a refinement $\hat{K}_{n,m}$ of $K_{n,m}$ ($m, n\geq 3$) and k even, the following assertions are equivalent:*

(1) *$\hat{K}_{n,m}$ is k-squared and it is a $(k, k/2)$-refinement.*

(2) *$\hat{K}_{n,m}$ is a proper $(k, k/2)$-refinement.*

(3) *$\hat{K}_{n,m}$ is k-cycled.*

Proof. First we show (1)⇒(2). Suppose $v, x\in M$, $w\in N$ and that the order of vw is equal to the order of wx. Then by (1), the order of vy is equal to the order of xy for any $y\in N$. If the order of vw is different from the order of wx, the orders of vy and xy must be different for any $y\in N$. Clearly a proper $(k, k/2)$-refinement is k-squared. It is easy to show by induction that a k-squared, $(k, k/2)$-refinement must be k-cycled. Thus (2)⇒(3).

If $\hat{K}_{n,m}$ is k-cycled then in particular it is k-squared and by Lemma 1 it is also a $(k, k/2)$-refinement, which shows (3)⇒(1), completing the proof of the theorem. □

The following is obvious for $K_{2,m}$ ($m\geq 3$). If $\{x, y\}$ is the set of two vertices, then either the sum of orders of xv and yv is $o(\text{mod } k)$ for any $v\in M$, or the sum of orders of xv and yv is $k/2\,(\text{mod } k)$ for any $v\in M$.

3. **K-Cycled refinements with minimal number of edges.** Define $m(G,k)=\text{Min}|E(\hat{G})|$, where $\hat{G}$ is a k-cycled refinement of G. To compute $m(K_n, k)$ when k is even and $n>3$, consider refinements of type $K^{(k)}_{l,n-l}$, where an edge of

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K_n contains exactly $k-1$ or $k/2-1$ refinement vertices. Clearly,

$$(1) \qquad m(K_n, k) \le \binom{l}{2}k + \binom{n-l}{2}k + l(n-l)\frac{k}{2}, \qquad 0 \le l \le \left[\frac{n}{2}\right].$$

Using Corollary 1 and the fact that the right hand side of (1) attains its minimum when $l=[n/2]$ we have

COROLLARY 2. *For any* $n>3$

$$m(K_n, k) = \begin{cases} \dfrac{n(n-1)}{2}k, & k \text{ odd} \\ \dfrac{n(n-1)}{2}k - \left[\dfrac{n^2}{4}\right]\dfrac{k}{2}, & k \text{ even.} \end{cases}$$

Furthermore, there is a unique (up to isomorphism) k-cycled refinement of K_n with $m(K_n, k)$ edges.

We shall call a refinement of K_n *triangle free* if each triangle in K_n contains at least one refinement vertex and denote by $t(n)$ the minimal number of edges of a triangle free refinement of K_n.

For $k=2$, Corollary 2 can be improved.

COROLLARY 3. *For* $n>3$, $t(n)=m(K_n, 2)$.

Furthermore, there is a unique (up to isomorphism) triangle free refinement of K_n with $t(n)$ edges.

Proof. Let $\hat{K}_n$ be a triangle free refinement of K_n with a minimal number of edges. We may assume that an edge of K_n contains at most a single refinement vertex. Remove from $\hat{K}_n$ all subdivided edges. The graph T obtained is triangle free but for any edge e, $T+e$ contains a triangle. By Turan's theorem ([1], [3]), $T=K_{[n/2],[n+1/2]}$. Hence $|E(\hat{K}_n)|=m(K_n, 2)$ and the unique minimal triangle free refinement of K_n is in class $K^{(2)}_{[n/2],[n+1/2]}$, where in the refinement, there is at most one refinement vertex on an edge. $\square$

Since in a bipartite graph all cycles are even, we obtain from Lemma 1:

COROLLARY 4. *For any* 3-*edge connected bipartite graph* B *and* k *even,*

$$m(B, k) = \frac{k}{2} \cdot |E(B)|.$$

Moreover, if $\hat{B}$ is a refinement of B with a minimal number of edges, then each edge of B contains $k/2-1$ refinement vertices.

By Corollary 1, the following result is obtained.

COROLLARY 5. *For any* 3-*edge connected graph* G *and* k *odd,*

$$m(G, k) = k\,|E(G)|.$$

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Moreover, if $\hat{G}$ is a refinement of G with a minimal number of edges, then each edge of G contains $k-1$ refinement vertices.

4. **An application.** In [2] the following result is proved

THEOREM 3. *any refinement of K_{n+1} which is a subgraph of Q^m $(m \geq n)$ has at least n^2 edges. Moreover, if the refinement has exactly n^2 edges it is unique (up to isomorphism).*

Theorem 1 is now applied to obtain a shorter and different proof of Theorem 3.

Clearly a refinement of K_{n+1} which is a subgraph of Q^m with a minimal number of edges must be a 2-cycled refinement with at most one refinement vertex on each edge of K_{n+1}. Since $K_{2,3}$ is not a subgraph of Q^m, the only two 2-cycled refinements of K_{n+1} which are subgraph of Q^m are in $K^{(2)}_{0,n+1}$ or $K^{(2)}_{1,n}$. The latter has fewer edges (n^2) and is the only desired refinement of K_{n+1}.

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