A NOTE ON GENERALISED WREATH PRODUCT GROUPS

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(Received 3 January 1984)

Communicated by D. E. Taylor

Abstract

Generalised wreath products of permutation groups were discussed in a paper by Bailey and us. This note determines the orbits of the action of a generalised wreath product group on \(m\)-tuples \((m \geq 2)\) of elements of the product of the base sets on the assumption that the action on each component is \(m\)-transitive. Certain related results are also provided.


1. Introduction

In an earlier paper with R. A. Bailey [3] we discussed a number of properties of the generalised wreath product group (over a poset \((I, \rho)\), denoted by \((G, \Delta) = \prod_{(I, \rho)}(G_i, \Delta_i)\), and, in particular, determined the orbits of the action of \(G\) on \(\Delta \times \Delta\). These orbits take a particularly simple form if \((G_i, \Delta_i)\) is 2-transitive for each \(i \in I\). One of the purposes of this note is to derive the corresponding result for \(G\) acting on \(\Delta^m\) for \(m \geq 2\), under the assumption that \((G_i, \Delta_i)\) is \(m\)-transitive for each \(i \in I\). We go on to discuss the action of certain subgroups of \(G\) on certain subsets of the orbits so determined.

The results of this note are required for a discussion of cumulants and \(k\)-statistics, of order higher than 2, of families of random variables labelled by the index sets which arise in complicated analyses of variance.
2. Preliminaries

The notation and terminology of Bailey et al. [3] will be used without comment. The poset \((I, \leq)\) is assumed finite throughout this note. For any natural number \(m\), we write \(m = \{1, \ldots, m\}\) and, if \(h: m \rightarrow S\) is any map defined on \(m\), we write \(\ker h\) for the partition of \(m\) induced by \(h\), i.e. \(x\) and \(y\) in \(m\) are in the same block of \(\ker h\) if and only if \(xh = yh\). The lattice of all partitions of \(m\) is denoted by \(\mathcal{P}(m)\); see Aigner [1] for many properties of these lattices.

We write \(\text{Hom}(I, \mathcal{P}(m))\) for the set of all monotone maps \(\phi: I \rightarrow \mathcal{P}(m)\); this is a lattice under the pointwise operations. Now, any map \(h: m \rightarrow \Delta\) defines an element \(\phi^h \in \text{Hom}(I, \mathcal{P}(m))\) by the formula \(\phi^h(i) = \bigwedge_{j \geq i} \ker h_j\), where \(h_j = h \upharpoonright J\). Note that

(a) for all \(x, y \in m\), we have that \(x\) and \(y\) are in the same block of \(\phi^h(i)\) if and only if \(xh \sim \bigwedge_{A(i)} yh\),
(b) \(\phi^h(i) = \ker h_{\upharpoonright I} \land \ker h_i\),
(c) we have \(\phi^h = \phi^k\) if and only if \(\bigwedge_{j \in J} \ker h_j = \bigwedge_{j \in J} \ker k_j\) for all ancestral sets \(J\).

For \(\phi \in \text{Hom}(I, \mathcal{P}(m))\), we write \(\mathcal{O}_{\phi} = \{ h \in \Delta^m: \phi^h = \phi \}\).

3. The main result

Our main result is the following.

**THEOREM.** If \((G_i, \Delta_i)\) is \(m\)-transitive for each \(i \in I\), then \(\{ \mathcal{O}_{\phi}: \phi \in \text{Hom}(I, \mathcal{P}(m))\}\) is exactly the set of orbits of the generalized wreath product group \(G\) acting on \(\Delta^m\).

The proof is contained in the following lemmas.

**LEMMA 1.** \(\mathcal{O}_{\phi}\) is \(G\)-invariant.

**PROOF.** For each \(i \in I\) and \(h \in \Delta^m\), Theorem B of [2] shows that, if \(x, y \in m\),

\[xh \sim yh \quad \text{if and only if} \quad xhf \sim yhf.\]

Thus, by note (a) above, \(\phi^h = \phi^{hf}\).

**LEMMA 2.** If \((G_i, \Delta_i)\) is \(m\)-transitive for each \(i \in I\), then \(G\) acts transitively on \(\mathcal{O}_{\phi}\).
PROOF. Fix \( i \in I \) and \( h, k \in \mathcal{O}_\phi \), and suppose that \( \ker h\pi^i \) has blocks \( B_1, \ldots, B_r \). Then, for all \( r \leq s \) and \( x, y \in B_r \), we have \( xh\pi^i = yh\pi^i \) if and only if \( xk\pi^i = yk\pi^i \) and, consequently, \( xh\pi^i = yh\pi^i \) if and only if \( xk\pi^i = yk\pi^i \). Since each \( |B_r| \leq m \), our assumptions imply that, for all \( r \leq s \), there exists \( g_r \in G_i \) such that, for all \( x \in B_r \), we have \((xh)g_r = (xk)g_r \). Also, by the definition of \( \ker h\pi^i \), there is a map \( f_i: \Delta^i \to G_i \) such that, for all \( r \leq s \) and \( x \in B_r \), we have \((xh\pi^i)f_i = g_r \).

Carrying out this process for each \( i \in I \) produces an element \( f = (f_i) \in G \) such that \( h^f = k \).

The proof of Lemma 2 shows more, namely that if, for each \( i \in I \), we have \( G_i \) being \( m_i \)-transitive with \( m_i \geq \sup \{|B|: B \text{ is a block of } \phi(i)\} \), then \( G \) is transitive on \( \mathcal{O}_\phi \).

These two lemmas show that, when all the \( (G_i, \Delta_i) \) are \( m \)-transitive, the orbits of \( G \) on \( \Delta^m \) are labelled by the elements of \( \text{Hom}(I, \mathcal{P}(m)) \) (a result which is well known when \( |I| = 1 \), as follows: the \( |O_\phi| \) are disjoint, and each is non-empty since, for \( \phi \in \text{Hom}(I, \mathcal{P}(m)) \), we can define an \( h: m \to \Delta \) such that \( \phi^h = \phi \) by arbitrarily choosing its component maps \( h_i: m \to \Delta_i \), subject only to \( \ker h_i = \phi(i) \) for each \( i \in I \).

The following reformulation of the definition of \( \phi^h \) is of some interest.

**Lemma 3.** \( \phi^h = \bigvee \{ \phi \in \text{Hom}(I, \mathcal{P}(m)): (\forall i \in I)(\phi(i) \leq \ker h_i) \} \).

**Proof.** Denote the right-hand side of the above expression by \( \psi^h \). If \( i \leq j \), then \( \psi^h(i) \leq \psi^h(j) \leq \ker h_j \) and thus, if \( x \) and \( y \) belong to the same block of \( \psi^h(i) \), then \( xh_j = yh_j \) for all \( j > i \). But this means that \( xh\pi^i = yh\pi^i \) and so \( \psi^h \leq \phi^h \). On the other hand, \( \phi^h \leq \psi^h \) by definition, and so \( \phi^h = \psi^h \).

**Remark.** When \( m = 2 \), the lattice \( \mathcal{P}(m) \) is just the 2-element chain and in this case \( \text{Hom}(I, \mathcal{P}(m)) \) is isomorphic to the distributive lattice of all ancestral sets (i.e. dual ideals or filters) of \( I \). Thus these conclusions are consistent with Theorem C of Bailey et al. [2].

As an illustration of our conclusion for \( m > 2 \), we depict in Figure 1 the lattice \( \text{Hom}(I, \mathcal{P}(m)) \) where \( I \) is the poset \( \{1, 2: 2 \leq 1\} \) and \( m = 3 \). This lattice labels the orbits of \( S_n \text{wr} S_t \) acting on triples of elements from \( n \times t \).

In Speed and Bailey [5] it was shown that the poset \( (I, \rho) \) defines an association scheme on \( \Delta \). Theorem C of Bailey et al. [3] shows that the associate classes coincide with the orbits of \( G \) if each \( G_i \) is 2-transitive. The above proof gives the following stronger result: if each \( G_i \) is 2-transitive then the poset-defined association scheme is \( t \)-transitive for all \( t \), in the sense of Cameron [4, p. 103], and hence \( t \)-regular for all \( t \), in the sense of Babai [2, p. 2].
4. Related results

For certain results in statistics, which will be published elsewhere, it is necessary to have information concerning the actions of some subgroups of $G$ on certain subsets of $\mathbb{R}^n$.

Let $j \in I$ be fixed and write $(G^j, \Delta)$ for the generalized wreath product $\prod_{(I, \leq)}(\tilde{G}_i, \Delta_i)$ where, for $i \neq j$, the group $\tilde{G}_i$ contains the identity permutation alone, whilst $\tilde{G}_j = G_j$. Thus $G^j$ is the subgroup of $G$ corresponding to an action which moves only the $j$th coordinate; see Lemma 4 below. For $h \in \Delta^m$ and $\phi \in \text{Hom}(I, \mathcal{P}(\mathbb{R}))$, we write

$$\mathcal{O}_\phi^h,j = \{ k \in \mathcal{O}_\phi : k_i = h_i \text{ for all } i \neq j \},$$

$$\mathcal{O}_\phi^{h,j} = \{ k \in \mathcal{O}_\phi : k_i = h_i \text{ for all } i > j \}.$$

**Lemma 4.** If $f \in G^j$ and $h \in \Delta^m$ then $(hf)_i = h_i$ for all $i \neq j$.
**Proof.** This is an immediate consequence of the definition of $G^j$ and the action of generalized wreath product groups: if $f = (f_i)$, where $f_i : \Delta' \to \hat{G}$, for each $i \in I$, and $x \in \mathfrak{m}$, then, for $i \neq j$,

$$(xhf)_i = (xh)_i((xh\pi^i)f_i) = (xh)_i1_i = (xh)_i,$$

where we have denoted the identity permutation on $\Delta_i$ by $1_i$.

**Corollary.** $G^j$ fixes both $\mathcal{O}^{h,j}_\phi$ and $\mathcal{O}^{h,j}_\phi$ setwise.

**Lemma 5.** If $G_j$ is $m$-transitive the $G^j$ is transitive on $\mathcal{O}^{h,j}_\phi$.

**Proof.** Take $k \in \mathcal{O}^{h,j}_\phi$. It is sufficient to find $f \in G^j$ so that $kf = h$, and by Lemma 4 we need only consider the $j$th coordinates.

We denote the blocks of kernel $h\pi^j$ by $B_1, \ldots, B_s$ and, by the reasoning in the proof of Lemma 2, we see that, for each $r = 1, \ldots, s$, we can choose $g_r \in G$, such that, for each $x \in B_r$, we have $(xk)g_r = (xh)_j$. Continuing the line of reasoning of Lemma 2, we choose $f_r$ arbitrarily subject only to the requirement that, for each $r = 1, \ldots, s$ and $x \in B_r$, we have $(xh\pi^j)f_r = g_r$. The definition of $f$ is now completed by defining $f_i (i \neq j)$ in the only way possible and we have found an $f$ with $kf = h$.

**Remark.** The proof has in fact shown that, if $h, k \in \mathcal{O}_\phi$ and $h_i = k_i$ for all $i > j$, then there exists an element $f \in G^j$ such that $(kf)_i = h_i$ for all $i > j$. This shows that the orbits of $G^j$ on $\mathcal{O}^{h,j}_\phi$ are labelled by the elements of $\{ \{ k_i : i \neq j \} : k \in \mathcal{O}^{h,j}_\phi \}$ and are exactly the sets

$$\{ l \in \mathcal{O}_\phi : l_i = h_i, i > j, l_i = k_i, i \neq j \}.$$

Our final result shows that, for $h, k \in \mathcal{O}_\phi$, we can find an $f \in G$ such that $kf = h$, having the form

(1) $f = f_1f_2 \cdots f_u$, with $f_i \in G^{j_i}$ $(t = 1, \ldots, u),$

where $I = \{ j_1, \ldots, j_u \}$. Loosely speaking, we can "move over" $\mathcal{O}_\phi$ using elements from the subgroups $G^j$ of $G$. This is the only result for which $f$ must be finite.

**Lemma 6.** If $(G_i, \Delta_i)$, for each $i \in I$, is $m$-transitive then, for $h, k \in \mathcal{O}_\phi$, there exists $f \in G$ of the form (1) such that $kf = h$.

**Proof.** We number the elements of $I$, beginning with the maximal ones, in such a way that if $i > j$ in $I$, then the number that $j$ is assigned is larger than that assigned to $i$. 

https://doi.org/10.1017/S1446788700026173 Published online by Cambridge University Press
By the remark following the proof of Lemma 5, we can find \( f_1 \in G^j \) such that 
\[
(kf_1)_j = h_j.
\]
Assume now that this has been done for \( j_1, \ldots, j_{t-1}, t \geq 2 \), and so \( k(f_1 \cdots f_{t-1}) \) agrees with \( h \) at \( j_1, \ldots, j_{t-1} \). Then we have \( k' = k(f_1 \cdots f_{t-1}) \in \mathcal{P}^{h,j} \) and, by the last remark, once more there exists \( f_t \in G^j \) which sends \( k' \) to 
\[
k(f_1 \cdots f_t) \in \mathcal{P}^{h,j}.
\]
Thus \( k(f_1 \cdots f_t) \) agrees with \( h \) at \( j_1, \ldots, j_t \), and the induction proof is complete.

References