THE FUNDAMENTAL GROUPOID AS A TOPOLOGICAL GROUPOID

by R. BROWN and G. DANESH-NARUIE

Introduction

Let X be a topological space. Then we may define the fundamental groupoid πX and also the quotient groupoid $(\pi X)/N$ for N any wide, totally disconnected, normal subgroupoid N of πX (1). The purpose of this note is to show that if X is locally path-connected and semi-locally 1-connected, then the topology of X determines a "lifted topology" on $(\pi X)/N$ so that it becomes a topological groupoid over X. With this topology the subspace $\mathrm{St}_{(\pi X)/N}x$ which is the fibre of the initial point map $\partial' : (\pi X)/N \to X$ over x in X, is the usual covering space $\tilde{X}_{N,x}$ of X determined by the normal subgroup $N\{x\}$ of the fundamental group $\pi(X, x)$.

One advantage of considering $(\pi X)/N$ rather than the collection of spaces $\tilde{X}_{N,x}$, $x \in X$, is that, in a suitable sense, $(\pi X)/N$ is functorial. For example, if G is a topological group acting on X, and N is G-invariant, then G acts also on $(\pi X)/N$. This gives us in Section 3 a formulation of the theory of the fundamental group of a transformation group due to F. Rhodes (8, 9) simply as an object group of the semi-direct product groupoid $G \ltimes (\pi X)/N$.

In Section 2 the topology on $(\pi X)/N$ is related to Ehresmann's work on locally trivial groupoids (5).

A final algebraic section shows that any extension of a group G arises as the exact sequence of the projection $G \ltimes A \to G$ of a semi-direct product of G with some groupoid A.

The genesis of this paper should be described. The main ideas of producing a lifted topology on $(\pi X)/N$, of describing its fundamental group, and of exploring the relationship with the work of (5, 8, 9) are to be found in (4). The present formulation of these results was worked out while the second author visited Bangor in 1972.†

1. The lifted topology on $(\pi X)/N$

A *topological groupoid* Γ *over* X is a groupoid such that $\mathrm{Ob}\,(\Gamma) = X$, and with topologies on both X and on the set Γ of arrows such that all the structure functions are continuous; these structure functions are: the initial and final maps $\partial', \partial : \Gamma \to X$; the unit map $u : X \to \Gamma$ which sends $x \mapsto 1_x$; the composition $\theta : \Gamma \ltimes \Gamma \to \Gamma$, $(a, b) \mapsto ba$, whose domain is the set of (a, b) such that $\partial a = \partial' b$; and the inverse map $\Gamma \to \Gamma$, $a \mapsto a^{-1}$. Thus a topological groupoid is a natural generalisation of a topological group.

The topological groupoid Γ is *locally trivial* (5) if each x in Ob (Γ) has a neighbourhood U such that there is a continuous function $\lambda : U \to \Gamma$ such that $\lambda(y) \in \Gamma(x, y)$ for all $y \in U$.

Now let X be a topological space and N a totally disconnected, wide, normal subgroupoid of πX, the fundamental groupoid of X. Thus N assigns to each x in X a subgroup $N\{x\}$ of $\pi(X, x)$ so that if $a \in \pi X(x, y)$, then $aN\{x\}a^{-1} = N\{y\}$. The quotient groupoid $\pi/N = (\pi X)/N$ can then be defined to have object set X and arrows $x \to y$ the cosets $\alpha N\{x\}$ for $\alpha \in \pi X(x, y)$. In particular the object group of π/N at x is the quotient group $\pi(X, x)/N\{x\}$.

Let $p : \pi X \to \pi/N$ be the projection.

Theorem 1. *Let X be locally-path connected and semi-locally 1-connected.*

(a) *There is a topology on π/N so that it becomes a locally trivial topological groupoid over X with (topologically) discrete object groups.*

(b) *For each x in X, the subspace $\mathrm{St}_{\pi/N}x$ of π/N is the regular covering space of X based at x and determined by the subgroup $N\{x\}$ of $\pi(X, x)$.*

(c) *The fundamental group of π/N at 1_x is isomorphic to the subgroup of $\pi(X, x) \times \pi(X, x)$ of pairs (a, b) such that $aN\{x\} = bN\{x\}$.*

For the proof of Theorem 1, we consider the groupoid Q such that Ob (Q) consists of the arrows of π/N, and if $\alpha, \beta \in \pi/N$ then $Q(\alpha, \beta)$ consists of triples (a, α, b) such that a, b are arrows of πX and $p(b)\alpha p(a^{-1})$ is defined and equal to β. Composition in Q is given by $(a', \alpha', b')(a, \alpha, b) = (a'a, \alpha, b'b)$. (Thus Q is isomorphic to the comma category $(p \downarrow p)$, (7) p. 47.) There is a projection $q : Q \to \pi X \times \pi X$ given by $(a, \alpha, b) \mapsto (a, b)$; it is easy to check that q is a covering morphism of groupoids. But $\pi X \times \pi X = \pi(X \times X)$ ((1) p. 189). By (1) p. 309 the topology of $X \times X$ lifts uniquely to a topology on Ob $(Q) = \pi/N$ such that Ob $(q) : \pi/N \to X \times X$ is a covering map and there is an isomorphism $r : \pi(\pi/N) \to Q$ making the following diagram commutative

$$\pi(\pi/N) \xrightarrow{\ r\ } Q$$
$$\searrow \qquad \swarrow q$$
$$\pi(X \times X)$$

It follows that $\pi(\pi/N, 1_x)$ is isomorphic to the object group $Q\{1_x\}$, which consists of triples $(a, 1_x, b)$ such that $p(a)1_x p(b^{-1})$ is defined and equal to 1_x. This verifies (c) of Theorem 1.

In order to prove (a) and (b) of Theorem 1, we need to describe in detail the lifted topology on Q.

Let $\mathscr{U}$ be the open cover of X consisting of all open, path-connected subsets U of X such that the inclusion $i : U \to X$ maps each fundamental group of U to the trivial group. The elements of U will be called *canonical neighbourhoods*. For each U in $\mathscr{U}$ and x in U, define $\lambda_x : U \to \pi X$ by choosing for each x' in U a path in U from x to x' and letting $\lambda_x(x')$ be the class in $\pi X(x, x')$ of this path—

the conditions on U imply that $\lambda_x(x')$ is independent of the choice of path in U from x to x'. Let $\mu_x = p\lambda_x : U \rightarrow \pi/N$, and let $\tilde{U}_x = \mu_x(U)$.

Lemma 2. *Let $\alpha \in \pi/N(x, y)$. Then the sets $\tilde{V}_y \alpha \tilde{U}_x^{-1}$ for all U, V in $\mathcal{U}$ such that $x \in U$, $y \in V$, form a set of basic neighbourhoods for the lifted topology on π/N.*

The proof follows easily from the description of the lifted topology in **(1)** pp. 307-8.

We can now prove that π/N is a topological groupoid. In fact, the only non-trivial part of the proof is continuity of composition $\theta : \pi/N \tilde{\times} \pi/N \rightarrow \pi/N$. Let $\theta(\beta, \alpha) = \beta\alpha$, where $\alpha \in \pi/N(x, y)$, $\beta \in \pi/N(y, z)$. Let $\tilde{W}_z \beta\alpha \tilde{U}_x^{-1}$ be a basic neighbourhood of $\beta\alpha$. Then for any V in $\mathcal{U}$ such that $y \in V$,

$$\theta(\tilde{W}_z \beta \tilde{V}_y^{-1} \tilde{\times} \tilde{V}_y \alpha \tilde{U}_x^{-1}) = \tilde{W}_z \beta\alpha \tilde{U}_x^{-1}$$

and so θ is continuous.

The mappings μ_x give the local trivialisations of π/N.

If $\alpha \in \pi/N\{x\}$, then the intersection of $\pi/N\{x\}$ with a basic neighbourhood of α is $\{\alpha\}$—so $\pi/N\{x\}$ has the discrete topology.

If $\alpha \in \pi/N(x, y)$, then a basic neighbourhood of α in $\mathrm{St}_{\pi/N}x$ is $\tilde{U}_y \alpha$, and these are the usual basic neighbourhoods in the theory of covering spaces. This completes the proof of Theorem 1.

The topological groupoid $(\pi X)/N$ is functorial in the following sense.

Let $f : Y \rightarrow X$ be a map of locally path-connected and semi-locally 1-connected spaces. Let M, N be wide, totally disconnected, normal subgroupoids of πY, πX respectively such that $(\pi f)(M) \subseteq N$.

Proposition 3. *The morphism $g : (\pi Y)/M \rightarrow (\pi X)/N$ induced by πf is continuous.*

Proof. Let $\alpha : y \rightarrow y'$ be an arrow of $(\pi Y)/M$ and let $\beta = g(\alpha) : x \rightarrow x'$. Let U, U' be canonical neighbourhoods of x, x' respectively. Then there are canonical neighbourhoods V, V' of y, y' respectively such that $f(V) \subseteq U$, $f(V') \subseteq U'$. Clearly

$$g(\tilde{V}'_{y'} \alpha \tilde{V}_y^{-1}) \subseteq \tilde{U}'_{x'} \beta \tilde{U}_x^{-1}$$

and so g is continuous.

We now consider *topological covering morphisms* $p : G \rightarrow H$ of topological groupoids. A definition of this given in **(4)**, that each induced map $\mathrm{St}_G x \rightarrow \mathrm{St}_H p(x)$ is a homeomorphism, turns out not to be strong enough; the definition which follows is due to R. Brown and J. P. L. Hardy **(6)**, **(10)**.

First of all let $q : G \rightarrow H$ be a morphism of topological groupoids and consider the pull-back diagram

$$
\begin{array}{ccc}
H \tilde{\times} \mathrm{Ob}(G) & \longrightarrow & \mathrm{Ob}(G) \\
\downarrow & & \downarrow q \\
H & \xrightarrow{\ \partial'\ } & \mathrm{Ob}(H)
\end{array}
$$

Then $q: G \to H$ and $\partial' : G \to \mathrm{Ob}\,(G)$ determine a continuous function

$$\tilde{q} : G \to H \,\widetilde{\times}\, \mathrm{Ob}\,(G).$$

Definition. The morphism $q: G \to H$ is a *topological covering morphism* if $\tilde{q} : G \to H \,\widetilde{\times}\, \mathrm{Ob}\,(G)$ is a homeomorphism.

Let X, Y, M, N satisfy the conditions of Proposition 3.

Theorem 4. *If $f: Y \to X$ is a covering map and $(\pi f)\,(M) = N$ then the morphism $g : (\pi Y)/M \to (\pi X)/N$ induced by f is a topological covering morphism of topological groupoids.*

Proof. By 9.2.1 of (1), $\pi f: \pi Y \to \pi X$ is (abstractly) a covering morphism, and it is easy to prove that so also is $g : (\pi Y)/M \to (\pi X)/N$. Certainly

$$(g, \partial') : (\pi Y)/M \to (\pi X)/N \,\widetilde{\times}\, Y$$

is continuous and bijective. We prove (g, ∂') is an open mapping.

Let $\alpha : y \to y'$ be an arrow of $(\pi Y)/M$. Then we can choose canonical neighbourhoods V, V' of y, y' respectively such that $U = f(V)$, $U' = f(V')$ are canonical neighbourhoods of $x = f(y)$, $x' = f(y')$ respectively. If

$$W = \tilde{V}_y \alpha \tilde{V}_{y'}'^{-1},$$

then $g(W)$ is a basic neighbourhood of $g(\alpha)$, while $(g, \partial')(W) = g(W) \,\widetilde{\times}\, V$, which is open in $(\pi X)/N \,\widetilde{\times}\, Y$. This completes the proof.

We now examine products. Let X, Y be locally path-connected and semi-locally 1-connected. Let N, M be totally disconnected, wide, normal sub-groupoids of πX, πY respectively.

Proposition 5. *The lifted topology on $\pi(X \times Y)/N \times M$ makes it isomorphic as topological groupoid to $(\pi X)/N \times (\pi Y)/M$.*

Proof. This follows easily from an examination of basic neighbourhoods, those in $\pi(X \times Y)/N \times M$ being of the form $(\tilde{U}_{x'} \times \tilde{V}_{y'})(\alpha, \beta)(\tilde{U}_x^{-1} \times \tilde{V}_y^{-1})$ for $\alpha : x \to x'$ in $(\pi X)/N$, $\beta : y \to y'$ in $(\pi Y)/M$.

2. The Ehresmann theory of locally trivial groupoids

Throughout this section, G will be a locally trivial, transitive†, topological groupoid over X. The study of such groupoids was initiated by Ehresmann, who has proved (5):

Theorem 5. (i) *The initial point map $\partial' : G \to X$ is the projection of a fibre bundle with fibre $\mathrm{St}_G x$ over x and group $G\{x\}$.*

(ii) *If $x \in X$, the final point map $\partial : \mathrm{St}_G x \to X$ is the projection of a principal fibre bundle with group $G\{x\}$.*

It is also true that $(\partial', \partial) : G \to X \times X$ is a fibre bundle with fibre $G\{x\}$ and group $G\{x\} \times G\{x\}$ (5), but this result will not concern us here.

† It is convenient, for topological groupoids to replace the term "connected groupoid" of (1) by "transitive groupoid".

As a consequence of Theorem 5 (i) we have a transgression function $\Delta : \pi(X, x) \to G\{x\}$.

Corollary 6. *Let X be path-connected and $x \in X$. The following conditions are equivalent.*

(i) $\operatorname{St}_G x$ *is path-connected.*
(ii) $\Delta : \pi(X, x) \to G\{x\}$ *is surjective.*
(iii) $\partial'_* : \pi(G, 1_x) \to \pi(X, x)$ *is surjective.*

Proof. These follow immediately from the exact homotopy sequences of the fibrations described in Theorem 5.

There is an inverse to the operator taking G to the principal bundle $\operatorname{St}_G x$. Let A be a topological group, and $p : E \to X$ a principle A-bundle. Then the groupoid $\mathscr{G}(E)$ of admissible maps of E is defined—if $x, y \in X$, then $\mathscr{G}(E)(x, y)$ is the set of admissible maps $E_x \to E_y$. The local triviality of $p : E \to X$ enables one to define a topology on $\mathscr{G}(E)$ so that it becomes a locally trivial, transitive topological groupoid over X.

The further results of Ehresmann are:

Theorem 7. *If $p : E \to X$ is a principal A-bundle, and $x \in X$, then there is an isomorphism of A-bundles, $E \cong \operatorname{St}_{\mathscr{G}(E)} x$.*

Theorem 8. *If $\operatorname{St}_G x$ is given its structure as a principal $G\{x\}$-bundle, then there is an isomorphism of topological groupoids $G \cong \mathscr{G}(\operatorname{St}_G x)$.*

We can now state a classification theorem for certain topological groupoids over a path-connected space X.

Theorem 9. *Suppose G is transitive, locally trivial, and has topologically discrete vertex groups, that X is path-connected, and $x \in X$. Then the following conditions are equivalent:*

(i) $\operatorname{St}_G x$ *is path connected,*
(ii) G *is isomorphic to $(\pi X)/N$ for some wide, normal totally disconnected subgroupoid N of πX.*

Proof. That (ii)$\Rightarrow$(i) is clear, since $\operatorname{St}_G x$ is then a connected covering space of X; so we prove (i)$\Rightarrow$(ii).

The given conditions imply that $\operatorname{St}_G x$ is a path-connected regular covering space over X, corresponding to a normal subgroup $N\{x\}$ of $\pi(X, x)$. Since X is path-connected, we can define $N\{y\}$ for any $y \in X$ by $N\{y\} = \alpha N\{x\}\alpha^{-1}$ for any $\alpha \in \pi X(x, y)$. Then $N\{y\}$ is independent of choice of α, by normality, and the collection $N\{y\}$, $y \in X$, is a wide, normal, totally disconnected normal subgroupoid of πX.

So we have isomorphisms of principal $G\{x\}/N\{x\}$ bundles over X

$$\operatorname{St}_G x \cong \tilde{X}_{N, x} \cong \operatorname{St}_{(\pi X)/N} x.$$

Hence $G \cong \mathscr{G}(\operatorname{St}_G x) \cong \mathscr{G}(\operatorname{St}_{(\pi X)/N} x) \cong (\pi X)/N$. This completes the proof.

The isomorphism of groupoids $(\pi X)/N = \mathscr{G}(\tilde{X}_{N,x})$ and Ehresmann's topology on the latter groupoid gives an alternative method of topologising the groupoid $(\pi X)/N$. The disadvantages of this method are the dependence on the base point, and that further work is needed to describe the fundamental group of the space $(\pi X)/N$.

3. Transformation groups

Let G be a topological transformation group of the space X, and assume X has the local conditions of Theorem 1. Let π/N be as in Section 1, and assume further that N is invariant under G in that $gN\{x\} = N\{^gx\}$ for all $g \in G$, $x \in X$. Then G acts also on the groupoid π/N and it is easy to see that if π/N has the lifted topology then this action is continuous.

We now form the groupoid $G \bar{\times} \pi/N$ considered in (3). (It is preferable to call this the *semi-direct product* rather than split-extension as in (3).) The object set of $G \bar{\times} \pi/N$ is X, the elements in $G \bar{\times} \pi/N$ from x to y are pairs (g, a) with $g \in G$, $a \in \pi/N(^gx, y)$, and composition is

$$(h, b)(g, a) = (hg, b\,^ga).$$

Let X have its given topology, and let $G \bar{\times} \pi/N$ have its topology as a subset of the product. Then it is easy to check that $G \bar{\times} \pi/N$ is a topological groupoid.

The object groups of $G \bar{\times} \pi/N$ have been considered by F. Rhodes (9). He defined a group $\sigma_\rho(X, x_0, G)$ for the case ρ is a G-invariant normal subgroup of $\pi(X, x_0)$, the elements of this group being equivalence classes $[f;\ g]_\rho$ of pairs $(f;\ g)$ such that $g \in G$, f is a path from x_0 to $^gx_0$, and two pairs $(f;\ g), (f';\ g')$ are equivalent if and only if $g = g'$ and the class $[f' - f] \in \rho$.

Proposition 10. *If $x_0 \in X$ and $\rho = N\{x_0\}$, the map*

$$(G \bar{\times} \pi/N)\{x_0\} \rightarrow \sigma_\rho(X, x_0, G)$$

given by $(g, [f]) \mapsto [-f;\ g]$, *is an isomorphism.*

The proof is obvious. If $\rho = 1$, then $\sigma_\rho(X, x_0, G)$ is written $\sigma(X, x_0, G)$.

The exact sequences given in (9) p. 906 can be derived as follows. The projection $G \bar{\times} \pi/N \rightarrow G$ is a fibration of groupoids (Proposition 2.4 of (3)). By Theorem 4.3 of (2), and Proposition 4 there is an exact sequence

$$1 \rightarrow \pi(X, x_0)/\rho \rightarrow \sigma_\rho(X, x_0, G) \rightarrow G \rightarrow \pi_0 X \rightarrow \pi_0(G \bar{\times} \pi/N) \rightarrow 1.$$

Further the quotient mappings $\pi X \rightarrow \pi/N$ is a fibration of groupoids and so determines a fibration $G \bar{\times} \pi X \rightarrow G \bar{\times} \pi/N$ (Proposition 2.7 of (3)). The exact sequence of this fibration at x_0 is

$$1 \rightarrow \rho \rightarrow \sigma(X, x_0, G) \rightarrow \sigma_\rho(X, x_0, G) \rightarrow 1.$$

Results about actions of groups on covering spaces also fall within the present framework. The topological groupoid $G \bar{\times} \pi/N$ acts on the space π/N via $\partial : \pi/N \rightarrow X$ by the rule

$$(g, b).a = b\,^ga.$$

In particular, if $x_0 \in X$, then the group $\sigma_\rho(X, x_0, G) = (G \barwedge \pi/N)\{x_0\}$ acts on $\partial^{-1}(x_0) = \mathrm{Cost}_{\pi/N}x_0$. But $\mathrm{Cost}_{\pi/N}x_0$ is, like $\mathrm{St}_{\pi/N}x_0$, the covering space of X determined by the subgroup $\rho = N\{x_0\}$ of $\pi(X, x_0)$. More generally, if N' is another normal, G-invariant subgroupoid of πX such that $N' \subseteq N$, then $G \barwedge \pi/N'$ acts on π/N by the above rule, and so $\sigma_\rho(X, x_0, G)$ acts on the covering space $\mathrm{Cost}_{\pi/N}x_0$ of X—this is Proposition 2 of (9).

4. An algebraic remark

Let G be a group and A a transitive groupoid which is a G-module. If $x \in \mathrm{Ob}(A)$ and $B = A\{x\}$ then the fibration $G \barwedge A \to G$ determines an exact sequence of groups

$$1 \to B \to (G \barwedge A)\{x\} \to G \to 1, \tag{4.1}$$

i.e. an extension of G by B.

Conversely, suppose given an extension

$$1 \to B \to E \overset{p}{\to} G \to 1 \tag{4.2}$$

we show that this extension is isomorphic to (4.1) for some groupoid A—thus *every extension of groups derives from a semi-direct product of groupoids.*

For the proof, we take A to be the covering groupoid $A = E \barwedge G$ determined by the action of E on G on the right. Thus $\mathrm{Ob}(A) = G$, and

$$A(g, g') = \{(e, g) \in E \times G : g = g'e\}$$

with composition $(e', g')(e, g) = (e'e, g)$. An action of G on A is then defined by left multiplication of objects, and on arrows by $h.(e, g) = (e, hg)$.

Now $(G \barwedge A)\{1\}$ consists of triples (h, e, g) such that $(e, g) \in A(h, 1)$, whence $h = g = 1.e$. So we have an isomorphism $(G \barwedge A)\{1\} \to E$, $(p(e), e, p(e)) \mapsto e$, which maps the subgroup $A\{1\}$ isomorphically to B.

REFERENCES

(1) R. Brown, *Elements of Modern Topology* (McGraw-Hill, Maidenhead, 1968).

(2) R. Brown, Fibrations of groupoids, *J. Algebra* **15** (1970), 103-32.

(3) R. Brown, Groupoids as coefficients, *Proc. London Math. Soc.* (3) **25** (1972), 413-426.

(4) G. Danesh-Naruie, *Topological groupoids*, Ph.D. Thesis, Southampton University (1970).

(5) Ch. Ehresmann, Catégories topologiques et catégories différentiables, *Coll. Géom. Diff. Glob. Bruxelles* (1959).

(6) J. P. L. Hardy, *Topological groupoids*, M.A. Dissertation, University of Wales (1972).

(7) S. MacLane, *Categories for the Working Mathematician* (Springer-Verlag, Berlin, 1971).

(8) F. RHODES, On the fundamental group of a transformation group, *Proc. London Math. Soc.* (3) **16** (1966), 635-650.

(9) F. RHODES, On lifting transformation groups, *Proc. Amer. Math. Soc.* **19** (1968), 905-908.

(10) R. BROWN, G. DANESH-NARUIE, J. P. L. HARDY, Topological groupoids II: covering morphisms and G-spaces, *Math. Nachr.* (to appear).

SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY COLLEGE OF NORTH WALES
BANGOR, GWYNEDD.
U.K.
AND
DANESHSARAYE-ALI
ROOZVELT AVE.
TEHRAN
IRAN