BULL. AUSTRAL. MATH. SOC. 46B20
VOL. 33 (1986), 457-470.

UNIFORM ASYMPTOTIC SMOOTHNESS OF NORMS

T. LEWIS, J. WHITFIELD AND V. ZIZLER

We study a notion of smoothness of a norm on a Banach space X which generalizes the notion of uniform differentiability and is formulated in terms of unicity of Hahn Banach extensions of functionals on block subspaces of a fixed Schauder basis S in X. Variants of this notion have already been used in estimating moduli of convexity in some spaces or in fixed point theory. We show that the notion can also be used in studying the convergence of expansions coefficient of elements of X^* along the dual basis S^*.

It often happens that in using the notion of uniform smoothness of a norm on a Banach space X one does not actually need its full strength. This is why many variants of it have arisen and been studied (see for example [3], [4], [7], [10]). To consider the notion studied in this note let us suppose that $S = \{x_k, h_k\}$, $k = 1,2,\ldots$ is a Schauder basis in a Banach space X (h_k are biorthogonal functions associated with the basis). If $x = \sum h_k(x)x_k$ then

$$\text{supp } x = \{k, h_k(x) \neq 0\} .$$

Let us recall that it is easy to check that a norm on a Banach space X with Schauder basis S is uniformly smooth (that is uniformly Frechet

Received 3 September 1985. Research supported by NSREC (Canada) and University of Alberta grants.

differentiable) if and only if given a sequence f_n , g_n of norm one elements of the dual space X^* for which there are some $z_n \in X$, supp $z_n < \infty$ with $\|f_n\|_{sp(z_n)} \to 1$, $\|f_n - g_n\|_{sp(z_n)} \to 0$ we must have $\|f_n - g_n\| \to 0$. The notion studied in this note arises if we replace the $sp(z_n)$ by block subspaces $sp\{x_1, x_2, \ldots, x_{k_n}\}$ (see Definition 1). Since the norm on a space X is often constructed in (nice) terms of a given Schauder basis in X , the class of such spaces is quite wide (see Proposition 2). On the other hand variants of this notion are known to be useful in Renorming theory (see [15] where they were apparently considered for the first time in the literature), in fixed point theory ([8], [5]) or in estimating moduli of convexity in some spaces ([1], [2], [6]). In this note we study connections of this notion with other geometrical properties of the space (Propositions 2,3) and show that it can also be used in studying the convergence of expansion coefficients of the elements of the dual space X^* along the dual basis S^* (results of Orlicz-Kadec-Figiel-Pisier type). We encountered this notion in our attempt to find a nice characterization of dual spaces which can be given an equivalent locally uniformly convex norm.

We will use the following notation throughout: B_1 , S_1 (B_1^*, S_1^*) will denote the closed unit ball and sphere in $X(X^*)$ respectively. We will consider in X^* its usual dual supremum norm. If $S = \{x_k, h_k\}$, $k = 1,2,\ldots$ is a Schauder basis of X , then the basis projections are

$P_n x = \sum_{k=1}^{n} h_k(x)\, x_k$, and we define $T_n = I - P_n$, $D_n = P_{n+1} - P_n$ where I stands for the identity operator in X . Their dual projections are denoted by P_n^* , T_n^* , D_n^* respectively. If $S = \{x_k, h_k\}$ is a basis of X , then the basic sequence formed by $h_k \in X^*$ is denoted by S^* . If $f \in X^*$ and $n \in N$ (N denotes the set of all natural numbers), then $\|f\|_n$ $(\|f\|_{-n})$ denote the norm of the restriction of f to the space $P_n X (T_n X)$ respectively, that is the norms of the restrictions in

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

$(P_nX)^*((T_nX)^*)$, where $P_nX(T_nX)$ are provided with the norm from X.

Variants of the following definition were considered in [15], [8], [1], [2], [6].

DEFINITION 1. Let X be a Banach space with a Schauder basis $S = \{x_k, h_k\}$, $k = 1,2,\ldots$. We say that X is uniformly asymptotically smooth along the basis S(UASAS) if for every $\varepsilon > 0$,

$$\delta(\varepsilon) = \inf\{1-\|\tfrac{f+g}{2}\|_n \ , \ f \ , \ g \in B_1^* \ , \ n \in N \ , \ \|f-g\|_n = 0 \ , \ \|f-g\| \geq \varepsilon\}$$

is positive. We will need the following elementary

PROPOSITION 1. *Let X be a Banach space with a Schauder basis $S = \{x_k, h_k\}$, $k = 1,2,\ldots$. Then the following properties are equivalent.*

(i) *X is UASAS.*

(ii) *For every $\varepsilon > 0$,*

$$\delta_1(\varepsilon) = \inf\{\max(\|f-g\|_n \ , \ 1-\|f\|_n) \ , \ f \ , \ g \in B_1^* \ , \ n \in N \ , \ \|f-g\| \geq \varepsilon\}$$

is positive.

(iii) *If for some f_n, $g_n \in X^*$, $\{f_n\}$ bounded and some $k_n \in N$*

$$\lim_n 2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n+g_n\|_{k_n}^2 = 0$$

and

$$\lim_n \|f_n-g_n\|_{k_n} = 0 ,$$

then

$$\lim_n \|f_n-g_n\| = 0 .$$

Proof. (i) ⇒ (ii). If X does not satisfy (ii), then there is an $\varepsilon > 0$ and sequence f_n, $g_n \in B_1^*$ and $k_n \in N$ with

$$\lim_n \|f_n-g_n\|_{k_n} = 0 \ , \ \lim_n \|f_n\|_{k_n} = 1 \ \text{ and } \ \|f_n-g_n\| > \varepsilon .$$

Denote then by $h_n' = f_n - g_n$ restricted to $P_{k_n}X$ and let $h_n = P_{k_n}^* h_n' \in X^*$. Finally, let $\lambda_n = \max(\|g_n+h_n\|, 1)$ and $f_n' = \lambda_n^{-1} f_n$,

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

$g_n' = \lambda_n^{-1}(g_n+h_n)$. Then $\lim \|h_n\| = 0$, $\lim_n \lambda_n = 1$, f_n' , $g_n' \in B_1^*$,

$\|f_n'-g_n'\| \geq \frac{\varepsilon}{2}$ for $n > n_0$ and $\|f_n'\|_{k_n} \longrightarrow 1$, $\|f_n'-g_n\|_{k_n} = 0$.

So, X then is not UASAS.

(ii) ⇒ (iii). If (iii) does not hold, then there are $\varepsilon > 0$, $k_n \in N$ and f_n , $g_n \in X^*$, $\{f_n\}$ bounded with $\|f_n-g_n\| \geq \varepsilon$,

$$\lim 2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n+g_n\|_{k_n}^2 = 0 \text{ and } \lim \|f_n-g_n\|_{k_n} = 0 .$$

Using the triangle inequality, we then have

$$\begin{aligned} 2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n+g_n\|_{k_n}^2 &= 2\|f_n\|_{k_n}^2 + 2\|g_n\|_{k_n}^2 - \|f_n+g_n\|_{k_n}^2 \\ &\quad + 2(\|f_n\|^2-\|f_n\|_{k_n}^2) + 2(\|g_n\|^2-\|g_n\|_{k_n}^2) \\ &\geq (\|f_n\|_{k_n}-\|g_n\|_{k_n})^2 + 2(\|f_n\|^2-\|f_n\|_{k_n}^2) \\ &\quad + 2(\|g_n\|^2-\|g_n\|_{k_n}^2) . \end{aligned}$$

Since $\{f_n\}$ is bounded, suppose (without loss of generality) that $\lim \|f_n\| = 1$.

Then we have subsequently:

$$\|f_n\|_{k_n} \to 1 , \|g_n\|_{k_n} \to 1 , \|g_n\| \to 1 .$$

Thus, letting

$$\lambda_n = \max(\|f_n\|,\|g_n\|) ,$$

$$f_n' = \lambda_n^{-1} f_n \text{ and } g_n' = \lambda_n^{-1} g_n ,$$

we have

$$f_n' , g_n' \in B_1^* \subset X^* , \|f_n'-g_n'\| \geq \frac{\varepsilon}{2} \text{ for } n > n_0 , \|f_n'-g_n'\|_{k_n} \to 0$$

and $\|f_n'\|_{k_n} \to 1$. Therefore (ii) does not hold for $\frac{\varepsilon}{2}$.

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

(iii) $\Rightarrow$ (i). If X is not UASAS, then there are $\varepsilon > 0$, f_n , $g_n \in B_1^*$ and $k_n \in N$ such that

$$\|f_n-g_n\|_{k_n} = 0 \ , \ \|f_n\|_{k_n} \to 1 \ , \ \|f_n-g_n\| \geq \varepsilon \ .$$

Then obviously $1 \geq \|f_n\| \geq \|f_n\|_{k_n} = \|g_n\|_{k_n} \leq \|g_n\| \leq 1$ and therefore

$$2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n+g_n\|_{k_n}^2 = 2\|f_n\|^2 + 2\|g_n\|^2 - 4\|f_n\|_{k_n}^2 \to 0 \ .$$

Furthermore, $\|f_n-g_n\|_{k_n} = 0$, $\|f_n-g_n\| \geq \varepsilon$. Therefore (iii) does not hold.

To place the notion studied here into the (large) family of its relatives, let us first recall a notion introduced by Huff in [9] and proved to be useful in the fixed point theory in [5], where a w^* modification of it was formulated.

DEFINITION 2. (Huff) X^* is said to have uniformly w^* Kadec-Klee property (UW^*KK) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$\{f_n\} \subset B_1^* \ , \ n = 1,2,\ldots \ , \ \|f_n-f_m\| \geq \varepsilon \quad \text{for} \quad n \neq m$$

and

$$\lim f_n = f \quad \text{in the } w^* \text{ topology of } X^* \ ,$$

then

$$\|f\| \leq 1 - \delta \ .$$

The following definition is related for example to [6], [2].

DEFINITION 3. Let X be a Banach space with a Schauder basis $S = \{x_k,h_k\}$, $k = 1,2,\ldots$. We shall say that X is uniformly smooth along the basis S (USAS) if

$$\lim_n \frac{\|z_n+t_n\| + \|z_n-t_n\| - 2}{\|t_n\|} = 0$$

whenever $z_n \in S_1 \subset X$, $t_n \in X$, $\|t_n\| \to 0$ are such that for every n ,

$$\max\{\text{supp } z_n\} < \min\{\text{supp } t_n\} \ .$$

Furthermore, US will denote a uniformly smooth space, that is the space

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

whose norm is uniformly Frechet differentiable on the unit sphere.

PROPOSITION 2. *Let* X *be a Banach space with a Schauder basis* $S = \{x_k, h_k\}$, $k = 1,2,\ldots$. *Then*

(i) X *is* $US \Rightarrow X$ *is* $USAS \Rightarrow X$ *is* $UASAS \Rightarrow X^*$ *is* $UW^*KK \Rightarrow$ *the norm and* w^* *convergence of sequences on the unit sphere of* X^* *coincide.*

(ii) *If* $X = c_0$ *with its usual unit vector basis, then* X *is* USAS *with* $\delta(\varepsilon) = \frac{\varepsilon}{2}$.

(iii) *If* $X = \ell_p$, $p \in (1,\infty)$ *with its usual unit vector basis, then* $\delta(\varepsilon) = 1 - \left(1 - (\frac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$, *where* q *is a dual index to* p , *that is* $q = \frac{p}{p-1}$.

Proof. (i) The first implication is obvious, the fourth can be found in [9], so it remains to show that the second and third implications are true. If X is USAS, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if x , $t \in X$, $x \in S_1$, $\|t\| \le \delta$ and $\max\{\text{supp } x\} < \min\{\text{supp } t\}$, then

$$\|x+t\| + \|x-t\| \le 2 + \varepsilon\|t\| .$$

We show that X is then UASAS.
Let K be the basis constant of S , that is $K = \sup\{\|P_n\| , n \in N\}$. Then, given $\varepsilon > 0$ and f_n , $g_n \in B_1^* \subset X^*$ and $k_n \in N$ with $\|f_n\|_{k_n} \to 1$, $\|f_n - g_n\|_{k_n} \to 0$, we show that starting with some n_0 , $\|f_n - g_n\| < 2\varepsilon(1+K)$. To do so, choose a $\delta > 0$ from USAS as above in this proof. Furthermore choose $z_n \in (P_{k_n} X) \cap S_1$ such that $f_n(x_n) \to 1$. Then $\max\{\text{supp } z_n\} \le k_n$ and $g_n(z_n) \to 1$. Moreover, if $t_n \in X$, $\|t_n\| = \delta$, $\min\{\text{supp } t_n\} > k_n$, then by the property of δ from USAS,

$$f_n(z_n+t_n) + g_n(z_n-t_n) \le \|z_n+t_n\| + \|z_n-t_n\| \le 2 + \varepsilon\|t_n\|$$

and therefore

$$(f_n-g_n)(t_n) \le 2 + \varepsilon\|t_n\| - f_n(x_n) - g_n(x_n) \le 2\varepsilon\delta$$

for $n > n_0$. Thus

$$\limsup \|f_n - g_n\|_{-k_n} \le 2\varepsilon .$$

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

Since

$$\limsup \|f_n - g_n\|_{k_n} = 0 ,$$

we have

$$\limsup \|f_n - g_n\| \le \limsup \|P^*_{k_n}(f_n - g_n)\| + \limsup \|(I - P^*_{k_n})(f_n - g_n)\|$$

$$\le K \limsup \|f_n - g_n\|_{k_n} + (1+K)\limsup \|f_n - g_n\|_{-k_n}$$

$$\le 2\varepsilon(1+K) .$$

To show that the third implication in (i) is true, suppose that X is UASAS but does not possess UW^*KK. The fact that X^* is not UW^*KK means that there is an $\varepsilon > 0$ such that for every $n \in N$ there is a sequence

$$\{f^n_k\} , k = 1,2,\ldots \text{ with } \|f^n_k - f^n_\ell\| \ge \varepsilon \text{ for } k \ne \ell$$

and with $w^* \lim_k f^n_k = f^n$ existing and $\|f^n\| > 1 - \frac{1}{n}$. Let $\delta_1 = \delta_1(\varepsilon)$, where $\delta_1(\varepsilon)$ is as in Proposition 1 (ii). Choose $n > \frac{1}{\delta_1}$, $n \in N$ and fix it. Moreover choose and fix an $m \in N$ so that $\|f^n\|_m > 1 - \frac{1}{2n}$. Then if k_0 is large enough, we have from $w^* \lim_\ell f^n_\ell = f^n$ that

$$\|f^n_k - f^n\|_m < \frac{1}{n} \text{ and } \|f^n_k\|_m > 1 - \frac{1}{n} \text{ for } k,\ell > k_0 .$$

Then since $n > \frac{1}{\delta_1}$, by the definition of δ_1 we have

$\|f^n_k - f^n_e\| < \varepsilon$ for $k,\ell > k_0$; a contradiction.

(ii) Observe that if $X = c_0$ and S is the usual unit vector basis in X and if $z,t \in X$, $\|z\| = 1$, $\|t\| \le 1$, $\max\{\text{supp } z\} < \min\{\text{supp } t\}$, then

$$\|z+t\| + \|z-t\| - 2 = 0 .$$

The evaluation of $\delta(\varepsilon)$ *for this case will follow from the proof of (iii).*

(iii) First observe that

$$\delta(\varepsilon) \ge 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}} .$$

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

For otherwise we would have for some $f, g \in B_1^*$, $\|f-g\| \geq \varepsilon$ and $f = g$ on some P_nX that $1 - \|f\|_n < 1 - \left(1-(\frac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$. This would mean that

$$\|f\|_n > \left(1-(\tfrac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$$

and therefore

$$1 - \|f\|_n^q < 1 - \left(1-(\tfrac{\varepsilon}{2})^q\right) = (\tfrac{\varepsilon}{2})^q .$$

Then,

$$\|f-g\|^q = \|f-g\|_n^q + \|f-g\|_{-n}^q = \|f-g\|_{-n}^q$$

$$\leq (\|f\|_{-n}+\|g\|_{-n})^q = \left((\|f\|^q-\|f\|_n^q)^{\frac{1}{q}} + (\|g\|^q-\|g\|_n^q)^{\frac{1}{q}}\right)^q$$

$$\leq \left((1-\|f\|_n^q)^{\frac{1}{q}}+(1-\|f\|_n^q)^{\frac{1}{q}}\right)^q = \left(2(1-\|f\|_n^q)^{\frac{1}{q}}\right)^q$$

$$= 2^q (1-\|f\|_n^q) < 2^q \cdot (\tfrac{\varepsilon}{2})^q = \varepsilon^q \quad \text{, a contradiction}$$

Therefore $\delta(\varepsilon) \geq 1 - \left(1-(\frac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$.

On the other hand, putting $f = \left((1-(\frac{\varepsilon}{2})^q)^{\frac{1}{q}}, \varepsilon_1, \varepsilon_2, \ldots,\right)$, $g = \left((1-(\frac{\varepsilon}{2})^q)^{\frac{1}{q}}, -\varepsilon_1, -\varepsilon_2, \ldots\right.$ where $\varepsilon_i > 0$ and $\sum \varepsilon_i^q = (\frac{\varepsilon}{2})^q$, we have $\|f\| = \|g\| = 1$, $\|f-g\| = \varepsilon$, $\|f\|_1 = \left(1-(\frac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$, $\|f-g\|_1 = 0$.

Therefore $\delta(\varepsilon) \leq 1 - \left(1-(\frac{\varepsilon}{2})^q\right)^{\frac{1}{q}}$. The following remarks compared with [9] are helpful in showing that none of the implications in Proposition 2 (i) have their true converse.

REMARK 1. Consider $X = R^3$ with its usual maximum norm and let $S = \{x_1 = (1,0,0), x_2 = (0,0,1), x_3 = (0,1,1)\}$. By an easy geometrical argument it follows that if $f_1, f_2 \in X^*$, $\|f_1\| = \|f_2\| = 1$ with $f_1 = f_2$

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

on $sp\{x_1, x_2\}$, then $f_1 = f_2$. The same applies to $sp\{x_1\}$. From this and a compactness argument it follows that X is UASAS with respect to the basis S . However the norm of X is not differentiable at $x_1 + x_2$ with respect to the direction x_3 , so X is not USAS.

REMARK 2. ℓ_∞^2 is UASAS with respect to the usual unit vector basis but not UASAS with respect to the basis $S = \{(1,1),(1,0)\}$.

REMARK 3. A simple example of a 3-dimensional ball (a truncated octahedron) shows that the space need not be UASAS if all of its basic vectors (usual unit vectors here) are points of Frechet smoothness.

REMARK 4. If $X = R^3$ with its usual maximum norm, then X is UASAS with respect to the basis $S = \{(1,0,0), (1,0,1), (0,1,0)\}$ and $(1,0,1)$ is not a point of smoothness of the norm.

We will finish by showing some of the applications of the notion of UASAS.

First, let us mention that the notion of UW^*KK has already been used by D. van Dulst and B. Sims in [5] in the fixed theory of nonexpansive mappings. Namely they proved a result which has the following Corollary (for terminology see [5]).

PROPOSITION 3. (D. van Dulst and B. Sims). *If* X^* *is* UW^*KK , *then Chebyshev centres with respect to* w^**-compact convex sets are nonempty compact and convex.*

We will now discuss estimations on the speed of convergence of expansions of elements of X^* along the dual basis S^* for a UASAS space X , in terms of its modulus. The first part of Proposition 4 gives (via Bishop-Phelps theorem) a qualitative form to a well-known fact that a monotone basis S of a space X (that is, all $\|P_n\| \le 1$) for which the norm and w^* convergence of sequences on the unit sphere of X^* coincide, is necessarily shrinking, that is S^* is the basis of X^* . The second part concerns a version of a Kadec result that for an unconditionally convergent series $\sum z_i$ in a uniformly convex space X , $\sum \delta(z_i) < \infty$, where δ denotes the modulus of convexity of X ([11], [2], [6], [16]).

PROPOSITION 4. *Let* S *be a monotone Schauder basis in an UASAS space* X . *Then*

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

(i) *If* $f \in S_1^* \subset X^*$ *attains its norm at* $z \in S_1 \subset X$ *, then for every* $n \in N$ *,*

$$\delta(\|T_n^* f\|) \le \|T_n z\| ,$$

(where $T_n = I - P_n$ *).*

(ii) *If* $f \in S_1^* \subset X^*$ *, then*

$$\sum \delta(\|D_n^* f\|) < \infty ,$$

(where $D_n = P_{n+1} - P_n$ *).*

Proof. The proof of (i) is similar to that of M.I. Kadec for the main Theorem in [*11*] and the proof of (ii) is an adjustment of the proof of T. Figiel [*6*] or G. Pisier ([*16*], Proposition 2.1).

(i) Given such an $f \in S_1^* \subset X$, $z \in S_1 \subset X$ and $n \in N$, observe that the elements f and $f - T_n^* f = P_n^* f$ both lie in the unit ball $B_1^* \subset X^*$, are equal on $P_n X$ and their distance is $\|T_n^* f\|$. Furthermore observe that because of the monotonicity of S ,

$$(P_n X) \cap B_1 = P_n(X \cap B_1) .$$

Thus we have

$$\|f\|_n = \sup\{|f(y)| , y \in (P_n X) \cap B_1 = \sup\{|f(P_n y)| , y \in B_1\}$$

$$= \|P_n^* f\| \ge (P_n^* f)z = (f - T_n^* f)z = 1 - f(T_n z) \ge 1 - \|T_n z\| .$$

Hence by definition of $\delta(\varepsilon)$,

$$\delta(\|T_n^* f\|) \le \|T_n z\| .$$

For the proof of (ii) we will need the following variant of the result of Figiel ([*6*]).

LEMMA 1. *If* X *is an* UASAS *space with respect to a monotone basis* S *and* $\delta(\varepsilon)$ *is as in Definition* 1 *, then*

$$\frac{\delta(\varepsilon)}{\varepsilon} \text{ is a nondecreasing function on } \varepsilon \in (0,2] .$$

Proof. The proof is similar to that given for Figiel's result in [*14*], p.66. Let $f , g \in B_1^*$ be such that for some $n \in N$ and $\varepsilon > 0$

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

$$P_n^*(f-g) = 0 \ , \ \|f-g\| \geq \varepsilon \ .$$

Let $\eta \in (0,\varepsilon)$.

Denote by $d = P_n^* f = P_n^* g$. Observe that $\|d\| < 1$ (use UASAS of X) . First suppose that $d \neq 0$. Then consider the halfline $0d$ originating at the origin 0 . Let c be the intersection of $0d$ with the unit sphere $S_1^* \subset X^*$. Now choose on the halflines fc and gc points f' and g' respectively so that $f'g'$ is parallel to fg and $\|f'-g'\| = \eta$.

Suppose that d does not lie on the line fg . Then consider the plane p which passes through $f'g'$ and is parallel to the plane going through f , g and d . Let d' be the intersection of p and $0c$. Then $f'd'$ is parallel to fd and $g'd'$ is parallel to gd . Since $P_n^*(f-g) = 0$, this means that $P_n^*(f'-g') = 0$ and therefore

$$P_n^* f' = P_n^* d' = d' \ .$$

Similarly $P_n^* g' = d'$.

Using similar triangles and monotonicity of the basis S we have

$$\frac{1 - \|\frac{f+g}{2}\|_n}{\varepsilon} = \frac{1 - \|P_n^* f\|}{\varepsilon} \geq \frac{1 - \|P_n^* f\|}{\|f-g\|} = \frac{\overline{dc}}{\overline{fg}}$$

$$= \frac{\overline{d'c}}{\overline{f'g'}} = \frac{1 - \|P_n^* f'\|}{\|f'-g'\|} = \frac{1 - \|\frac{f'+g'}{2}\|_n}{\eta}$$

$$\geq \frac{\delta(\eta)}{\eta} \ .$$

There is no difficulty in adapting the arguments above to the case where $d = 0$ (take an arbitrary $c \in P_n^* X^* \subset S_1^*$) or where d lies on the line fg (the picture then becomes two dimensional).

Taking infimum over all f , g and n as above in the formula on the left hand side of the series of inequalities, we get that

$$\frac{\delta(\varepsilon)}{\varepsilon} \geq \frac{\delta(\eta)}{\eta} \ .$$

Having Lemma 1 proved we can finish the proof of Proposition 6(ii) by

https://doi.org/10.1017/S0004972700004032 Published online by Cambridge University Press

following that of Figiel ([6]) or Pisier ([16], proof of Proposition 2.1):

Fix an $f \in S_1^* \subset X^*$. Notice that from the monotonicity of the basis S we have that for every $n \in N$, $1 \geq \|P_{n+1}^* f\| \geq \|P_n^* f\|$. Since $\dfrac{P_n^* f}{\|P_{n+1}^* f\|}$ and $\dfrac{P_{n+1}^* f}{\|P_{n+1}^* f\|}$ both lie in the unit ball $B_1^* \subset X^*$ and are equal on $P_n X$, we have by definition of $\delta(\varepsilon)$, by the monotonicity of basis and by the monotonicity of $\dfrac{\delta(\varepsilon)}{\varepsilon}$ that

$$\frac{\delta(\|D_n^* f\|)}{\|P_{n+1}^* f\|} \leq \delta\left(\frac{\|D_n^* f\|}{\|P_{n+1}^* f\|}\right) \leq 1 - \frac{1}{2}\left\|\frac{P_{n+1}^* f + P_n^* f}{\|P_{n+1}^* f\|}\right\|_n = 1 - \left\|\frac{P_n^* f}{\|P_{n+1}^* f\|}\right\| .$$

Multiplying, we get

$$\delta(\|D_n^* f\|) \leq \|P_{n+1}^* f\| - \|P_n^* f\| ,$$

and finally, summing,

$$\sum_1^k \delta(\|D_n^* f\|) \leq \|P_{k+1}^* f\| - \|P_1^* f\| \leq \|P_{k+1}^* f\| \leq 1 .$$

References

[1] Z. Altshuler, "Uniform convexity in Lorentz sequence spaces", *Israel J. Math.* 20 (1975), 260-274.

[2] Bui-Minh-Chi and V.I. Guraii, "Some characteristics of normed spaces and their applications to the generalization of Parseval's inequality for Banach spaces", *Teor. Funkciǐ Funkcional. Anal. i Priložen* 8 (1969), 74-91 (Russian).

[3] M.M. Day, *Normed linear spaces* (Springer Verlag, 1975).

[4] J. Diestel, *Geometry of Banach spaces - selected topics* (Lecture Notes in Mathematics 485. Springer Verlag, 1975).

[5] D. van Dulst, and B. Sims, *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type (KK), in Banach space theory and its applications,* Proceedings, Bucharest 1981, (Lecture Notes in Mathematics 991. Springer Verlag, 1983).

[6] T. Figiel, "On the moduli of convexity and smoothness", *Studia Math.* 56 (1976), 121-155.

[7] J. Giles, *Convex analysis with applications in differentiation of convex functions,* (Res. Notes in Math. 58. Pitnam, 1982).

[8] J.-P. Gossez and E. Lami Dozo, "Structure normale et base de Schauder", *Acad. Roy. Belg. Bull. Cl. Sci.* 55 (1969), 673-681.

[9] R. Huff, "Banach spaces which are nearly uniformly convex", *Rocky Mountain J. Math.* 10 (1980), 743-749.

[10] V. Istratescu, *Strict convexity and complex strict convexity, theory and applications.* (Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, 1984).

[11] M.I. Kadec, "Unconditional convergence of series in uniformly convex spaces", *Uspekhi Mat. Nauk* (N.S.) 11 (1956), 185-190 (Russian).

[12] G. Kothe, *Topological vector spaces I* (English translation), (Springer Verlag, 1969).

[13] J. Linderstrauss and L. Tzafriri, *Classical Banach spaces I, sequence spaces* (Springer Verlag, 1977).

[14] J. Linderstrauss and L. Tzafrinzi, *Classical Banach spaces II, function spaces* (Springer Verlag, 1979).

[15] A.R. Lovaglia, "Locally uniformly convex spaces", *Trans. Amer. Math. Soc.* 78 (1955), 225-238.

[16] G. Pisier, "Martingales with values in uniformly convex spaces", *Israel J. Math.* 20 (1975), 326-350.

T. Lewis, V. Zizler,
Department of Mathematics,
University of Alberta,
Edmonton, Alberta,
T6G 2G1, Canada.

J. Whitfield,
Department of Mathematical Sciences,
Lakehead University,
Thunder Bay, Ontario,
P7B 5E1, Canada.