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Intersection cohomology of moduli spaces of vector bundles over curves

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Abstract

We compute the intersection cohomology of the moduli spaces $\mathcal{M}_{r,d}$ of semistable vector bundles having rank r and degree d over a curve. We do this by relating the Hodge–Deligne polynomial of the intersection cohomology of $\mathcal{M}_{r,d}$ to the Donaldson–Thomas invariants of the curve. These invariants can be computed by methods going back to Harder, Narasimhan, Desale and Ramanan. More generally, we introduce Donaldson–Thomas classes in the Grothendieck group of mixed Hodge modules over $\mathcal{M}_{r,d}$, and relate them to the class of the intersection complex of $\mathcal{M}_{r,d}$. Our methods can be applied to the moduli spaces of semistable objects in arbitrary hereditary categories.

1. Introduction

Let C be a complex smooth projective curve of genus g. For $\gamma = (r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$, let $\mathcal{M}_\gamma$ (respectively $\mathcal{M}^{\mathsf{s}}_\gamma$) denote the moduli space of semistable (respectively stable) vector bundles of rank r and degree d over C. Then $\mathcal{M}_\gamma$ is projective and irreducible, $\mathcal{M}^{\mathsf{s}}_\gamma \subseteq \mathcal{M}_\gamma$ is open and smooth, and $\mathcal{M}^{\mathsf{s}}_\gamma = \mathcal{M}_\gamma$ if r and d are coprime. If $\mathcal{M}^{\mathsf{s}}_\gamma$ is nonempty (this is always the case for $g \geqslant 2$), then its dimension is $(g-1)r^2 + 1$. The history of the computation of various invariants of $\mathcal{M}_\gamma$ for coprime r and d is rather long. A recursive formula to determine the Betti numbers of $\mathcal{M}_\gamma$ was proved in [HN74, DR75] by applying the Weil conjectures and the Siegel formula [HN74], based on the computation of the Tamagawa number of L_n over function fields [Wei82]. An alternative proof of this recursive formula was obtained in [AB83] using gauge theory. An algebro-geometric method to prove the Siegel formula was developed in [BGL94, GL92], and this method was used in [dB01] to prove the recursive formula for the motivic classes and Hodge polynomials of $\mathcal{M}_\gamma$. Independently, the same recursive formula for the Hodge polynomials of $\mathcal{M}_\gamma$ was proved in [EK00] using equivariant cohomology and a refinement of the approach of Atiyah and Bott [AB83]. On the other hand, the above recursive formula was solved in [LR96, Zag96], and hence a rather explicit formula was obtained for the invariants of $\mathcal{M}_\gamma$ in the case of coprime r and d.

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In this paper, we compute the intersection cohomology of $\mathcal{M}_\gamma$ for arbitrary $\gamma = (r, d)$. These invariants were previously determined for $r = 2$ in [Kir86a, Kir92], using partial desingularizations of GIT quotients and related techniques developed in [Kir84, Kir85, Kir86b]. Based on these computations, as well as computations of the intersection cohomology for moduli spaces of low-rank vector bundles on surfaces [Yos95, Remark 4.6], it was proposed in [Man13] that intersection cohomology should be related to Donaldson–Thomas (DT) invariants, which are also known as BPS invariants. These invariants are usually defined for moduli spaces of sheaves on 3-Calabi–Yau varieties or, more generally, for moduli spaces of objects of 3-Calabi–Yau categories [KS08]. However, the construction can also be applied to hereditary categories (abelian categories with vanishing Ext^i for $i \geqslant 2$) and, in particular, to the category of coherent sheaves over a curve. The corresponding DT invariants DT^E_γ can be computed for arbitrary γ using the recursive formula mentioned earlier. After relating these invariants to the intersection cohomology of $\mathcal{M}_\gamma$, we obtain an effective method to compute the latter.

More precisely, the moduli space $\mathcal{M}_\gamma$ can be represented as a GIT quotient of a smooth variety $R_\gamma = R_{r,d}$ by an action of a general linear group $G_\gamma = G_{r,d}$ [New78]. The cohomology groups with compact support $H^*_c(R_\gamma, \mathbb{Q})$ and $H^*_c(G_\gamma, \mathbb{Q})$ carry mixed Hodge structures, and we can consider the corresponding Hodge–Deligne polynomials (or E-polynomials, see Section 5.6)

$$E(R_\gamma),\ E(G_\gamma) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].$$

For any $\mu \in \mathbb{Q}$, we consider the generating series

$$Q_\mu = 1 + \sum_{d/r=\mu} \mathbb{L}^{(1-g)r^2/2} \frac{E(R_{r,d})}{E(G_{r,d})} t^r \in \mathbb{Q}(u^{\frac{1}{2}}, v^{\frac{1}{2}})[\![t]\!], \tag{1}$$

where $\mathbb{L} = E(\mathbb{A}^1) = uv$ and $\mathbb{L}^{1/2} = -(uv)^{1/2}$. An explicit formula for Q_μ can be obtained using the methods from [LR96, Zag96] mentioned earlier (see Theorem 5.11). In particular, if r and d are coprime, then $\frac{E(\mathcal{M}_\gamma)}{\mathbb{L}-1} = \frac{E(R_\gamma)}{E(G_\gamma)}$, and hence we can determine Betti and Hodge numbers of $\mathcal{M}_\gamma$. The role of the other summands of the series Q_μ is less transparent. Motivated by the definition of DT invariants in other contexts (see e.g. [Moz13a, Moz13b]), we define the DT invariants $\mathrm{DT}^E_\gamma = \mathrm{DT}^E_{r,d} \in \mathbb{Q}(u^{\frac{1}{2}}, v^{\frac{1}{2}})$ by the formula

$$\sum_{d/r=\mu} \mathrm{DT}^E_{r,d}\, t^r = (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}) \mathrm{Log}(Q_\mu), \tag{2}$$

where Log is the plethystic logarithm (45). As Q_μ can be computed explicitly, we can also compute the DT invariants. These computations show that $\mathrm{DT}^E_{r,d}$ are polynomials with integer coefficients. A similar statement in the context of quiver representations was proved in [KS11, Efi12].

Now let us describe the relationship between DT invariants and the intersection cohomology of moduli spaces. Given a complex algebraic variety X of dimension d, let $\mathrm{IC}_X \in \mathrm{Perv}(\mathbb{Q}_X)$ be its intersection complex. This can be lifted to a mixed Hodge module $\mathbf{IC}_X$ of weight 0 (see Section 2.3) so that $\mathbf{IC}_X = \mathbb{Q}_X \langle d \rangle = \mathbb{Q}_X(d/2)[d]$ for smooth X. The intersection cohomology

$$\mathrm{IH}^*(X, \mathbb{Q}) = H^*(X, \mathbf{IC}_X \langle -d \rangle) \tag{3}$$

is a complex of mixed Hodge structures (pure of weight 0 if X is projective) and we may consider its Hodge–Deligne polynomial.

THEOREM 1.1 (See Theorem 5.10). *If* $\mathcal{M}^{\mathsf{s}}_\gamma = \varnothing$, *then* $\mathrm{DT}^E_\gamma = 0$. *If* $\mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing$, *then*

$$\mathrm{DT}^E_\gamma = E(H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma})) = \mathbb{L}^{-\dim \mathcal{M}_\gamma/2} E(\mathrm{IH}^*(\mathcal{M}_\gamma, \mathbb{Q})) \text{and} \tag{4}$$

$$\mathrm{DT}^E_\gamma(y, y) = (-y)^{-\dim \mathcal{M}_\gamma} \sum\nolimits_k \dim \mathrm{IH}^k(\mathcal{M}_\gamma, \mathbb{Q})(-y)^k. \tag{5}$$

The object $\mathbf{IC}_X$ is self-dual with respect to Verdier duality, and hence we obtain the following result.

COROLLARY 1.2. *We have:*

(i) $\mathrm{DT}^E_\gamma \in \mathbb{Z}[u, v, (uv)^{-\frac{1}{2}}]$;

(ii) $\mathrm{DT}^E_\gamma(u^{-1}, v^{-1}) = \mathrm{DT}^E_\gamma(u, v)$; *and*

(iii) $\mathrm{DT}^E_\gamma(-y, -y) \in \mathbb{N}[y^{\pm 1}]$.

In order to prove Theorem 1.1, we introduce relative DT classes $\mathbf{DT}_\gamma \in \mathbf{K}(\mathrm{MHM}(\mathcal{M}_\gamma))$ such that $E(a_!\,\mathbf{DT}_\gamma) = \mathrm{DT}^E_\gamma$ for the projection $a : \mathcal{M}_\gamma \to \mathbf{pt}$ (we also have $E(a_! \mathbf{IC}_{\mathcal{M}_\gamma}) = E(H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma}))$). We prove an analogue of Theorem 1.1 for these classes.

THEOREM 1.3 (See Corollary 5.8). *We have*

$$\mathbf{DT}_\gamma = \begin{cases} 0 & \text{if } \mathcal{M}^{\mathsf{s}}_\gamma = \varnothing, \\ [\mathbf{IC}_{\mathcal{M}_\gamma}] & \text{if } \mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing, \end{cases}$$

in $\mathbf{K}(\mathrm{MHM}(\mathcal{M}_\gamma))$.

For the proof of this result we will utilize the ideas in [MR19], which studies moduli spaces of quiver representations; similar extensions of the methods of [MR19] were developed independently in [Mei15]. In addition, we will clarify the technical issues in [Mei15, MR19] in Section 2. First, we will introduce the moduli space $\mathcal{M}^{\mathsf{f}}_\gamma$ of stable framed vector bundles. Under appropriate assumptions, this moduli space is smooth, and the canonical morphism $\pi : \mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma$ is projective. The fibres of this map over $\mathcal{M}^{\mathsf{s}}_\gamma$ are projective spaces, while the other fibres can be identified with moduli spaces of stable nilpotent quiver representations (see Theorem 4.10). This analysis of the fibres will be used in Theorem 4.11 to show that π is a virtually small map, which is a generalization of the notion of a small map (see Section 2.6). The properties of virtually small maps (see Theorem 2.21) imply that the leading term of $\pi_* \mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}$ (with respect to the weight degree on the Grothendieck group of mixed Hodge modules) can be related to $\mathbf{IC}_{\mathcal{M}_\gamma}$. On the other hand, we can express the class of $\pi_* \mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}$ in terms of DT classes (see Theorem 5.6), and we will show in Theorem 5.7 that the leading term in this expression is related to $\mathbf{DT}_\gamma$. While our main focus is on the moduli spaces of vector bundles on a curve, all our proofs generalize verbatim to moduli spaces in other hereditary categories.

A closely related result is proved in [FST25]. More precisely, the authors consider the projection $\pi : \mathcal{M}^{\mathsf{f}} \to \mathcal{M}$, where $\mathcal{M} = \bigsqcup_{r \geqslant 0} \mathcal{M}_{r,0}$ and $\mathcal{M}^{\mathsf{f}} = \bigsqcup_{r \geqslant 0} \mathcal{M}^{\mathsf{f}}_{r,0}$ is the moduli space of stable framed objects for the parabolic framing functor (48) ($\mathcal{M}^{\mathsf{f}}_{r,0}$ is denoted by $\mathcal{P}_0(r)$ in [FST25]), and prove a specialization of Theorem 5.7 (see also Theorem 5.6) for Poincaré polynomials (see [FST25, Theorem 8.7]). As in our paper, the authors perform a careful analysis of the map π and its fibres in order to determine the pushforward of the intersection complex.

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2. Virtually small maps

2.1 Mixed t-categories

For a subcategory $\mathcal{S} \subseteq \mathcal{D}$ (or a collection of subcategories) of a triangulated category $\mathcal{D}$, let $\langle \mathcal{S} \rangle \subseteq \mathcal{D}$ be the minimal full subcategory that contains $\mathcal{S} \cup \{0\}$ and is closed under extensions. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geqslant 0})$ [BBD82] on a triangulated category $\mathcal{D}$ is called bounded if $\mathcal{D} = \langle \mathcal{A}[n] : n \in \mathbb{Z} \rangle$ for the heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geqslant 0}$. For $M \in \mathcal{D}$, we define the cohomology object $H^i M = \tau_{\leq 0} \tau_{\geqslant 0}(M[i]) \in \mathcal{A}$. A (bounded) t-category is a triangulated category equipped with a (bounded) t-structure.

Definition 2.1. A bounded t-structure on a triangulated category $\mathcal{D}$ is called mixed if its heart $\mathcal{A}$ is equipped with strictly full subcategories $\mathcal{A}_n \subseteq \mathcal{A}$, for $n \in \mathbb{Z}$, satisfying:

(i) $\operatorname{Hom}^i(\mathcal{A}_m, \mathcal{A}_n) = \operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n[i]) = 0$ for $m < n + i$; and

(ii) $\mathcal{A} = \langle \mathcal{A}_n : n \in \mathbb{Z} \rangle$.

Objects of $\mathcal{A}_n$ are called pure of weight n. An object $M \in \mathcal{D}$ is called pure of weight n if $H^i M$ is pure of weight $n + i$ for all $i \in \mathbb{Z}$. A triangulated category $\mathcal{D}$ equipped with a mixed t-structure is called a mixed t-category. An abelian category $\mathcal{A}$ such that the standard t-structure of $D^b(\mathcal{A})$ is mixed is called an (abelian) mixed category.

Remark 2.2. The above notion is closely related to ft-categories [BGS88], mixed abelian categories [BGS96], weight structures [Bon07], and co-t-structures [Pau07].

Remark 2.3. A structure as above will be called weakly mixed if the first axiom is substituted by the requirement $\operatorname{Hom}^i(\mathcal{A}_m, \mathcal{A}_n) = 0$ for $m < n + i$, $m \neq n$ and $\operatorname{Hom}^i(\mathcal{A}_n, \mathcal{A}_n) = 0$ for $i \geqslant 2$.

Example 2.4. For an algebraic variety X over $\mathbb{C}$, the abelian category $\operatorname{MHM}(X)$ of mixed Hodge modules is mixed and of finite length. The category of pure weight n objects of $\operatorname{MHM}(X)$ is the category $\operatorname{HM}^p(X, n)$ of polarizable pure Hodge modules of weight n over X. We have $\operatorname{HM}^p(\mathbf{pt}, n) = \operatorname{HS}^p(\mathbb{Q}, n)$, the category of polarizable $\mathbb{Q}$-Hodge structures of weight n, and $\operatorname{MHM}(\mathbf{pt}) = \operatorname{MHS}^p(\mathbb{Q})$, the category of (graded-)polarizable mixed $\mathbb{Q}$-Hodge structures.

Example 2.5. Let X_0 be an algebraic variety over a finite field. Then the category $D^b_m(X_0, \bar{\mathbb{Q}}_\ell) \subset D^b_c(X_0, \bar{\mathbb{Q}}_\ell)$ of mixed complexes [BBD82, Section 5.1] is a weakly mixed category. More precisely, it has a bounded (perverse) t-structure with the heart $\mathcal{A}$ of finite length equipped with a weight filtration. For any simple, non-isomorphic objects $M, N \in \mathcal{A}$ one can prove that $\operatorname{Hom}^i(M, N) = 0$ if $w(M) < w(N) + i$ (the vanishing is proved for $w(M) + 1 < w(N) + i$ in [BBD82] and a similar argument can be used for our slightly stronger statement). One can also prove that $\operatorname{Hom}^i(M, M) = 0$ for $i \geqslant 2$, but $\operatorname{Hom}^1(M, M)$ is always nonzero for simple M. While most of the results in this paper are formulated in terms of mixed Hodge modules, they can be similarly formulated in the context of ℓ-adic sheaves.

Lemma 2.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_n \subseteq \mathcal{A}$, $n \in \mathbb{Z}$, be strictly full subcategories such that $\operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n) = \operatorname{Ext}^1(\mathcal{A}_m, \mathcal{A}_n) = 0$ for $m < n$ and $\mathcal{A} = \langle \mathcal{A}_n : n \in \mathbb{Z} \rangle$. Then $\mathcal{A}_n \subseteq \mathcal{A}$ are Serre subcategories of $\mathcal{A}$ (closed under taking extensions, quotients, and subobjects). Every object $M \in \mathcal{A}$ has a unique increasing filtration $W_\bullet M$ (called the weight filtration) such that:*

(i) *$W_{-n} M = 0$ and $W_n M = M$ for $n \gg 0$; and*

(ii) *$\operatorname{Gr}^W_n M = W_n M / W_{n-1} M \in \mathcal{A}_n$ for all $n \in \mathbb{Z}$.*

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Proof. Left to the reader. □

Lemma 2.7 (Decomposition theorem). *For every pure object $M \in \mathcal{D}$ there is a (non-canonical) isomorphism $M \simeq \bigoplus_{i\in\mathbb{Z}} H^i(M)[-i]$. If $\mathcal{A}_n$ is of finite length, then it is semisimple.*

Proof. Left to the reader. □

For $M \in \mathcal{A}$, we write $w(M) \leqslant n$ if $\operatorname{Gr}_k^W M = 0$ for $k > n$ and we write $w(M) \geqslant n$ if $\operatorname{Gr}_k^W M = 0$ for $k < n$. We define

$$w(M) = \sup\left\{n \in \mathbb{Z} \,\middle|\, \operatorname{Gr}_n^W M \neq 0\right\} \in \mathbb{Z} \sqcup \{-\infty\}.$$

For $M \in \mathcal{D}$, we write $w(M) \leqslant n$ if $w(H^i M) \leqslant n+i$ for all $i \in \mathbb{Z}$ and we write $w(M) \geqslant n$ if $w(H^i M) \geqslant n+i$ for all $i \in \mathbb{Z}$. We define

$$w(M) = \max\left\{w(H^i M) - i \,\middle|\, i \in \mathbb{Z}\right\} \in \mathbb{Z} \sqcup \{-\infty\}.$$

Let

$$\mathcal{D}_{\leqslant n} = \{M \in \mathcal{D} \mid w(M) \leqslant n\} = \langle \mathcal{A}_m[i] : m+i \leqslant n\rangle,$$

$$\mathcal{D}_{\geqslant n} = \{M \in \mathcal{D} \mid w(M) \geqslant n\} = \langle \mathcal{A}_m[i] : m+i \geqslant n\rangle.$$

Then $\mathcal{D}_{\leqslant n}[1] = \mathcal{D}_{\leqslant n+1}$, $\mathcal{D}_{\geqslant n}[1] = \mathcal{D}_{\geqslant n+1}$ and

$$\operatorname{Hom}(\mathcal{D}_{\leqslant m}, \mathcal{D}_{\geqslant n}) = 0 \quad \forall m < n.$$

2.2 Exactness

An additive functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between triangulated categories is called exact (or triangulated) if it commutes with translations and preserves distinguished triangles. For an exact functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between t-categories with hearts $\mathcal{A}_1, \mathcal{A}_2$, we define

$$^{\mathrm{p}}F : \mathcal{A}_1 \to \mathcal{A}_2, \quad M \mapsto H^0 FM. \tag{6}$$

An exact functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between t-categories is called right (respectively left) t-exact if $F(\mathcal{D}_1^{\leqslant 0}) \subseteq \mathcal{D}_2^{\leqslant 0}$ (respectively $F(\mathcal{D}_1^{\geqslant 0}) \subseteq \mathcal{D}_2^{\geqslant 0}$). We say that F has t-amplitude $[a, b]$ if $F(\mathcal{D}_1^{\leqslant 0}) \subseteq \mathcal{D}_2^{\leqslant b}$ and $F(\mathcal{D}_1^{\geqslant 0}) \subseteq \mathcal{D}_2^{\geqslant a}$.

Theorem 2.8 (See [BBD82]). *Let $f : X \to Y$ be a morphism between algebraic varieties with fibres of dimension $\leqslant d$. Then (for the perverse t-structure on $D_c^b(\mathbb{Q}_X)$ or the standard t-structure on $D^b\mathrm{MHM}(X)$):*

(i) *$f_!, f^*$ have t-amplitude $\leqslant d$, meaning that $f_![d], f^*[d]$ are right t-exact; and*
(ii) *$f_*, f^!$ have t-amplitude $\geqslant -d$, meaning that $f_*[-d], f^![-d]$ are left t-exact.*

If f is affine, then f_* is right t-exact and $f_!$ is left t-exact.

An exact functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between mixed t-categories is called right (respectively left) w-exact if $F(\mathcal{D}_{1,\leq 0}) \subseteq \mathcal{D}_{2,\leq 0}$ (respectively $F(\mathcal{D}_{1,\geqslant 0}) \subseteq \mathcal{D}_{2,\geqslant 0}$). The following result is Deligne's generalization of Weil's conjectures in the ℓ-adic context [Del80] and Saito's theorem in the MHM context [Sai90].

Theorem 2.9 (See [Del80, Sai90]). *Let $f : X \to Y$ be a morphism between algebraic varieties. Then:*

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(i) $f_!, f^*$ *are right w-exact; and*

(ii) $f_*, f^!$ *are left w-exact.*

Remark 2.10. The external tensor product [Sai90]

$$\boxtimes : \mathrm{MHM}(X) \times \mathrm{MHM}(Y) \to \mathrm{MHM}(X \times Y)$$

is w-exact, meaning that if $M \in \mathrm{MHM}(X)$ and $N \in \mathrm{MHM}(Y)$ are pure, then $M \boxtimes N$ is pure of weight $w(M) + w(N)$. The induced external tensor product on the derived categories is automatically t-exact (and w-exact). On the other hand, the tensor product $\otimes^*$ on $D^b\mathrm{MHM}(X)$, defined by $M \otimes^* N = \Delta^*(M \boxtimes N)$ for the diagonal $\Delta : X \to X \times X$, is right t-exact and right w-exact (cf. [BBD82, 5.1.14]). The duality functor $\mathbb{D} : \mathrm{MHM}(X) \to \mathrm{MHM}(X)^{\mathrm{op}}$ maps objects of weight n to objects of weight $-n$ (actually $\mathbb{D}M \simeq M(n)$ for any pure object M of weight n).

2.3 Mixed Hodge modules

For an algebraic variety X over $\mathbb{C}$, let $d_X = \dim X$ denote the maximal dimension of its irreducible components. Let $\mathrm{MHM}(X)$ be the category of mixed Hodge modules over X. Let $D^b_c(\mathbb{Q}_X)$ be the derived category of bounded $\mathbb{Q}$-complexes with constructible cohomology, and let $\mathrm{Perv}(\mathbb{Q}_X) \subset D^b_c(\mathbb{Q}_X)$ be the subcategory of perverse sheaves. The exact functor $\mathrm{rat} : D^b\mathrm{MHM}(X) \to D^b_c(\mathbb{Q}_X)$ maps $\mathrm{MHM}(X)$ to $\mathrm{Perv}(\mathbb{Q}_X)$ and is compatible with Grothendieck's six operations on both sides. We have $\mathrm{MHM}(\mathbf{pt}) = \mathrm{MHS}^p(\mathbb{Q})$, the category of (graded-)polarizable mixed $\mathbb{Q}$-Hodge structures.

Let $\mathbb{Q}(n) \in \mathrm{MHM}(\mathbf{pt})$ be the Tate object of weight $-2n$ for $n \in \mathbb{Z}$. It induces the Tate twist functor $\mathrm{MHM}(X) \to \mathrm{MHM}(X)$, $M \mapsto M(n) = M \otimes \mathbb{Q}(n)$. We consider the Lefschetz object

$$\mathbb{L} = H^*_c(\mathbb{A}^1, \mathbb{Q}) = \mathbb{Q}(-1)[-2] \in D^b\mathrm{MHM}(\mathbf{pt}) \tag{7}$$

of weight 0 and the corresponding functor

$$\mathbb{L} : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(X), \quad M \mapsto M(-1)[-2]. \tag{8}$$

Note that $H^*_c(\mathbb{P}^n, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n$ for $n \geqslant 0$.

In what follows, we will require the class $\mathbb{L}^{1/2}$ at the level of Grothendieck groups (see Section 2.4), but it will also be convenient to have a related object at the level of categories. We have two approaches to this problem. The first one is to embed $\mathrm{MHM}(X)$ into the category of monodromic mixed Hodge modules $\mathrm{MMHM}(X)$ (see [KS11, DM20]), so that one has the objects $\mathbb{Q}(1/2) \in \mathrm{MMHM}(\mathbf{pt})$ and $\mathbb{L}^{1/2} = \mathbb{Q}(-1/2)[-1] \in D^b\mathrm{MMHM}(\mathbf{pt})$ and the corresponding twist functors. The second approach is to construct the root category $\mathcal{A}^{1/2}$ and the functor $\mathbb{T}^{1/2} = \mathbb{Q}(1/2) : \mathcal{A}^{1/2} \to \mathcal{A}^{1/2}$ for the category $\mathcal{A} = \mathrm{MHM}(X)$ and the functor $\mathbb{T} = \mathbb{Q}(1) : \mathcal{A} \to \mathcal{A}$. More generally, we construct the root category $\mathcal{A}^{1/r}$ and the functor $\mathbb{T}^{1/r} : \mathcal{A}^{1/r} \to \mathcal{A}^{1/r}$ for any $r \geqslant 1$ as follows. We define $\mathcal{A}^{1/r} = \bigoplus_{i=0}^{r-1} \mathcal{A}^{(i)}$ with $\mathcal{A}^{(i)} = \mathcal{A}$ and the functor $\mathbb{T}^{1/r}$ sending $\mathcal{A}^{(i)} \ni M \mapsto M \in \mathcal{A}^{(i+1)}$ for $0 \leqslant i < r-1$ and $\mathcal{A}^{(r-1)} \ni M \mapsto \mathbb{T}M \in \mathcal{A}^{(0)}$. We embed $\mathcal{A} = \mathcal{A}^{(0)} \hookrightarrow \mathcal{A}^{1/r}$ so that $(\mathbb{T}^{1/r})^r = \mathbb{T}$. The weight of $M \in \mathcal{A}^{(i)} \subset \mathcal{A}^{1/2}$ is defined to be $w(M) - i$ (if $M \in \mathcal{A}$ is pure) so that $\mathbb{T}^{1/2}$ has weight -1 and $\mathbb{T} = \mathbb{Q}(1)$ has weight -2 as before. We can also define the functor $\mathbb{Q}(n/2)$ for any $n \in \mathbb{Z}$. The same construction can be applied to $D^b\mathrm{MHM}(X)$ and we define $\mathbb{L}^{1/2} = \mathbb{Q}(-1/2)[-1]$. By abuse of notation we will continue to write $\mathrm{MHM}(X)$ instead of $\mathrm{MHM}(X)^{1/2}$. We define

$$M\langle n \rangle = M(n/2)[n] = \mathbb{L}^{-n/2} M, \qquad M \in D^b\mathrm{MHM}(X). \tag{9}$$

Note that the weight of $M\langle n \rangle$ is equal to the weight of M.

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Let X be an irreducible algebraic variety of dimension d. The object

$$\mathbb{Q}_X = a_X^*\mathbb{Q} \in D^b\mathrm{MHM}(X), \quad a_X : X \to \mathbf{pt},$$

has weight ≤ 0. If X is smooth, then $\mathbb{Q}_X$ has weight 0 and $\mathbb{Q}_X[d] \in \mathrm{MHM}(X)$. Generally, we define the intersection complex

$$\mathbf{IC}_X = j_{!*}(\mathbb{Q}_U \langle d \rangle) \in \mathrm{MHM}(X), \tag{10}$$

where $j : U \hookrightarrow X$ is an embedding of a nonempty, open, smooth subvariety and

$$j_{!*}(M) = \Im({}^{\mathrm{p}}j_! M \to {}^{\mathrm{p}}j_* M) \in \mathrm{MHM}(X), \quad M \in \mathrm{MHM}(U). \tag{11}$$

The object $\mathbf{IC}_X$ is self-dual (meaning that $\mathbb{D}\mathbf{IC}_X \simeq \mathbf{IC}_X$) and has weight 0. The object $\mathrm{IC}_X = \mathrm{rat}(\mathbf{IC}_X)$ is the perverse intersection complex in $\mathrm{Perv}(\mathbb{Q}_X)$. We define the intersection cohomology

$$\mathrm{IH}^*(X,\mathbb{Q}) = H^*(X, \mathbf{IC}_X \langle -d \rangle) \tag{12}$$

so that $\mathrm{IH}^*(X,\mathbb{Q}) = H^*(X,\mathbb{Q})$ for smooth X.

Remark 2.11. Some authors define the intersection complex $\mathbf{IC}'_X = j_{!*}(\mathbb{Q}_U[d]) \in \mathrm{MHM}(X)$. It is pure of weight d and satisfies $\mathbb{D}\mathbf{IC}'_X \simeq \mathbf{IC}'_X(d)$ (the same is true for any pure object of weight d). One has $\mathbf{IC}'_X = \mathrm{Gr}_d^W H^d\mathbb{Q}_X$ [Sai89], where H^d is the dth cohomology in $D^b\mathrm{MHM}(X)$.

2.4 Degrees

For a graded abelian group $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and $x = \sum_{i\in\mathbb{Z}} x_i \in V$ with $x_i \in V_i$, let

$$\deg(x) = \sup\{i \mid x_i \neq 0\} \in \mathbb{Z} \sqcup \{-\infty\}. \tag{13}$$

If $n = \deg(x)$ is finite, we call x_n the leading term of x. If A is a graded integral domain and V is a graded torsion-free A-module, then

$$\deg(ax) = \deg(a) + \deg(x) \quad \forall a \in A,\ x \in V. \tag{14}$$

We extend the degree map to $\mathcal{F}(A) \otimes_A V$, where $\mathcal{F}(A)$ is the fraction field of A, by the formula

$$\deg(x/a) = \deg(x) - \deg(a) \quad \forall a \in A \setminus \{0\},\ x \in V. \tag{15}$$

For an abelian category $\mathcal{A}$ (respectively a triangulated category $\mathcal{D}$), let $K(\mathcal{A})$ (respectively $K(\mathcal{D})$) denote its Grothendieck group. Let $[M] \in K(\mathcal{D})$ denote the class of $M \in \mathcal{D}$. If $\mathcal{D}$ is a bounded t-category with the heart $\mathcal{A}$, then the canonical map $K(\mathcal{A}) \to K(\mathcal{D})$ is an isomorphism. If $\mathcal{D}$ is a mixed t-category with the heart $\mathcal{A}$, then the Grothendieck group $K(\mathcal{A}) \simeq K(\mathcal{D})$ is graded, where we define $\deg[M] = n$ for a pure object $M \in \mathcal{A}$ of weight n. If $\mathcal{A}$ is of finite length and C_n denotes the set of isomorphism classes of simple objects in $\mathcal{A}_n$, then $K(\mathcal{A}) \simeq \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}C_n$.

In particular, the Grothendieck group $K(\mathrm{MHM}(X)) \simeq K(D^b\mathrm{MHM}(X))$ is a graded module over the graded ring $K(\mathrm{MHM}(\mathbf{pt}))$ (cf. Section 3.3). Let

$$\mathbb{L} = [H_c^*(\mathbb{A}^1,\mathbb{Q})] = [\mathbb{Q}(-1)] \in K(\mathrm{MHM}(\mathbf{pt}))\text{and} \tag{16}$$

$$\mathbf{K}(\mathrm{MHM}(X)) = K(\mathrm{MHM}(X)) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{Z}[\mathbb{L}^{\pm 1/2}, (1-\mathbb{L}^n)^{-1} \mid n \geqslant 1]. \tag{17}$$

Note that $\mathbb{L}$ has degree 2. We extend the degree function to $\mathbf{K}(\mathrm{MHM}(X))$ so that $\mathbb{L}^{1/2}$ has degree 1. For $M \in D^b(\mathrm{MHM}(X))$, let $\deg(M) = \deg[M]$. For example, $\deg(\mathbb{Q}[n]) = 0$, while $w(\mathbb{Q}[n]) = n$ for $n \in \mathbb{Z}$. The duality functor $\mathbb{D}$ on $\mathrm{MHM}(X)$ induces the group homomorphism $\mathbb{D}$:

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$\mathbf{K}(\mathrm{MHM}(X)) \to \mathbf{K}(\mathrm{MHM}(X))$ such that $\mathbb{D}(\mathbb{L}^n a) = \mathbb{L}^{-n}\mathbb{D}(a)$ for $n \in \frac{1}{2}\mathbb{Z}$ and $a \in \mathbf{K}(\mathrm{MHM}(X))$. The class $[\mathbf{IC}_X] \in \mathbf{K}(\mathrm{MHM}(X))$ is self-dual (meaning that $\mathbb{D}[\mathbf{IC}_X] = [\mathbf{IC}_X]$).

2.5 Decompositions

We say that an object $M \in \mathrm{MHM}(X)$ is supported on a locally closed subvariety $Z \subseteq X$ if $M = j_{!*}{}^{\mathrm{p}}j^*M$ for the embedding $j: Z \hookrightarrow X$, meaning that the canonical map $M \to {}^{\mathrm{p}}j_*{}^{\mathrm{p}}j^*M$ induces an isomorphism $M \xrightarrow{\sim} j_{!*}{}^{\mathrm{p}}j^*M \hookrightarrow {}^{\mathrm{p}}j_*{}^{\mathrm{p}}j^*M$. For a simple object M, this means that Z contains a nonempty open subset of $\operatorname{supp} M$. We say that $M \in D^b\mathrm{MHM}(X)$ is supported on Z if every H^iM is supported on Z. In what follows, a partition of X means a finite collection $\mathcal{S}$ of locally closed subsets of X such that $X = \bigsqcup_{S \in \mathcal{S}} S$.

Lemma 2.12. *Let $\mathcal{S}$ be a partition of an algebraic variety X and $M \in \mathrm{MHM}(X)$ be a pure object. Then there is a unique decomposition $M = \bigoplus_{S \in \mathcal{S}} M_S$, where M_S is supported on S. We have $M_S = j_{S!*}{}^{\mathrm{p}}j_S^*M$.*

Proof. We can assume that M is simple. There exists a smooth, irreducible, locally closed $U \subset X$ such that $M = j_{!*}{}^{\mathrm{p}}j^*M$ for $j: U \hookrightarrow X$. Let $S \in \mathcal{S}$ be such that the dimension $d_{S\cap U}$ of $S \cap U$ is maximal. Then $S \cap U$ is open in U. We can substitute U by $U \cap S$ and assume that $U \subset S$. Then $M = j_{S!*}{}^{\mathrm{p}}j_S^*M$.

To prove uniqueness, we need to show that ${}^{\mathrm{p}}j^*j_{!*} = \mathrm{id}$ and ${}^{\mathrm{p}}i^*j_{!*} = 0$ for two embeddings $j: S \hookrightarrow X$ and $i: T \hookrightarrow X$ such that $S \cap T = \varnothing$. The first equation is clear. As $j_{!*}N = \Im({}^{\mathrm{p}}j_!N \to {}^{\mathrm{p}}j_*N)$ and ${}^{\mathrm{p}}i^*$ is right exact, it is enough to show that ${}^{\mathrm{p}}i^*{}^{\mathrm{p}}j_! = 0$. We have ${}^{\mathrm{p}}i^*{}^{\mathrm{p}}j_! = \tau_{\geqslant 0}i^*\tau_{\geqslant 0}j_! = \tau_{\geqslant 0}i^*j_! = 0$ as $i^*j_! = 0$ and $\tau_{\geqslant 0}i^*\tau_{\geqslant 0} = \tau_{\geqslant 0}i^*$. $\square$

Lemma 2.13. *Let $\mathcal{S}$ be a partition of an algebraic variety X and $M \in D^b\mathrm{MHM}(X)$ be a pure object. Then there is a decomposition $M = \bigoplus_{S \in \mathcal{S}} M_S$, where M_S is supported on S. We have $M_S \simeq \bigoplus_i (j_{S!*}{}^{\mathrm{p}}j_S^*H^iM)[-i]$.*

Proof. Let $e_S = j_{S!*}{}^{\mathrm{p}}j_S^*$ and $\bar{e}_S(M) = \bigoplus_i (e_SH^iM)[-i]$ for $S \in \mathcal{S}$. We have $e_S^2 = e_S$ and $e_Se_T = 0$ for $S \neq T$. Therefore $\bar{e}_S^2 = \bar{e}_S$ and $\bar{e}_S\bar{e}_T = 0$ for $S \neq T$. Using $M \simeq \bigoplus_i (H^iM)[-i]$ and applying the previous result to every H^iM, we obtain $M \simeq \bigoplus_S \bar{e}_S(M)$. Let $M = \bigoplus_S M_S$ be another decomposition with M_S supported on S. We have $\bar{e}_S(M_T) = 0$ for $S \neq T$, hence $\bar{e}_S(M) = \bar{e}_S(M_S)$. By the same argument as before, we have $M_S \simeq \bigoplus_T \bar{e}_T(M_S) = \bar{e}_S(M_S) = \bar{e}_S(M)$. $\square$

2.6 Defects

Let $\pi: X \to Y$ be a projective morphism between algebraic varieties. For $i \in \mathbb{N} \sqcup \{-\infty\}$, let $Y_i = \left\{y \in Y \,\middle|\, \dim \pi^{-1}(y) = i\right\}$ (note that $\dim \varnothing = -\infty$). The subsets $Y_{\leqslant k} = \bigsqcup_{i \leqslant k} Y_i$ are open in Y. For a subvariety $Z \subseteq Y$, we define the (fibre) defect (cf. [GM88, dCM03])

$$\delta(\pi, Z) = \max\left\{2i + d_{Z\cap Y_i} - d_X \mid i \geqslant 0\right\}. \tag{18}$$

If $\mathcal{S}$ is a finite partition of Y such that the fibres of π over $S \in \mathcal{S}$ have dimension c_S, then

$$\delta(\pi, Z) = \max_{S \in \mathcal{S}} \delta(\pi, Z \cap S), \quad \delta(\pi, Z \cap S) = 2c_S + d_{Z\cap S} - d_X. \tag{19}$$

Note that $\delta(\pi) = \delta(\pi, Y) = d_{X\times_Y X} - d_X \geqslant 0$. A (surjective) morphism π is called semismall if $\delta(\pi) = 0$ and small if $\delta(\pi) = 0$ and $\delta(\pi, Y_i) = 0$ only for $i = 0$.

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DEFINITION 2.14. A projective morphism $\pi : X \to Y$ is called n-small (or virtually small if n is clear from the context) if there exists an open subset $U \subseteq Y$ such that $\delta(\pi, Y\backslash U) < n$ and $U \subseteq Y_n$. Note that $\delta(\pi) \leqslant n$ if π is n-small.

Remark 2.15. If $\delta(\pi) < n$, we can assume that the open set U above is empty. If $\delta(\pi) = n$, we can assume that U is smooth and equidimensional (of dimension $d_X - n$). If X is smooth, we can assume that π is smooth over U.

LEMMA 2.16. *If $\pi : X \to Y$ is a projective surjective morphism and X, Y are irreducible, then $\pi : X \to Y$ is n-small if and only if $\delta(\pi, Y_i) < n$ for $i \neq n$ and $\delta(\pi, Y_n) \leqslant n$. If $\delta(\pi, Y_n) = n$, then $Y_n \subseteq Y$ is open and $n = d_X - d_Y$.*

Proof. Let π be n-small and $U \subseteq Y_n$ be such that $\delta(\pi, Y\backslash U) < n$. Then $Y_i \subseteq Y\backslash U$ for $i \neq n$, hence $\delta(\pi, Y_i) < n$. We also have $\delta(\pi, Y_n) \leq \delta(\pi) \leqslant n$. Conversely, assume that $\delta(\pi, Y_i) < n$ for $i \neq n$ and $\delta(\pi, Y_n) \leqslant n$. If $\delta(\pi, Y_n) < n$, then $\delta(\pi) < n$ and we can take $U = \varnothing$. If $\delta(\pi, Y_n) = n$, then $d_X = n + d_{Y_n} = d_{\pi^{-1}(Y_n)}$, hence $\pi^{-1}(Y_n) \subseteq X$ is open. The subset $\pi^{-1}(Y_{<n})$ is also open, hence $\pi^{-1}(Y_{<n}) = \varnothing$ and $Y_{<n} = \varnothing$. Therefore $Y_n \subseteq Y$ is open and we can take $U = Y_n$. We have $d_X = n + d_{Y_n} = n + d_Y$, hence $n = d_X - d_Y$. □

In what follows, let $\mathcal{D}$ denote the t-category $D^b\mathrm{MHM}(X)$ with the standard t-structure or the t-category $D^b_c(\mathbb{Q}_X)$ with the perverse t-structure. These t-structures are compatible under the functor $\mathrm{rat} : D^b\mathrm{MHM}(X) \to D^b_c(\mathbb{Q}_X)$. For $K \in D^b_c(\mathbb{Q}_X)$, let $\mathsf{H}^i K \in \mathrm{Sh}(\mathbb{Q}_X)$ denote the ith cohomology sheaf and ${}^{\mathrm{p}}H^i K \in \mathrm{Perv}(\mathbb{Q}_X)$ denote the ith cohomology object with respect to the perverse t-structure.

LEMMA 2.17. *Let $M \in \mathcal{D} = D^b\mathrm{MHM}(Y)$ and $\mathcal{S}$ be a partition of Y such that $j_S^* M \in \mathcal{D}^{\leqslant n}$ for the embeddings $j_S : S \hookrightarrow Y$, $S \in \mathcal{S}$. Then $M \in \mathcal{D}^{\leqslant n}$.*

Proof. Let $U \subseteq Y$ be open and $j : U \hookrightarrow Y$, $i : Y\backslash U \hookrightarrow Y$ be the corresponding embeddings. For every $M \in D^b\mathrm{MHM}(Y)$ there is a triangle $j_! j^* M \to M \to i_! i^* M \to$. If $i^* M \in \mathcal{D}^{\leqslant n}$ and $j^* M \in \mathcal{D}^{\leqslant n}$, then $M \in \mathcal{D}^{\leqslant n}$ as $i_!$, $j_!$ are right t-exact. Given a partition, we can refine it so that it satisfies the frontier condition: if $S \cap \bar{T} \neq \varnothing$, then $S \subseteq \bar{T}$ for $S, T \in \mathcal{S}$. Note that we still have $j_S^* M \in \mathcal{D}^{\leqslant n}$ for all S as the restriction functor is right exact. We can apply the previous argument inductively and deduce that $M \in \mathcal{D}^{\leqslant n}$. □

LEMMA 2.18. *Let $\pi : X \to Y$ be a projective morphism such that X is smooth. For a locally closed embedding $j : Z \hookrightarrow Y$ we have $j^* \pi_* \mathbf{IC}_X \in D^{\leqslant \delta(\pi, Z)}\mathrm{MHM}(X)$.*

Proof. By Lemma 2.17 we can assume that $Z \subset Y_k$ for some $k \geqslant 0$. Consider the following Cartesian square.

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{j} & Y \end{array}$$

An object $K \in D^b_c(\mathbb{Q}_X)$ is contained in ${}^{\mathrm{p}}D^{\leqslant n}_c(\mathbb{Q}_X) \iff \dim(\mathrm{supp}\mathsf{H}^i K) \leqslant -i + n$ for all $i \in \mathbb{Z}$. We have $j^* \pi_* \mathrm{IC}_X = \pi'_* j'^* \mathbb{Q}_X[d_X]$. The object $K = j'^* \mathbb{Q}_X[d_X]$ satisfies $\dim(\mathrm{supp}\mathsf{H}^i K) = d_{X'} = -i + (d_{X'} - d_X)$ for $i = -d_X$ and $\mathsf{H}^i K = 0$ otherwise. This implies that $K \in {}^{\mathrm{p}}D^{\leqslant d_{X'} - d_X}_c(\mathbb{Q}_X)$. The functor π'_* has amplitude $[-k, k]$. Therefore $j^* \pi_* \mathrm{IC}_X = \pi'_* K \in {}^{\mathrm{p}}D^{\leqslant k + d_{X'} - d_X} = {}^{\mathrm{p}}D^{\leq \delta(\pi, Z)}(\mathbb{Q}_X)$.

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The functor $\mathrm{rat}: D^b\mathrm{MHM}(X) \to D^b_c(\mathbb{Q}_X)$ sends $j^*\pi_*\mathbf{IC}_X$ to $j^*\pi_*\mathrm{IC}_X$ and preserves t-structures, and hence $j^*\pi_*\mathbf{IC}_X \in D^{\leqslant \delta(\pi,Z)}\mathrm{MHM}(X)$. $\square$

Lemma 2.19. *Let $f: X \to Y$ be a morphism between algebraic varieties and $M \in \mathcal{D} = D^b\mathrm{MHM}(Y)$ be a pure object of weight 0 such that $f^*M \in \mathcal{D}^{\leqslant n}$. Then $\deg({}^{\mathrm{p}}f^*H^iM) \leqslant n$ for all $i \in \mathbb{Z}$.*

Proof. Since M is pure, we have $M \simeq \bigoplus_{i\in\mathbb{Z}}(H^iM)[-i]$. Therefore, $f^*(H^iM)[-i] \in \mathcal{D}^{\leqslant n}$ and $f^*(H^iM) \in \mathcal{D}^{\leqslant n-i}$. If ${}^{\mathrm{p}}f^*(H^iM) \neq 0$, then $i \leqslant n$. As H^iM has weight i, the object ${}^{\mathrm{p}}f^*(H^iM)$ has weight at most $i \leqslant n$. $\square$

Lemma 2.20. *Let $\pi: X \to Y$ be a projective morphism with smooth X. Let $\mathcal{S}$ be a partition of Y and $\pi_*\mathbf{IC}_X = \bigoplus_{S\in\mathcal{S}} M_S$ be a decomposition such that M_S is supported on S. Then $\deg M_S \leq \delta(\pi, S)$ for all $S \in \mathcal{S}$.*

Proof. We have $M_S \simeq \bigoplus_i (j_{S!*}{}^{\mathrm{p}}j_S^*H^iM)[-i]$ for $M = \pi_*\mathbf{IC}_X$, and hence we need to show that $\deg {}^{\mathrm{p}}j_S^*H^iM \leq \delta(\pi, S)$. The object M is pure of weight 0 and by Lemma 2.18 we have $j_S^*M \in \mathcal{D}^{\leq \delta(\pi,S)}$. By Lemma 2.19 we obtain $\deg {}^{\mathrm{p}}j_S^*H^iM \leq \delta(\pi, S)$. $\square$

2.7 The leading term

If $\pi: X \to Y$ is a projective morphism and is a (topological) fibre bundle with all fibres irreducible and of dimension n, then (cf. [GS93])

$$H^n\pi_*(\mathbb{Q}_X[d_X]) \simeq \mathbb{Q}_Y[d_Y](-n). \tag{20}$$

If X and Y are smooth (and irreducible), we obtain (cf. Appendix A)

$$H^n\pi_*\mathbf{IC}_X \simeq \mathbf{IC}_Y(-n/2). \tag{21}$$

Theorem 2.21. *Let $\pi: X \to Y$ be an n-small morphism with X smooth, and let $U \subseteq Y$ be a smooth equidimensional open subset such that $\delta(\pi, Y\backslash U) < n$, $\delta(\pi, U) = n$, and the fibres of π over U are smooth, connected, and of dimension n. Then $\pi_*\mathbf{IC}_X$ has degree n, and its leading term is $\mathbf{IC}_{\overline{U}}\langle -n\rangle$ (see Section 2.4).*

Proof. Let $j: U \hookrightarrow Y$ and $i: Z = Y\backslash U \hookrightarrow Y$ be the embeddings. For $M = \pi_*\mathbf{IC}_X$, we have $M = M_U \oplus M_Z$, where $M_U \simeq \bigoplus_k (j_{!*}j^*H^kM)[-k]$ and $M_Z \simeq \bigoplus_k (i_{!*}{}^{\mathrm{p}}i^*H^kM)[-k]$. We have $\deg(M_Z) \leq \delta(\pi, Z) < n$ by Lemma 2.20. On the other hand, $j^*H^kM = H^kj^*M = H^k\pi'_*\mathbf{IC}_{X'}$ for $\pi': X' = \pi^{-1}(U) \to U$. The object $H^k\pi'_*\mathbf{IC}_{X'}$ is zero for $k > n$ and has degree $< n$ for $k < n$. Morever, we have $H^n\pi'_*\mathbf{IC}_{X'} = \mathbf{IC}_U(-n/2)$, and hence the leading term of M_U is $(j_{!*}\mathbf{IC}_U(-n/2))[-n] = \mathbf{IC}_{\overline{U}}(-n/2)[-n]$. $\square$

We prove the following well-known result (cf. [GS93, Theorem 5]) for completeness.

Theorem 2.22. *Let $\pi: X \to Y$ be a semismall morphism with smooth X. Let $\mathcal{S}$ be a smooth partition of Y such that π is a (topological) fibre bundle with irreducible fibres over every $S \in \mathcal{S}$. Then $\pi_*\mathbf{IC}_X = \bigoplus_{\delta(\pi,S)=0}\mathbf{IC}_{\bar{S}}$.*

Proof. We have $M = \pi_*\mathbf{IC}_X \in \mathcal{D}^{\leq 0}$ by Lemma 2.18. As M is self-dual, we conclude that $M \in \mathrm{MHM}(X)$. By Lemma 2.12 we have $M = \bigoplus_S {}^{\mathrm{p}}j_{S!*}{}^{\mathrm{p}}j_S^*M$. If $\delta(\pi, S) < 0$, then $j_S^*M \in \mathcal{D}^{<0}$, and hence ${}^{\mathrm{p}}j_S^*M = 0$. Let us assume that $\delta(\pi, S) = 0$ and let $n = d_X - d_{X'}$ be the fibre dimension,

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where $X' = \pi^{-1}(S)$. Then

$$^{\mathrm{p}}j_S^* M = H^0(\pi'_* \mathbb{Q}_{X'}[d_X](d_X/2)) = H^n(\pi'_* \mathbb{Q}_{X'}[d_{X'}])(d_X/2) \simeq \mathbb{Q}_S[d_S](d_X/2 - n) = \mathbf{IC}_S,$$

where we applied (20) to $\pi' : X' \to S$. $\square$

3. λ-rings over commutative monoids

3.1 Pre-λ-rings of motivic functions

Let $\mathrm{Sch} = \mathrm{Sch}_K$ be the category of algebraic schemes over a field K of characteristic zero, by which we mean schemes X equipped with a finite type morphism $a_X : X \to \mathbf{pt} = \mathbf{Spec}(K)$. For a scheme (or an Artin stack) S locally of finite type over K, let Sch/S be the category of morphisms $X \to S$ over K, where $X \in \mathrm{Sch}$. For a morphism $f : S \to T$ we have the functor

$$f_! : \mathrm{Sch}/S \to \mathrm{Sch}/T, \quad [X \to S] \mapsto [X \to S \to T].$$

If $f : S \to T$ is of finite type, then $f_!$ has the right adjoint functor

$$f^* : \mathrm{Sch}/T \to \mathrm{Sch}/S, \quad [X \to T] \mapsto [X \times_T S \to S].$$

We also have the exterior product

$$\boxtimes : \mathrm{Sch}/S \times \mathrm{Sch}/T \to \mathrm{Sch}/(S \times T), \qquad [X \to S] \boxtimes [Y \to T] = [X \times Y \to S \times T].$$

Let $K(\mathrm{Sch}/S)$ be the Grothendieck group of algebraic schemes over S (see e.g. [Joy07, Bri12]). The above functors induce morphisms between Grothendieck groups. The Grothendieck group $K(\mathrm{Sch}) = K(\mathrm{Sch}/\mathbf{pt})$ has a commutative ring structure defined by $[X] \cdot [Y] = [X \times Y]$. The Grothendieck group $K(\mathrm{Sch}/S)$ is a module over $K(\mathrm{Sch})$. We define (cf. (17))

$$\mathbb{L} = [\mathbb{A}^1] \in K(\mathrm{Sch}), \text{and} \tag{22}$$

$$\mathbf{K}(\mathrm{Sch}/S) = K(\mathrm{Sch}/S) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{Z}[\mathbb{L}^{\pm 1/2}, (1 - \mathbb{L}^n)^{-1} : n \geqslant 1]. \tag{23}$$

In particular, the elements

$$[\mathbb{P}^n] = \frac{1 - \mathbb{L}^{n+1}}{1 - \mathbb{L}}, \quad [\mathbf{GL}_n] = \prod_{i=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^i) \tag{24}$$

are invertible in $\mathbf{K}(\mathrm{Sch}/\mathbf{pt})$. Consider a morphism $f : \mathfrak{X} \to S$, where $\mathfrak{X}$ is a finite type Artin stack with affine stabilizers. Then there exists an algebraic variety X with an action of the group $\mathbf{GL}_n$ and a geometric bijection $X/\mathbf{GL}_n \to \mathfrak{X}$ (see e.g. [Bri12]). The class

$$[\mathfrak{X} \to S] = [X \to \mathfrak{X} \to S]/[\mathbf{GL}_n] \in \mathbf{K}(\mathrm{Sch}/S) \tag{25}$$

depends only on the morphism f (see e.g. [Bri12]).

Let (S, μ, η) be a commutative monoid in the category of schemes (or Artin stacks) over K. We equip Sch/S with a symmetric monoidal structure having the tensor product

$$[X \xrightarrow{f} S] \otimes [Y \xrightarrow{g} S] = \mu_!(f \boxtimes g) = [X \times Y \to S \times S \xrightarrow{\mu} S] \tag{26}$$

and the unit object $\mathbb{1} = [\mathbf{pt} \xrightarrow{\eta} S]$. This tensor product induces the ring structure on $K(\mathrm{Sch}/S)$. We equip $K(\mathrm{Sch}/S)$ with a pre-λ-ring structure having σ-operations

$$\sigma^n[X \to S] = [S^n(X) \to S^n(S) \xrightarrow{\mu} S], \quad n \geqslant 1, \tag{27}$$

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where $S^n(X) = X^n/S_n$. We extend the pre-λ-ring structure to $\mathbf{K}(\mathrm{Sch}/S)$ by the formula

$$\sigma^n(y^m a) = y^{mn}\sigma^n(a), \quad y = -\mathbb{L}^{1/2},\ a \in \mathbf{K}(\mathrm{Sch}/S). \tag{28}$$

In particular, $\mathbf{pt}$ is a commutative monoid, and hence $K(\mathrm{Sch})$ and $\mathbf{K}(\mathrm{Sch})$ are pre-λ-rings. The maps $\eta_! : K(\mathrm{Sch}) \to K(\mathrm{Sch}/S)$ and $a_{S!} : K(\mathrm{Sch}/S) \to K(\mathrm{Sch})$ (induced by the unit $\eta : \mathbf{pt} \to S$ and the projection $a_S : S \to \mathbf{pt}$) are homomorphisms of pre-λ-rings.

3.2 Graded pre-λ-rings and completions

More generally, let Λ be a commutative monoid and $S = \bigsqcup_{\gamma\in\Lambda} S_\gamma$ be a Λ-graded commutative monoid in the category of schemes over K. The pre-λ-rings $K(\mathrm{Sch}/S) = \bigoplus_{\gamma\in\Lambda} K(\mathrm{Sch}/S_\gamma)$ and $\mathbf{K}(\mathrm{Sch}/S) = \bigoplus_{\gamma\in\Lambda} \mathbf{K}(\mathrm{Sch}/S_\gamma)$ are Λ-graded pre-λ-rings, meaning that

$$\deg(ab) = \deg(a) + \deg(b), \quad \deg\sigma^n(a) = n\deg a. \tag{29}$$

In particular, we consider the commutative monoid $\mathbf{\Lambda} = \bigsqcup_{\gamma\in\Lambda} \mathbf{pt}_\gamma$ with $\mathbf{pt}_\gamma = \mathbf{Spec}(K)$ in the category of schemes over K. Then $K(\mathrm{Sch}/\mathbf{\Lambda}) = K(\mathrm{Sch})[\Lambda] = \bigoplus_{\gamma\in\Lambda} K(\mathrm{Sch})t^\gamma$ is a Λ-graded pre-λ-ring with the product and σ-operations

$$[X]t^\gamma \cdot [Y]t^{\gamma'} = [X\times Y]t^{\gamma+\gamma'}, \quad \sigma^n([X]t^\gamma) = [S^nX]t^{n\gamma}.$$

The degree map

$$\deg : S \to \mathbf{\Lambda}, \quad S_\gamma \ni x \mapsto \mathbf{pt}_\gamma \tag{30}$$

is a homomorphism of commutative monoids and induces a homomorphism of pre-λ-rings $\deg_! : K(\mathrm{Sch}/S) \to K(\mathrm{Sch}/\mathbf{\Lambda}) = K(\mathrm{Sch})[\Lambda]$.

Assume that there exists a filtration $\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \ldots$ such that

$$\Lambda_i + \Lambda_j \subseteq \Lambda_{i+j}, \quad |\Lambda\backslash\Lambda_i| < \infty, \quad \bigcap_{i\geqslant 0}\Lambda_i = \varnothing. \tag{31}$$

For example, if $\Lambda \subseteq \mathbb{N}^n$ for some $n \geqslant 1$, we can define $\Lambda_k = \{\gamma\in\Lambda \mid \sum_{i=1}^n \gamma_i \geqslant k\}$. For a Λ-graded ring $A = \bigoplus_{\gamma\in\Lambda} A_\gamma$, we define its completion

$$\widehat{A} = \varprojlim_k \Big(A/\bigoplus_{\gamma\in\Lambda_k} A_\gamma\Big) \simeq \prod_{\gamma\in\Lambda} A_\gamma, \tag{32}$$

where the last isomorphism is an isomorphism of abelian groups. In particular, the completion $\widehat{\mathbf{K}}(\mathrm{Sch}/S) \simeq \prod_{\gamma\in\Lambda} \mathbf{K}(\mathrm{Sch}/S_\gamma)$ inherits the structure of a pre-λ-ring. Let us assume that $\Lambda_1 = \Lambda_+ = \Lambda\setminus\{0\}$, and consider the ideal $\widehat{\mathbf{K}_+}(\mathrm{Sch}/S) = \prod_{\gamma\in\Lambda_+} \mathbf{K}(\mathrm{Sch}/S_\gamma)$. The plethystic exponential

$$\mathrm{Exp} : \widehat{\mathbf{K}_+}(\mathrm{Sch}/S) \to 1 + \widehat{\mathbf{K}_+}(\mathrm{Sch}/S), \quad a \mapsto \sum_{n\geqslant 0}\sigma^n(a), \tag{33}$$

where $\sigma^0(a) = 1$, is a group isomorphism (between the additive and the multiplicative groups).

3.3 λ-rings of mixed Hodge modules

Let (S, μ, η) be a commutative monoid in the category of complex algebraic varieties such that the product $\mu : S\times S \to S$ is a finite map. We will equip the category $\mathrm{MHM}(S)$ with a symmetric monoidal structure in a similar way to the case of Sch/S considered earlier (see (26)).

Theorem 3.1. *The category* $\mathrm{MHM}(S)$ *with the tensor product*

$$\otimes : \mathrm{MHM}(S)\times\mathrm{MHM}(S) \to \mathrm{MHM}(S), \qquad E\otimes F = \mu_!(E\boxtimes F),$$

and the unit object $\mathbb{1} = \eta_! \mathbb{Q}$ *is a symmetric monoidal category. The tensor product is exact and w-exact.*

Proof. Let $\sigma : S \times S \to S \times S$ be the permutation of factors. By [MSS11, Theorem 1.9] (applied to the variety $X = S \sqcup S$), there exists a canonical isomorphism

$$\sigma^{\#} : E \boxtimes F \to \sigma_*(F \boxtimes E)$$

satisfying $\sigma_* \sigma^{\#} \circ \sigma^{\#} = id$. Applying the pushforward $\mu_!$ and using commutativity of the monoid, we obtain an isomorphism $\sigma_{EF} : E \otimes F \to F \otimes E$ satisfying $\sigma_{EF} \circ \sigma_{FE} = id$. This is the required braiding for the tensor product. We have $\mathbb{1} \otimes F = \mu_!(\eta_! \mathbb{Q} \boxtimes F) = \mu_!(\eta \times \mathrm{id})_!(\mathbb{Q} \boxtimes F) = F$. The tensor product is exact (respectively w-exact) as both $\mu_!$ and $\boxtimes$ are exact (respectively w-exact). □

The above tensor product $\otimes$ on $\mathrm{MHM}(S)$ should not be confused with the tensor product $\otimes^*$ on $D^b\mathrm{MHM}(S)$ defined in Remark 2.10, which is only right w-exact.

Remark 3.2. Without the assumption that μ is finite, we can similarly equip the category $D^b\mathrm{MHM}(S)$ with a symmetric monoidal structure. There is a monoidal functor

$$\chi_c : \mathrm{Sch}/S \to D^b\mathrm{MHM}(S), \quad [X \xrightarrow{f} S] \mapsto f_! \mathbb{Q}_X. \tag{34}$$

Let $\mathcal{A} = \mathrm{MHM}(S)$ (or any other Karoubian symmetric monoidal category linear over $\mathbb{Q}$). By [Get96, Hei07], the split Grothendieck group $\overline{K}(\mathcal{A})$ (with relations induced by direct sums) is a (special) λ-ring, with the product and σ-operations defined by

$$[E] \cdot [F] = [E \otimes F], \tag{35}$$

$$\sigma^n[E] = [S^n(E)], \quad S^n(E) = \Im\left(\frac{1}{n!} \sum_{\sigma \in S_n} \sigma\right) \subseteq E^{\otimes n}, \tag{36}$$

where $e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \mathrm{End}(E^{\otimes n})$ is an idempotent and $\Im(e)$ is obtained by its splitting.

Remark 3.3. More generally, for any partition $\lambda \vdash n$, let V_λ be the corresponding simple representation of the group S_n. We define the Schur functor (cf. [Del02])

$$S_\lambda : \mathcal{A} \to \mathcal{A}, \quad E \mapsto \mathrm{Hom}_{S_n}(V_\lambda, E^{\otimes n}), \tag{37}$$

where we first construct $\mathrm{Hom}(V_\lambda, E^{\otimes n})$ as a direct sum of $\dim V_\lambda$ copies of $E^{\otimes n}$ and then take the S_n-invariant subobject of $\mathrm{Hom}(V_\lambda, E^{\otimes n})$ by splitting the idempotent $e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$. In particular, for the trivial representation we obtain $S^n(E) = S_{(n)}(E)$ and for the alternating representation we obtain

$$\Lambda^n(E) = S_{(1^n)}(E) = \Im\left(\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma\right) \subseteq E^{\otimes n}. \tag{38}$$

We have $s_\lambda[E] = [S_\lambda(E)]$, where s_λ is the symmetric Schur function. In particular, $\lambda^n[E] = [\Lambda^n(E)]$.

THEOREM 3.4. *The λ-ring structure on the split Grothendieck group $\overline{K}(\mathrm{MHM}(S))$ descends to a λ-ring structure on $K(\mathrm{MHM}(S))$ with the unit element $1 = [\eta_* \mathbb{Q}]$. This λ-ring is $\mathbb{Z}$-graded (see (29)) by weight.*

Proof. The above product on $\overline{K}(\mathrm{MHM}(S))$ descends to $K(\mathrm{MHM}(S))$ as the tensor product is exact. The λ-structure descends to $K(\mathrm{MHM}(S))$ by [MS12, Lemma 2.1] and [Big10, Lemma 4.1].

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We have seen in Section 2.4 that $K(\mathrm{MHM}(S))$ is a $\mathbb{Z}$-graded abelian group. The product preserves degrees since the tensor product $\otimes$ on $\mathrm{MHM}(S)$ is w-exact. To see that the λ-structure respects the grading (see (29)), we note that if $E\in\mathrm{MHM}(S)$ is pure of weight m, then $E^{\otimes n}$ is pure of weight mn and so is $S^n(E)\subseteq E^{\otimes n}$. □

The monoidal functor defined in (34) induces a morphism of pre-λ-rings (cf. [Sch09, MS12])

$$\chi_c: K(\mathrm{Sch}/S)\to K(\mathrm{MHM}(S)),\quad [X\xrightarrow{f}S]\mapsto[f_!\mathbb{Q}_X]. \tag{39}$$

We extend the λ-ring structure to $\mathbf{K}(\mathrm{MHM}(S))$ using (28). Then χ_c extends to a morphism of pre-λ-rings $\chi_c:\mathbf{K}(\mathrm{Sch}/S)\to\mathbf{K}(\mathrm{MHM}(S))$.

Remark 3.5. Let $\mathcal{D}$ be a (Karoubian) bounded t-category with a t-exact symmetric monoidal structure (meaning that it preserves the heart $\mathcal{A}$). Then the λ-ring structure on $\overline{K}(\mathcal{D})$ descends to a λ-ring structure on the Grothendieck group $K(\mathcal{D})$ (with relations induced by distinguished triangles, see [Big10, Section 4]). The canonical isomorphism $K(\mathcal{A})\xrightarrow{\sim}K(\mathcal{D})$ is an isomorphism of λ-rings. We can apply this statement to $\mathcal{D}=D^b\mathrm{MHM}(S)$, where (S,μ,η) is a commutative monoid in the category of complex algebraic varieties with finite maps μ,η.

Note that we use the notation $\mathbb{L}$ both for the class $\chi_c[\mathbb{A}^1]=[H_c^*(\mathbb{A}^1,\mathbb{Q})]\in K(\mathrm{MHM}(\mathbf{pt}))$ (16) and for the class $[\mathbb{A}^1]\in K(\mathrm{Sch})$ (22), depending on the context. For a smooth (equidimensional) algebraic variety X of dimension d (or a smooth Artin stack with affine stabilizers), we define its virtual class

$$[X]_{\mathrm{vir}}=\mathbb{L}^{-d/2}[X]\in\mathbf{K}(\mathrm{Sch}). \tag{40}$$

For example,

$$[\mathbb{P}^n]_{\mathrm{vir}}=\mathbb{L}^{-n/2}(1+\mathbb{L}+\cdots+\mathbb{L}^n)=\mathbb{L}^{-n/2}+\mathbb{L}^{-n/2+1}+\cdots+\mathbb{L}^{n/2},\text{ and} \tag{41}$$

$$[\mathbf{GL}_n]_{\mathrm{vir}}=\mathbb{L}^{-n^2/2}\prod_{i=0}^{n-1}(\mathbb{L}^n-\mathbb{L}^i)=\mathbb{L}^{-n/2}\prod_{i=1}^{n}(\mathbb{L}^i-1). \tag{42}$$

Note that $\mathbf{IC}_X=\mathbb{Q}_X\langle d\rangle=\mathbb{L}^{-d/2}\mathbb{Q}_X$, and hence the class of

$$H_c(X,\mathbf{IC}_X)=\mathbb{L}^{-d/2}H_c(X,\mathbb{Q})$$

is equal to $\chi_c[X]_{\mathrm{vir}}$, which we will also denote by $[X]_{\mathrm{vir}}$ by abuse of notation.

Let Λ be a commutative monoid admitting a filtration satisfying (31), and $\Lambda_1=\Lambda_+=\Lambda\setminus\{0\}$. Let $S=\bigsqcup_{\gamma\in\Lambda}S_\gamma$ be a Λ-graded commutative monoid in the category of complex algebraic varieties (so that the S_γ are of finite type) such that the product $\mu:S_\gamma\times S_{\gamma'}\to S_{\gamma+\gamma'}$ is finite. We define

$$K(\mathrm{MHM}(S))=\bigoplus\nolimits_{\gamma\in\Lambda}K(\mathrm{MHM}(S_\gamma))$$

and equip it with the $\mathbb{Z}$-graded (and Λ-graded) λ-ring structure using the same formulas as before. We extend this λ-ring structure to $\mathbf{K}(\mathrm{MHM}(S))=\bigoplus_{\gamma\in\Lambda}\mathbf{K}(\mathrm{MHM}(S_\gamma))$ using (28). There is a commutative diagram

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Sch}/S) \arrow[r, "\deg_!"] \arrow[d, "\chi_c"'] & \mathbf{K}(\mathrm{Sch}/\mathbf{\Lambda})=\mathbf{K}(\mathrm{Sch})[\Lambda] \arrow[d, "\chi_c"] \\
\mathbf{K}(\mathrm{MHM}(S)) \arrow[r, "\deg_!"] & \mathbf{K}(\mathrm{MHM}(\mathbf{\Lambda}))=\mathbf{K}(\mathrm{MHM}(\mathbf{pt}))[\Lambda]
\end{tikzcd} \tag{43}$$

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where χ_c is defined in (39), $\deg : S \to \mathbf{\Lambda}$ is the degree map (30), and all arrows are homomorphisms of Λ-graded pre-λ-rings. These maps induce homomorphisms between completions (32).

As in (33), we have the plethystic exponential

$$\operatorname{Exp}: \widehat{\mathbf{K}_+}(\mathrm{MHM}(S)) \to 1 + \widehat{\mathbf{K}_+}(\mathrm{MHM}(S)), \quad a \mapsto \sum_{n\geqslant 0} \sigma^n(a). \tag{44}$$

In terms of the Adams operations, Exp and its inverse (called the plethystic logarithm) are

$$\operatorname{Exp}(a) = \exp\left(\sum_{n\geqslant 1} \frac{1}{n}\psi^n(a)\right), \quad \operatorname{Log}(1+a) = \sum_{n\geqslant 1} \frac{\mu(n)}{n}\psi^n(\log(1+a)), \tag{45}$$

where μ is the Möbius function.

4. Moduli spaces of framed objects

Let $\mathcal{M}_\gamma$ (respectively $\mathcal{M}^{\mathsf{s}}_\gamma$) denote the moduli space of semistable (respectively stable) vector bundles over a curve C of type $\gamma = (r,d) \in \mathbb{Z}^2$. The moduli space $\mathcal{M}^{\mathsf{s}}_\gamma$ is smooth, while the moduli space $\mathcal{M}_\gamma$ can have singularities if r and d are not coprime. On the other hand, we can construct the moduli space $\mathcal{M}^{\mathsf{f}}_\gamma$ of stable framed vector bundles, consisting of pairs (E,s), where E is a vector bundle over C of type γ and $s \in \Gamma(C,E)$ is a section. Under certain conditions, this moduli space is smooth and the canonical morphism $\pi : \mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma$ is projective. We will show that the fibres of π can be identified with moduli spaces of nilpotent quiver representations, and we will use this result to prove that the morphism π is virtually small (see Definition 2.14). The properties of virtually small maps (see Theorem 2.21) will be used later to analyse the degrees of the components of $\pi_* \mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}$. In what follows, we introduce moduli spaces of framed objects in a setting more general than that of vector bundles on a curve. A closely related treatment of framed objects in abelian categories, together with their moduli spaces, can be found in [Moz13c].

4.1 Stability of framed objects

Let $\mathrm{Vec}_{\mathrm{f}}$ be the category of finite-dimensional vector spaces over a field K. Let $\mathcal{A}$ be an abelian K-linear category and $\Phi : \mathcal{A} \to \mathrm{Vec}_{\mathrm{f}}$ be a left exact functor, to be called a framing functor.

Definition 4.1. A framed object is a pair (E,s), where $E \in \mathcal{A}$ and $s \in \Phi(E)$. We call it stable if for every $F \subseteq E$ with $s \in \Phi(F)$ we have $F = E$.

Let $Z = -\mathsf{d} + \mathsf{r}\mathbf{i} : \Gamma = K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$ (see e.g. [Bri07]), where $\mathsf{d}, \mathsf{r} \in \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{R})$. For every $0 \neq E \in \mathcal{A}$, we have either $\mathsf{r}(E) > 0$ or $\mathsf{r}(E) = 0$ and $\mathsf{d}(E) > 0$. We define the slope of E

$$\mu_Z(E) = \frac{\mathsf{d}(E)}{\mathsf{r}(E)} \in \mathbb{R} \sqcup \{+\infty\}$$

and we say that an object E is Z-semistable (respectively Z-stable) if, for every proper $0 \neq F \subset E$, we have $\mu_Z(F) \leq \mu_Z(E)$ (respectively $\mu_Z(F) < \mu_Z(E)$). For $\mu \in \mathbb{R}$, let $\mathcal{A}^\mu \subseteq \mathcal{A}$ be the subcategory of Z-semistable objects having slope μ (including the zero object). This is a weakly Serre subcategory of $\mathcal{A}$, meaning an abelian subcategory closed under extensions, kernels and cokernels.

It will be convenient to extend the notion of framed objects (E,s) as follows. Let $\mathcal{A}_{\mathsf{f}}$ be the category consisting of triples (E,V,s), where $E \in \mathcal{A}$, $V \in \mathrm{Vec}_{\mathrm{f}}$ and $s : V \to \Phi(E)$ is linear. The category $\mathcal{A}_{\mathsf{f}}$ is an abelian category, called the category of framed objects. A pair (E,s)

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with $s \in \Phi(E)$ can be identified with the triple $(E,K,s) \in \mathcal{A}_{\mathsf{f}}$. For $\varepsilon > 0$, we define the stability function on $\mathcal{A}_{\mathsf{f}}$

$$Z_\varepsilon(E,V,s) = Z(E) - \varepsilon \cdot \dim V$$

and the corresponding notion of stability in $\mathcal{A}_{\mathsf{f}}$. A framed object (E,s) is Z_ε-stable for $0 < \varepsilon \ll |Z(E)|$ (we will say that it is Z_+-stable) if and only if:

(i) E is Z-semistable; and

(ii) for every proper $F \subset E$ with $s \in \Phi(F)$, we have $\mu_Z(F) < \mu_Z(E)$.

Lemma 4.2. *For a framed object (E,s) with $E \in \mathcal{C} = \mathcal{A}^\mu$ the following are equivalent:*

(i) *(E,s) is Z_+-stable in $\mathcal{A}_{\mathsf{f}}$;*

(ii) *(E,s) is Z_+-stable in $\mathcal{C}_{\mathsf{f}}$; and*

(iii) *for every $F \subseteq E$ in $\mathcal{C}$ with $s \in \Phi(F)$, we have $F = E$.*

Proof. If (E,s) is Z_+-stable in $\mathcal{A}_{\mathsf{f}}$, then it is automatically Z_+-stable in $\mathcal{C}_{\mathsf{f}}$. If (E,s) is Z_+-stable in $\mathcal{C}_{\mathsf{f}}$ and $s \in \Phi(F)$ for a proper $F \subset E$ in $\mathcal{C}$, then $\mu_Z(F) < \mu_Z(E)$. This contradicts $F \in \mathcal{C} = \mathcal{A}^\mu$. Assume that (E,s) satisfies the third condition. Let $s \in \Phi(F)$ for a proper $F \subset E$ in $\mathcal{A}$. As $E \in \mathcal{A}^\mu$, we have $\mu_Z(F) \le \mu$. If $\mu_Z(F) = \mu$, then $F \in \mathcal{A}^\mu = \mathcal{C}$ and, by our assumption, $F = E$. Otherwise $\mu_Z(F) < \mu_Z(E)$. This implies that (E,s) is Z_+-stable in $\mathcal{A}_{\mathsf{f}}$. □

The above result implies that the notion of Z_+-stability of framed objects in $\mathcal{A}^\mu$ is equivalent to the notion of stability of framed objects from Definition 4.1. To define such a framed object, it is enough to have a left exact functor $\Phi : \mathcal{A}^\mu \to \mathrm{Vec}_{\mathrm{f}}$. Usually, this functor will be exact.

Example 4.3. Let Q be a quiver and $\mathcal{A} = \mathrm{Rep}Q$ be the category of Q-representations over a field K. For a fixed (framing) vector $\mathbf{w} \in \mathbb{N}^{Q_0}$, we define the exact functor

$$\Phi : \mathcal{A} \to \mathrm{Vec}_{\mathrm{f}}, \quad M \mapsto \bigoplus_{i \in Q_0} \mathrm{Hom}(K^{\mathbf{w}_i}, M_i) \simeq \bigoplus_{i \in Q_0} M_i^{\oplus \mathbf{w}_i}. \tag{46}$$

The category $\mathcal{A}_{\mathsf{f}}$ of framed objects can be identified with the category of representations of the new quiver Q' obtained from Q by adding a new vertex $*$ and $\mathbf{w}_i$ arrows $* \to i$ for all $i \in Q_0$. An object $(M,V,s) \in \mathcal{A}_{\mathsf{f}}$ is identified with the representation $M' \in \mathrm{Rep}Q'$ such that $M'_i = M_i$ for $i \in Q_0$ and $M'_* = V$. The linear map $s : V \to \Phi(M) = \bigoplus_{i \in Q_0} M_i^{\oplus \mathbf{w}_i}$ induces linear maps $M'_* \to M'_i$ corresponding to the arrows $* \to i$ in Q'. A framed object (M,s) corresponds to $M' \in \mathrm{Rep}Q'$ with $M'_* = K$. It is stable if and only if M'_* generates the whole representation.

Example 4.4. Let C be a smooth projective curve of genus g and $\mathcal{A} = \mathrm{Coh}C$ be the category of coherent sheaves over C. We consider the stability function $Z(E) = -\deg E + \mathrm{rk}E \cdot \mathbf{i}$ on $\mathcal{A}$. For $\mu \in \mathbb{Q}$, let $\mathcal{A}^\mu = \mathrm{Coh}^\mu C \subset \mathrm{Coh}C$ be the category of semistable vector bundles of slope μ. For a fixed line bundle L (or any coherent sheaf) over C, we consider the left exact functor

$$\Phi : \mathcal{A} \to \mathrm{Vec}_{\mathrm{f}}, \quad E \mapsto \mathrm{Hom}(L,E). \tag{47}$$

If the line bundle L has degree $\ell < \mu - (2g-2)$, then the functor $\Phi : \mathcal{A}^\mu \to \mathrm{Vec}_{\mathrm{f}}$ is exact. Indeed, $\mathrm{Ext}^1(L,E) \simeq \mathrm{Hom}(E, L \otimes \omega_C)^* = 0$ for $E \in \mathcal{A}^\mu$ since $\mu > \ell + 2g - 2$.

On the other hand, for a fixed point $p \in C$, the fibre functor

$$\Phi : \mathcal{A}^\mu \to \mathrm{Vec}_{\mathrm{f}}, \quad E \mapsto E(p) = E_p \otimes_{\mathcal{O}_{C,p}} K, \tag{48}$$

is also exact. Framed objects in this situation can be identified with parabolic vector bundles.

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As before, let $\mathcal{A}^\mu \subseteq \mathcal{A}$ be the category of Z-semistable objects of $\mathcal{A}$ having slope $\mu \in \mathbb{R}$. The stable objects of $\mathcal{A}$ having slope μ are exactly the simple objects of $\mathcal{A}^\mu$. Let

$$E_1, \ldots, E_n \in \mathcal{A}^\mu$$

be a collection of (pairwise non-isomorphic) simple objects, and let $\mathcal{C} = \langle E_1, \ldots, E_n \rangle \subset \mathcal{A}^\mu$ be the abelian category generated by them. This is a Serre subcategory of $\mathcal{A}^\mu$, meaning an abelian subcategory closed under taking extensions, subobjects and quotients. Given a left exact functor $\Phi : \mathcal{A}^\mu \to \mathrm{Vec}_\mathrm{f}$, we define the category $\mathcal{C}_\mathrm{f}$ of framed objects in $\mathcal{C}$ in the same way as before.

We are going to describe the categories $\mathcal{C}$ and $\mathcal{C}_\mathrm{f}$ as categories of quiver representations. Let Q be the quiver with vertices $1, \ldots, n$ and the number of arrows from i to j equal to

$$a_{ij} = \dim \mathrm{Ext}^1(E_i, E_j). \tag{49}$$

Let Q' be the quiver obtained from Q by adding a new vertex $*$ and $\mathbf{w}_i = \dim \Phi(E_i)$ arrows $* \to i$ for all $i \in Q_0$. A representation of Q is called nilpotent if it has a filtration by sub-representations such that the arrows of Q act trivially on the factors.

Theorem 4.5.

(i) *If $\mathcal{A}^\mu$ is hereditary, then the category $\mathcal{C}$ is equivalent to the category* $\mathrm{Rep}^{\mathrm{nil}}(Q)$ *of nilpotent representations of Q.*

(ii) *If $\mathcal{A}^\mu$ is hereditary and $\Phi : \mathcal{A}^\mu \to \mathrm{Vec}_\mathrm{f}$ is exact, then the category of framed objects $\mathcal{C}_\mathrm{f}$ is equivalent to* $\mathrm{Rep}^{\mathrm{nil}}(Q')$.

Proof. For the first statement, see [DX98, Section 1.5]. For the second statement we can assume that $\mathcal{C} = \mathrm{Rep}^{\mathrm{nil}}(Q)$ and we will show that an exact functor $\Phi : \mathcal{C} \to \mathrm{Vec}_\mathrm{f}$ is uniquely determined (up to a non-unique natural transformation) by its values on the simple objects. This will imply the second statement, as $\mathrm{Rep}^{\mathrm{nil}}(Q')$ can be identified with the category of framed objects in $\mathrm{Rep}^{\mathrm{nil}}(Q)$ (see Example 4.3).

Let $A = \Bbbk Q$ be the path algebra, $\mathfrak{r} \subset A$ be the radical, and $e_i \in A$ be the idempotent corresponding to the trivial path at a vertex $i \in Q_0$. It is enough to show that Φ is uniquely determined on the subcategory $\mathcal{C}_t = \mathrm{Rep} A_t \subset \mathcal{C}$ (which is not an exact subcategory), where $A_t = A/\mathfrak{r}^{t+1}$ for $t \geqslant 0$. Let S_i be the simple modules and $P_i = A_t e_i$ their projective covers as A_t-modules. For the projective module $P = \bigoplus_i \Phi(S_i)^* \otimes_\Bbbk P_i$, where $\Phi(S_i)^*$ denotes the dual vector space of $\Phi(S_i)$, we consider the functor $h_P = \mathrm{Hom}(P, -) : \mathcal{C} \to \mathrm{Vec}_\mathrm{f}$. We will construct an isomorphism $a : h_P \to \Phi$ of functors over $\mathcal{C}_t$. The map

$$\mathrm{Fun}(h_P, \Phi) \simeq \Phi(P) = \bigoplus_i \Phi(S_i)^* \otimes_\Bbbk \Phi(P_i) \to \bigoplus_i \Phi(S_i)^* \otimes_\Bbbk \Phi(S_i)$$

is surjective, as $P_i \to S_i$ is surjective and Φ is exact. Consider a preimage $a : h_P \to \Phi$ of the sum of identity maps. This corresponds to $a_i : \Phi(S_i) \to \Phi(P_i)$ such that the composition $\Phi(S_i) \to \Phi(P_i) \to \Phi(S_i)$ is the identity for all i. This induces a chain of maps

$$\begin{aligned} h_P(M) = \mathrm{Hom}(P, M) = \bigoplus_i \Phi(S_i) \otimes_\Bbbk \mathrm{Hom}(P_i, M) \\ \to \bigoplus_i \Phi(P_i) \otimes_\Bbbk \mathrm{Hom}(P_i, M) \to \Phi(M). \end{aligned}$$

For $M = S_j$, the composition of the last two arrows is an isomorphism. This implies that $a : h_P \to \Phi$ induces an isomorphism on simple objects. By the exactness of h_P and Φ on $\mathcal{C}_t$, we conclude that $a : h_P \to \Phi$ is an isomorphism of functors on the whole category $\mathcal{C}_t$. $\square$

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The above theorem implies that the moduli space of stable pairs (E,s) with $E\in\mathcal{C}$ can be identified with the moduli spaces of stable nilpotent representations of Q' having dimension one at the vertex $*$ (see Example 4.3). Stability of a representation $M\in \operatorname{Rep}Q'$ means that M_* generates the whole representation.

Remark 4.6. The second statement of the theorem implies that $\mathcal{C}_{\mathsf{f}}$ is hereditary. More generally, one can show that if $\mathcal{A}$ is hereditary and $\Phi:\mathcal{A}\to\mathrm{Vec}_{\mathsf{f}}$ is exact, then $\mathcal{A}_{\mathsf{f}}$ is hereditary [Moz20].

Example 4.7. Let S be a smooth projective surface and H be an ample divisor such that $H\cdot K_S<0$, where K_S is the canonical divisor of S. Define the slope function

$$\mu(E)=\frac{H\cdot c_1(E)}{\operatorname{rk}E},\qquad E\in\operatorname{Coh}S.$$

For $\mu\in\mathbb{R}$, let $\mathcal{A}^\mu=\operatorname{Coh}^\mu S\subset\operatorname{Coh}S$ be the category of semistable vector bundles (also called Mumford semistable) having slope μ. This category is hereditary. Indeed, for any $E,F\in\mathcal{A}^\mu$, we have $\operatorname{Ext}^2(E,F)\simeq\operatorname{Hom}(F,E\otimes\omega_S)^*=0$ as both F and $E\otimes\omega_S$ are semistable and

$$\mu(E\otimes\omega_S)=\frac{H\cdot c_1(E)+\operatorname{rk}E\cdot H\cdot K_S}{\operatorname{rk}E}=\mu(E)+H\cdot K_S<\mu(F).$$

For $n\in\mathbb{Z}$, we consider the left exact functor

$$\Phi_n:\mathcal{A}^\mu\to\mathrm{Vec}_{\mathsf{f}},\qquad E\mapsto\operatorname{Hom}(\mathcal{O}(-nH),E).\tag{50}$$

This functor is not exact in general, but one can prove the following (cf. [HL10, Sections 1.7, 3.3]). For $E\in\operatorname{Coh}S$, let $\operatorname{cl}(E)=(\operatorname{rk}E,H\cdot c_1(E),c_2(E))\in\mathbb{Z}^3$. For any finite subset $C\subset\mathbb{Z}^3$, there exists $n>0$ such that $R^i\Phi_n(E)=0$ for all $i>0$ and $E\in\mathcal{A}^\mu$ with $\operatorname{cl}(E)\in C$.

4.2 Moduli spaces

Let $\mathcal{A}$ be an abelian ($\mathbb{C}$-linear) category and $\operatorname{cl}:K(\mathcal{A})\to\Gamma$ be a homomorphism to an abelian group Γ equipped with a bilinear form χ compatible with the Euler form on $\mathcal{A}$

$$\chi(\operatorname{cl}E,\operatorname{cl}E')=\chi(E,E')=\sum_{i\geqslant0}(-1)^i\dim\operatorname{Ext}^i(E,E').\tag{51}$$

Let $Z=-\mathsf{d}+\mathsf{r}\mathbf{i}:\Gamma\to\mathbb{C}$ be a homomorphism inducing a stability function $K(\mathcal{A})\xrightarrow{\operatorname{cl}}\Gamma\xrightarrow{Z}\mathbb{C}$ (also denoted by Z). For $\mu\in\mathbb{R}$, let $\mathcal{A}^\mu\subseteq\mathcal{A}$ be the category of Z-semistable objects having slope μ (including the zero object). We define the semigroups

$$\Lambda=\{\operatorname{cl}(E)\in\Gamma\mid E\in\mathcal{A}^\mu\},\qquad\Lambda_+=\Lambda\setminus\{0\}.\tag{52}$$

We assume that $\mathcal{A}^\mu$ is hereditary and we have an exact functor $\Phi:\mathcal{A}^\mu\to\mathrm{Vec}_{\mathsf{f}}$ and a homomorphism $\phi:\Lambda\to\mathbb{Z}$ such that

$$\dim\Phi(E)=\phi(\operatorname{cl}E)\quad\forall E\in\mathcal{A}^\mu.\tag{53}$$

Example 4.8. As in Example 4.4, let C be a smooth projective curve of genus g over $\mathbb{C}$, and let $\mathcal{A}=\operatorname{Coh}C$ and $\mathcal{A}^\mu=\operatorname{Coh}^\mu C\subset\operatorname{Coh}C$ for a fixed $\mu\in\mathbb{Q}$. We consider the Chern character map $\operatorname{ch}:K(\mathcal{A})\to\Gamma=\mathbb{Z}^2$, $[E]\mapsto(\operatorname{rk}E,\deg E)$. Let $\Phi=\operatorname{Hom}(L,-):\mathcal{A}^\mu\to\mathrm{Vec}_{\mathsf{f}}$ for a line bundle L of degree $\ell<\mu-(2g-2)$. For $\gamma=\operatorname{ch}(E)=(r,d)$ and $\gamma'=\operatorname{ch}E'=(r',d')$, we have

$$\chi(\gamma,\gamma')=\chi(E,E')=(1-g)rr'+(rd'-r'd),\tag{54}$$

$$\phi(\gamma)=\chi(L,E)=(1-g-\ell)r+d.\tag{55}$$

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Note that the Euler form is symmetric on Λ. Indeed, if $\gamma, \gamma' \in \Lambda$, then $rd' = r'd$, hence $\chi(\gamma, \gamma') = (1-g)rr' = \chi(\gamma', \gamma)$.

We assume that for every $\gamma \in \Lambda$ there exists a moduli space $\mathcal{M}_\gamma$ of Z-semistable objects in $\mathcal{A}$ having class γ, and a moduli space $\mathcal{M}^{\mathsf{f}}_\gamma$ of stable framed objects (E, s) with $E \in \mathcal{A}^\mu$ having class γ (see e.g. [HL95] in the case of framed objects on a curve). The moduli space $\mathcal{M}^{\mathsf{f}}_\gamma$ is smooth as all objects are stable and the second Ext-group vanishes (see e.g. [Tha94]). The moduli space $\mathcal{M}_\gamma$ parametrizes poly-stable objects of the form

$$E = \bigoplus_{i=1}^{n} E_i^{m_i},$$

where E_i are pairwise non-isomorphic stable objects in $\mathcal{A}$ with $\gamma_i = \operatorname{cl} E_i \in \Lambda_+$ such that $\gamma = \sum_i m_i \gamma_i$. Note that the objects E_i are simple objects in $\mathcal{A}^\mu$. We define the type of E to be the pair $(\boldsymbol{\gamma}, \mathbf{m})$ with

$$\boldsymbol{\gamma} = (\gamma_i)_{i=1}^n \in (\Lambda_+)^n, \quad \mathbf{m} = (m_i)_{i=1}^n \in \mathbb{N}^n.$$

The set $S_{\boldsymbol{\gamma},\mathbf{m}} \subset \mathcal{M}_\gamma$ of all objects having type $(\boldsymbol{\gamma}, \mathbf{m})$ is called the Luna stratum of type $(\boldsymbol{\gamma}, \mathbf{m})$. In our applications they form a finite partition of $\mathcal{M}_\gamma$ into locally closed subsets. If $n = 1$ and $m_1 = 1$, then $S_{\boldsymbol{\gamma},\mathbf{m}}$ is equal to the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma \subseteq \mathcal{M}_\gamma$ which is smooth and open.

The canonical projection $\pi : \mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma$ sends a pair (E, s) to the graded object associated with the Jordan–Hölder filtration of E in $\mathcal{A}^\mu$. This map is projective, and its fibre over any $E \in \mathcal{M}^{\mathsf{s}}_\gamma$ is isomorphic to the projective space $P(\Phi(E)) \simeq \mathbb{P}^{\phi(\gamma)-1}$.

Following (49), we define the quiver $Q_{\boldsymbol{\gamma}}$ with vertices $1, \dots, n$ and the number of arrows from i to j equal to

$$a_{ij} = \dim \operatorname{Ext}^1(E_i, E_j) = \delta_{ij} - \chi(\gamma_i, \gamma_j). \tag{56}$$

We define $Q'_{\boldsymbol{\gamma}}$ to be the quiver obtained from $Q_{\boldsymbol{\gamma}}$ by adding one vertex $*$ and adding

$$\mathbf{w}_i = \dim \Phi(E_i) = \phi(\gamma_i) \tag{57}$$

arrows $* \to i$ for all vertices i in $Q_{\boldsymbol{\gamma}}$.

Remark 4.9. In the case of vector bundles on a curve and $\gamma_i = \operatorname{ch} E_i = (r_i, d_i)$ with $d_i / r_i = \mu$, we obtain $a_{ij} = \delta_{ij} + (g-1) r_i r_j$. Note that the corresponding quiver is symmetric $(a_{ij} = a_{ji})$.

The Euler form of $Q_{\boldsymbol{\gamma}}$ is given by

$$\chi_{Q_{\boldsymbol{\gamma}}}(\mathbf{m}, \mathbf{m}') = \sum_i m_i m'_i - \sum_{i,j} a_{ij} m_i m'_j = \sum_{i,j} (\delta_{ij} - a_{ij}) m_i m'_j, \quad \mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n.$$

If $\gamma = \sum_i m_i \gamma_i$ and $\gamma' = \sum_i m'_i \gamma_i$, then

$$\chi(\gamma, \gamma') = \sum_{i,j} \chi(\gamma_i, \gamma_j) m_i m'_j = \chi_{Q_{\boldsymbol{\gamma}}}(\mathbf{m}, \mathbf{m}').$$

This means that the map $\mathbb{Z}^n \to \Gamma$, $\mathbf{m} \mapsto \sum_i m_i \gamma_i$ preserves the Euler forms.

Theorem 4.10. *For any object $E \in \mathcal{M}_\gamma$ of type $(\boldsymbol{\gamma}, \mathbf{m})$, the fibre $\pi^{-1}(E) \subset \mathcal{M}^{\mathsf{f}}_\gamma$ is isomorphic to the moduli space $\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}$ of stable nilpotent representations M of $Q'_{\boldsymbol{\gamma}}$ with $M|_{Q_{\boldsymbol{\gamma}}}$ having dimension vector $\mathbf{m}$ and $\dim M_* = 1$. Stability of M means that M_* generates the whole representation.*

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Proof. Let $E=\bigoplus_{i=1}^n E_i^{m_i}$, where $\mathrm{cl}E_i=\gamma_i\in\Lambda_+$. Let $\mathcal{C}\subset\mathcal{A}^\mu$ be the abelian category generated by $E_1,\dots,E_n$. We proved in Theorem 4.5 that $\mathcal{C}$ is equivalent to the category $\mathrm{Rep}^{\mathrm{nil}}(Q_{\boldsymbol{\gamma}})$ and $\mathcal{C}_{\mathsf{f}}$ is equivalent to the category $\mathrm{Rep}^{\mathrm{nil}}(Q'_{\boldsymbol{\gamma}})$.

If $(E',s)\in\pi^{-1}(E)$, then the factors of the Jordan–Hölder filtration of $E'\in\mathcal{A}^\mu$ contain m_i copies of E_i for all $1\leqslant i\leqslant n$. Therefore $E'\in\mathcal{C}$ and (E',s) is stable in $\mathcal{C}_{\mathsf{f}}$ (cf. Lemma 4.2). As $\mathcal{C}_{\mathsf{f}}\simeq\mathrm{Rep}^{\mathrm{nil}}(Q'_{\boldsymbol{\gamma}})$, we can identify (E',s) with a stable nilpotent representation M of $Q'_{\boldsymbol{\gamma}}$ such that $M|_{Q_{\boldsymbol{\gamma}}}$ has dimension vector $\mathbf{m}$ and $\dim M_*=1$. □

4.3 Virtual smallness

Let us now assume that the Euler form is symmetric on Λ.

Theorem 4.11. *The defect of $\pi:\mathcal{M}^{\mathsf{f}}_\gamma\to\mathcal{M}_\gamma$ over $S_{\boldsymbol{\gamma},\mathbf{m}}$ is $\leqslant\phi(\gamma)-1$, with equality only over the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma\subseteq\mathcal{M}_\gamma$ (if $\mathcal{M}^{\mathsf{s}}_\gamma\neq\varnothing$).*

Proof. Let $S_{\boldsymbol{\gamma},\mathbf{m}}\subseteq\mathcal{M}_\gamma$ be the stratum corresponding to $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_n)\in\Gamma^n$ and $\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n$ such that $\sum_i m_i\gamma_i=\gamma$. Let $Q=Q_{\boldsymbol{\gamma}}$ and $Q'=Q'_{\boldsymbol{\gamma}}$ be as in (56). By our assumptions, the quiver Q is symmetric (cf. Remark 4.9). By Theorem 4.10, for $E=\bigoplus_{i=1}^n E_i^{m_i}\in S_{\boldsymbol{\gamma},\mathbf{m}}$, we can identify the fibre $\pi^{-1}(E)$ with the moduli space $\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}$ of stable nilpotent representations of Q'. Let $R_{\mathbf{m}}=\bigoplus_{a:i\to j}\mathrm{Hom}(\mathbb{C}^{m_i},\mathbb{C}^{m_j})$ be the space of all representations of Q having dimension vector $\mathbf{m}$, and let $R^{\mathrm{nil}}_{\mathbf{m}}\subset R_{\mathbf{m}}$ be the subscheme of nilpotent representations. These schemes are equipped with the action of the group $G_{\mathbf{m}}=\prod_{i\in Q_0}\mathbf{GL}_{m_i}$. As Q is a symmetric quiver, we have (see [MR19])

$$\tfrac12\chi_Q(\mathbf{m},\mathbf{m})+\dim R^{\mathrm{nil}}_{\mathbf{m}}-\dim G_{\mathbf{m}}\leqslant\sum_{i=1}^n m_i\left(\tfrac12\chi_Q(e_i,e_i)-1\right),$$

where e_i is the standard basis vector of $\mathbb{Z}^n$. We have

$$\dim\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}=\dim R^{\mathrm{nil}}_{\mathbf{m}}(Q)-\dim G_{\mathbf{m}}+\mathbf{w}\cdot\mathbf{m}$$

as $G_{\mathbf{m}}$ acts freely on stable framed representations. Note that $\mathbf{w}\cdot\mathbf{m}=\phi(\gamma)$. Therefore,

$$\dim\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}\leqslant\sum_{i=1}^n m_i\left(\tfrac12\chi(\gamma_i,\gamma_i)-1\right)-\tfrac12\chi(\gamma,\gamma)+\phi(\gamma).$$

The dimension of the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma\subseteq\mathcal{M}_\gamma$ is $1-\chi(\gamma,\gamma)$, and hence $\dim S_{\boldsymbol{\gamma},\mathbf{m}}=\sum_{i=1}^n(1-\chi(\gamma_i,\gamma_i))$ (if $S_{\boldsymbol{\gamma},\mathbf{m}}$ is non-empty). The defect of π over $S_{\boldsymbol{\gamma},\mathbf{m}}$ is equal to (see (18))

$$\begin{aligned}\delta(\pi,S_{\boldsymbol{\gamma},\mathbf{m}})&=2\dim\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}+\dim S_{\boldsymbol{\gamma},\mathbf{m}}-\dim\mathcal{M}^{\mathsf{f}}_\gamma\\&=2\dim\mathcal{M}^{\mathsf{f},\mathrm{nil}}_{\mathbf{m}}+\sum_{i=1}^n(1-\chi(\gamma_i,\gamma_i))+(\chi(\gamma,\gamma)-\phi(\gamma))\\&\leq\sum_{i=1}^n m_i\left(\chi(\gamma_i,\gamma_i)-2\right)+\sum_{i=1}^n(1-\chi(\gamma_i,\gamma_i))+\phi(\gamma)\\&=\sum_{i=1}^n(m_i-1)\left(\chi(\gamma_i,\gamma_i)-1\right)-\sum_{i=1}^n m_i+\phi(\gamma)\leqslant-\sum_{i=1}^n m_i+\phi(\gamma)\leqslant\phi(\gamma)-1,\end{aligned}\tag{58}$$

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where we used the fact that $m_i \geqslant 1$ and $\chi(\gamma_i,\gamma_i) \leq 1$. The last inequality in (58) can be an equality only for $n=1$ and $m_1=1$. In this case the stratum $S_{\boldsymbol{\gamma},\mathbf{m}}$ is the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma$. The fibre over $E\in\mathcal{M}^{\mathsf{s}}_\gamma$ can be identified with the projective space $P(\Phi(E))\simeq\mathbb{P}^{\phi(\gamma)-1}$. Therefore, the defect of π over $\mathcal{M}^{\mathsf{s}}_\gamma$ is

$$\delta(\pi,\mathcal{M}^{\mathsf{s}}_\gamma)=2(\phi(\gamma)-1)+(1-\chi(\gamma,\gamma))+(\chi(\gamma,\gamma)-\phi(\gamma))=\phi(\gamma)-1. \qquad \square$$

5. DT invariants

5.1 Moduli spaces

We follow the conventions from Section 4.2. Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let $\mathcal{A}=\operatorname{Coh}C$ and $\mathcal{A}^\mu=\operatorname{Coh}^\mu C\subseteq\operatorname{Coh}C$ be the category of semistable vector bundles having a fixed slope $\mu\in\mathbb{Q}$. In a similar way to (52), we define

$$\Lambda_+=\{(r,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\mid d/r=\mu\}\,,\quad \Lambda=\Lambda_+\sqcup\{0\}\,.$$

Note that $\Lambda=\mathbb{N}\gamma_0\simeq\mathbb{N}$ for a unique element $\gamma_0=(r_0,d_0)\in\Lambda_+$.

For $\gamma=(r,d)\in\Lambda$, let $\mathcal{M}_\gamma$ (respectively $\mathcal{M}^{\mathsf{s}}_\gamma$) be the moduli space of semistable (respectively stable) vector bundles over C having class γ (rank r and degree d). Similarly, let $\mathfrak{M}_\gamma$ be the moduli stack of semistable vector bundles over C having class γ.

Remark 5.1. The moduli space $\mathcal{M}_\gamma$ is always irreducible [LeP97, Theorem 8.5.2]. If the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma\subseteq\mathcal{M}_\gamma$ is nonempty, then it is smooth of dimension $1-\chi(\gamma,\gamma)=(g-1)r^2+1$. Moreover, the following statements hold.

(i) If $g\geqslant 2$, then $\mathcal{M}^{\mathsf{s}}_\gamma\neq\varnothing$ [LeP97, Theorem 8.6.1].

(ii) If $g=1$, then $\mathcal{M}_\gamma\simeq S^k(C)$ for $k=\gcd(r,d)$ [LeP97, Theorem 8.6.2]. We have $\mathcal{M}^{\mathsf{s}}_\gamma=\mathcal{M}_\gamma$ if $k=1$ and $\mathcal{M}^{\mathsf{s}}_\gamma=\varnothing$ otherwise. Indeed, if $\mathcal{M}^{\mathsf{s}}_\gamma\neq\varnothing$, then $\dim S^k(C)=1$, hence $k=1$.

(iii) If $g=0$, then $\mathcal{M}_\gamma=\mathbf{pt}$ if $d/r\in\mathbb{Z}$ and $\mathcal{M}_\gamma=\varnothing$ otherwise. We have $\mathcal{M}^{\mathsf{s}}_\gamma=\mathcal{M}_\gamma$ if $r=1$ and $\mathcal{M}^{\mathsf{s}}_\gamma=\varnothing$ otherwise.

Remark 5.2. Note that the restriction of the Euler form (54) to Λ (or $\mathcal{A}^\mu$) is symmetric. All of the results below translate verbatim to moduli spaces in other hereditary categories under the assumption that the restriction of the Euler form to $\mathcal{A}^\mu$ is symmetric.

As in Example 4.8, we fix a line bundle L of degree $\ell<\mu-(2g-2)$, and consider the exact functor $\Phi=\operatorname{Hom}(L,-):\mathcal{A}^\mu\to\operatorname{Vec}_{\mathrm{f}}$ and the homomorphism $\phi:\Lambda\to\mathbb{Z}$ such that $\dim\Phi(E)=\phi(\operatorname{ch}E)$ for all $E\in\mathcal{A}^\mu$. Let $\mathcal{M}^{\mathsf{f}}_\gamma$ be the moduli space of stable framed objects (E,s) with $E\in\mathcal{A}^\mu$ having class γ and $s\in\Phi(E)$. We define

$$\mathcal{M}=\bigsqcup_{\gamma\in\Lambda}\mathcal{M}_\gamma,\quad \mathfrak{M}=\bigsqcup_{\gamma\in\Lambda}\mathfrak{M}_\gamma,\quad \mathcal{M}^{\mathsf{f}}=\bigsqcup_{\gamma\in\Lambda}\mathcal{M}^{\mathsf{f}}_\gamma,$$

and consider the commutative diagram

$$\begin{tikzcd} \mathcal{M}^{\mathsf{f}} \arrow[rr, "q"] \arrow[dr, "\pi"'] & & \mathfrak{M} \arrow[dl, "p"] \\ & \mathcal{M} & \end{tikzcd}$$

where the projective map $\pi:\mathcal{M}^{\mathsf{f}}\to\mathcal{M}$ was introduced in Section 4.2 and the map $q:\mathcal{M}^{\mathsf{f}}\to\mathfrak{M}$ sends a pair (E,s) to E. The moduli space $\mathcal{M}$ is a Λ-graded commutative monoid in the category

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of algebraic schemes over $\mathbb{C}$, where the product map $\mu:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is given by the direct sum of vector bundles and is a finite map. The unit map is $\eta:\mathbf{pt}=\mathcal{M}_0\hookrightarrow\mathcal{M}$. By the results of Section 3, we have a Λ-graded pre-λ-ring $\mathbf{K}(\mathrm{Sch}/\mathcal{M})$ and a Λ-graded (and $\mathbb{Z}$-graded) λ-ring $\mathbf{K}(\mathrm{MHM}(\mathcal{M}))$. As in (43), we have a commutative diagram

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Sch}/\mathcal{M}) \arrow[r, "\deg_!"] \arrow[d, "\chi_c"'] & \mathbf{K}(\mathrm{Sch}/\mathbf{\Lambda})=\mathbf{K}(\mathrm{Sch})[\Lambda] \arrow[d, "\chi_c"] & \\
\mathbf{K}(\mathrm{MHM}(\mathcal{M})) \arrow[r, "\deg_!"] & \mathbf{K}(\mathrm{MHM}(\mathbf{\Lambda}))=\mathbf{K}(\mathrm{MHM}(\mathbf{pt}))[\Lambda] \arrow[r, "E"] & \mathbb{Q}(u^{\frac12},v^{\frac12})[\Lambda]
\end{tikzcd}$$

where $\deg:\mathcal{M}\to\mathbf{\Lambda}$ is the degree map (30), E is the E-polynomial map (see Section 5.6), and all arrows are homomorphisms of pre-λ-rings.

5.2 Motivic DT invariants

The (absolute) motivic DT invariants $\mathrm{DT}^{\mathfrak{m}}_\gamma\in\mathbf{K}(\mathrm{Sch})$ are defined by the formula in $\widehat{\mathbf{K}}(\mathrm{Sch}/\mathbf{\Lambda})\simeq\prod_{\gamma\in\Lambda}\mathbf{K}(\mathrm{Sch})t^\gamma$

$$\sum_{\gamma\in\Lambda}\mathbb{L}^{\frac12\chi(\gamma,\gamma)}[\mathfrak{M}_\gamma]t^\gamma=\mathrm{Exp}\left(\frac{\sum_{\gamma\in\Lambda_+}\mathrm{DT}^{\mathfrak{m}}_\gamma\,t^\gamma}{\mathbb{L}^{\frac12}-\mathbb{L}^{-\frac12}}\right).$$

Note that $\dim\mathfrak{M}_\gamma=-\chi(\gamma,\gamma)$, and hence $\mathbb{L}^{\frac12\chi(\gamma,\gamma)}[\mathfrak{M}_\gamma]=[\mathfrak{M}_\gamma]_{\mathrm{vir}}$ (40). Note also that $\mathbb{L}^{\frac12}-\mathbb{L}^{-\frac12}=[\mathbb{G}_{\mathrm{m}}]_{\mathrm{vir}}$. Similarly, we define the (relative) motivic DT invariants $\mathbf{DT}^{\mathfrak{m}}_\gamma\in\mathbf{K}(\mathrm{Sch}/\mathcal{M}_\gamma)$ by the formula in $\widehat{\mathbf{K}}(\mathrm{Sch}/\mathcal{M})$

$$\sum_{\gamma\in\Lambda}\mathbb{L}^{\frac12\chi(\gamma,\gamma)}[\mathfrak{M}_\gamma\to\mathcal{M}_\gamma]=\mathrm{Exp}\left(\frac{\sum_{\gamma\in\Lambda_+}\mathbf{DT}^{\mathfrak{m}}_\gamma}{\mathbb{L}^{\frac12}-\mathbb{L}^{-\frac12}}\right).\tag{59}$$

We define $\mathbf{DT}^{\mathfrak{m}}=\sum_{\gamma\in\Lambda_+}\mathbf{DT}^{\mathfrak{m}}_\gamma\in\widehat{\mathbf{K}}(\mathrm{Sch}/\mathcal{M})$. The following statement is a relative analogue of the result proved in [KS11, Section 6.2, Theorem 10] for quivers with potentials (see also [KS08, Section 7.3, Conjecture 9]).

Theorem 5.3 (Integrality conjecture [Mei15]). *The class $\mathbf{DT}^{\mathfrak{m}}_\gamma$ is contained in the image of the map $K(\mathrm{Sch}/\mathcal{M}_\gamma)[\mathbb{L}^{-1/2}]\to\mathbf{K}(\mathrm{Sch}/\mathcal{M}_\gamma)$.*

5.3 Mixed DT invariants

The (absolute) DT invariants are defined by

$$\mathrm{DT}_\gamma=\chi_c(\mathrm{DT}^{\mathfrak{m}}_\gamma)\in\mathbf{K}(\mathrm{MHS}^p),\tag{60}$$

where $\chi_c:\mathbf{K}(\mathrm{Sch})\to\mathbf{K}(\mathrm{MHS}^p)$ was defined in (39). Similarly, the relative DT invariants are

$$\mathbf{DT}_\gamma=\chi_c(\mathbf{DT}^{\mathfrak{m}}_\gamma)\in\mathbf{K}(\mathrm{MHM}(\mathcal{M}_\gamma)).\tag{61}$$

Using the fact that $\chi_c:\mathbf{K}(\mathrm{Sch}/\mathbf{\Lambda})\to\mathbf{K}(\mathrm{MHM}(\mathbf{\Lambda}))$ and $\chi_c:\mathbf{K}(\mathrm{Sch}/\mathcal{M})\to\mathbf{K}(\mathrm{MHM}(\mathcal{M}))$ are morphisms of pre-λ-rings, we can write

$$\sum_{\gamma\in\Lambda}\mathbb{L}^{\frac12\chi(\gamma,\gamma)}\chi_c[\mathfrak{M}_\gamma]=\mathrm{Exp}\left(\frac{\sum_{\gamma\in\Lambda_+}\mathrm{DT}_\gamma}{\mathbb{L}^{\frac12}-\mathbb{L}^{-\frac12}}\right)\text{and}\tag{62}$$

$$\sum_{\gamma\in\Lambda}\mathbb{L}^{\frac12\chi(\gamma,\gamma)}\chi_c[\mathfrak{M}_\gamma\to\mathcal{M}_\gamma]=\mathrm{Exp}\left(\frac{\sum_{\gamma\in\Lambda_+}\mathbf{DT}_\gamma}{\mathbb{L}^{\frac12}-\mathbb{L}^{-\frac12}}\right).\tag{63}$$

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COROLLARY 5.4. *The class* $\mathbf{DT}_\gamma$ *is contained in* $K(\mathrm{MHM}(\mathcal{M}_\gamma))[\mathbb{L}^{1/2}]$.

Proof. By Theorem 5.3, the class $\mathbf{DT}_\gamma$ is contained in the image of the map (note that $\mathbb{L}$ is invertible) $K(\mathrm{MHM}(\mathcal{M}_\gamma))[\mathbb{L}^{1/2}] \to \mathbf{K}(\mathrm{MHM}(\mathcal{M}_\gamma))$. This map is injective, as $K(\mathrm{MHM}(\mathcal{M}_\gamma))$ is free over $\mathbb{Z}[\mathbb{L}^{\pm 1}]$ (the basis consists of isomorphism classes of simple objects in $\mathrm{MHM}(\mathcal{M}_\gamma)$ having weight 0 or 1). $\square$

Remark 5.5. The stack $\mathfrak{M}_\gamma$ can be represented as a global quotient X/G, where X is smooth and $G = \mathbf{GL}_n$ is a general linear group (see e.g. [HL10]). If $q : X \to Y$ is a principal G-bundle between smooth algebraic varieties, then $[\mathbf{IC}_Y] = [G]_{\mathrm{vir}}^{-1} \cdot q_![\mathbf{IC}_X]$. In our case, we consider the diagram

$$\begin{tikzcd} X \arrow[rr, "q"] \arrow[dr] & & \mathfrak{M}_\gamma \arrow[dl, "p"] \\ & \mathcal{M}_\gamma & \end{tikzcd}$$

and formally define

$$p_![\mathbf{IC}_{\mathfrak{M}_\gamma}] = [G]_{\mathrm{vir}}^{-1} \cdot (pq)_![\mathbf{IC}_X] \in \mathbf{K}(\mathrm{MHM}(M_\gamma)).$$

Then $p_![\mathbf{IC}_{\mathfrak{M}_\gamma}] = \mathbb{L}^{-d_X/2+d_G/2}[G]^{-1} \cdot (pq)_![\mathbb{Q}_X] = \mathbb{L}^{\frac{1}{2}\chi(\gamma,\gamma)} \cdot \chi_c[\mathfrak{M}_\gamma \to \mathcal{M}_\gamma]$. Therefore,

$$\sum_{\gamma\in\Lambda} p_![\mathbf{IC}_{\mathfrak{M}_\gamma}] = \mathrm{Exp}\left(\frac{\sum_{\gamma\in\Lambda_+}\mathbf{DT}_\gamma}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right).$$

5.4 DT invariants and framed moduli spaces

In this section we will express DT invariants using moduli spaces of stable framed objects. This expression is a version of a wall-crossing formula, also called a DT/PT correspondence. Recall that we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^{\mathsf{f}} \arrow[rr, "q"] \arrow[dr, "\pi"'] & & \mathfrak{M} \arrow[dl, "p"] \\ & \mathcal{M} & \end{tikzcd}$$

THEOREM 5.6. *We have*

$$\pi_!\left(\sum_{\gamma\in\Lambda}(-1)^{\phi(\gamma)}[\mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}]\right) = \mathrm{Exp}\left(\sum_{\gamma\in\Lambda_+}(-1)^{\phi(\gamma)}[\mathbb{P}^{\phi(\gamma)-1}]_{\mathrm{vir}}\,\mathbf{DT}_\gamma\right). \tag{64}$$

Proof. We will prove a similar formula in $\widehat{\mathbf{K}}(\mathrm{Sch}/\mathcal{M})$ and then apply the map $\chi_c : \widehat{\mathbf{K}}(\mathrm{Sch}/\mathcal{M}) \to \widehat{\mathbf{K}}(\mathrm{MHM}(\mathcal{M}))$. The wall-crossing formula for framed objects (cf. [ER09, Moz13c, Bri11]) implies that

$$\sum_\gamma \mathbb{L}^{\frac{1}{2}\chi(\gamma,\gamma)+\phi(\gamma)}[\mathfrak{M}_\gamma \to \mathcal{M}_\gamma] = \sum_\gamma \mathbb{L}^{\frac{1}{2}\chi(\gamma,\gamma)}[\mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma] \cdot \sum_\gamma \mathbb{L}^{\frac{1}{2}\chi(\gamma,\gamma)}[\mathfrak{M}_\gamma \to \mathcal{M}_\gamma].$$

Applying (59), we obtain

$$\mathrm{Exp}\left(\frac{\sum_\gamma \mathbb{L}^{\phi(\gamma)}\,\mathbf{DT}^{\mathsf{m}}_\gamma}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right) = \sum_\gamma \mathbb{L}^{\frac{1}{2}\chi(\gamma,\gamma)}[\mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma] \cdot \mathrm{Exp}\left(\frac{\sum_\gamma \mathbf{DT}^{\mathsf{m}}_\gamma}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right).$$

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Let $d_\gamma = \dim \mathcal{M}^{\mathsf{f}}_\gamma = -\chi(\gamma,\gamma) + \phi(\gamma)$ and $z_\gamma = \mathbb{L}^{-d_\gamma/2}[\mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma]$. Then

$$\sum_\gamma \mathbb{L}^{\frac{1}{2}\phi(\gamma)} z_\gamma = \operatorname{Exp}\left(\frac{\sum_\gamma (\mathbb{L}^{\phi(\gamma)}-1)\,\mathbf{DT}^{\mathfrak{m}}_\gamma}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\right) = \operatorname{Exp}\left(\sum_\gamma \mathbb{L}^{\frac{1}{2}}[\mathbb{P}^{\phi(\gamma)-1}]\,\mathbf{DT}^{\mathfrak{m}}_\gamma\right).$$

We apply the pre-λ-ring morphism $\sum_\gamma x_\gamma \mapsto \sum_\gamma (-\mathbb{L}^{1/2})^{-\phi(\gamma)} x_\gamma$ to both sides and obtain

$$\sum_\gamma (-1)^{\phi(\gamma)} z_\gamma = \operatorname{Exp}\left(\sum_\gamma (-1)^{\phi(\gamma)}[\mathbb{P}^{\phi(\gamma)-1}]_{\mathrm{vir}}\,\mathbf{DT}^{\mathfrak{m}}_\gamma\right).$$

Finally, we apply χ_c to both sides and note that $\chi_c(z_\gamma) = [\pi_!(\mathbb{Q}_{\mathcal{M}^{\mathsf{f}}_\gamma}\langle d_\gamma\rangle)] = \pi_![\mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}]$. □

5.5 DT invariants and intersection complexes

For $\gamma = (r,d) \in \Lambda$, the stable locus $\mathcal{M}^{\mathsf{s}}_\gamma \subseteq \mathcal{M}_\gamma$ is open, smooth and has dimension $1-\chi(\gamma,\gamma)$ (if $\mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing$). We consider the object $\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}} \in \operatorname{MHM}(\mathcal{M}_\gamma)$, which is defined to be zero if $\mathcal{M}^{\mathsf{s}}_\gamma = \varnothing$.

THEOREM 5.7. *We have* $\mathbf{DT}_\gamma = [\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}]$ *for* $\gamma \in \Lambda$.

Proof. The formula (64) can be written in the form

$$(-1)^{\phi(\gamma)}\pi_*[\mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}] = \sum_{\substack{m:\Lambda_+\to\mathbb{N}\\ \sum_\alpha m_\alpha\alpha=\gamma}} \prod_{\alpha\in\Lambda_+} \sigma^{m_\alpha}((-1)^{\phi(\alpha)}[\mathbb{P}^{\phi(\alpha)-1}]_{\mathrm{vir}}\,\mathbf{DT}_\alpha). \tag{65}$$

We will compare the highest degree terms on both sides. By Theorem 4.11, the map $\pi : \mathcal{M}^{\mathsf{f}}_\gamma \to \mathcal{M}_\gamma$ is n-small, where $n = \phi(\gamma)-1$. It is a $\mathbb{P}^n$-fibration over $\mathcal{M}^{\mathsf{s}}_\gamma$. By Theorem 2.21, the object $\pi_*\mathbf{IC}_{\mathcal{M}^{\mathsf{f}}_\gamma}$ has degree $\leqslant n$ and the degree n component $\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}\langle -n\rangle = \mathbb{L}^{n/2}\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}$.

Let us consider the summand on the right corresponding to $m : \Lambda_+ \to \mathbb{N}$ that is different from the delta-function $\delta_\gamma : \Lambda_+ \to \mathbb{N}$. By induction on γ, the DT invariants in this summand satisfy $\mathbf{DT}_\alpha = [\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\alpha}}]$, and hence have degree zero (if $\mathbf{DT}_\alpha \neq 0$). Therefore, $z_\alpha = (-1)^{\phi(\alpha)}[\mathbb{P}^{\phi(\alpha)-1}]_{\mathrm{vir}}\,\mathbf{DT}_\alpha$ has degree $\leqslant \phi(\alpha)-1$. As $\mathbf{K}(\operatorname{MHM}(\mathcal{M}))$ is a weight-graded λ-ring (see Theorem 3.4), we conclude that $\sigma^{m_\alpha}(z_\alpha)$ has degree $\leqslant m_\alpha(\phi(\alpha)-1)$. Therefore, the summand $\prod_\alpha \sigma^{m_\alpha}(z_\alpha)$ has degree $\leq \sum_\alpha m_\alpha(\phi(\alpha)-1) = \phi(\gamma) - \sum_\alpha m_\alpha < \phi(\gamma)-1 = n$.

The summand on the right corresponding to the delta-function $\delta_\gamma : \Lambda_+ \to \mathbb{N}$ is equal to $z_\gamma = (-1)^{\phi(\gamma)}[\mathbb{P}^{\phi(\gamma)-1}]_{\mathrm{vir}}\,\mathbf{DT}_\gamma$. By Corollary 5.4, we have $\mathbf{DT}_\gamma \in K(\operatorname{MHM}(\mathcal{M}_\gamma))[\mathbb{L}^{\frac{1}{2}}]$, and hence we can write $\mathbf{DT}_\gamma = \sum_{i\in\mathbb{Z}} x_i$, where x_i is homogeneous of degree i. We can prove by induction that the class $\mathbf{DT}_\gamma$ is self-dual, and hence $\mathbf{DT}_\gamma$ has degree $r \geqslant 0$ (if $\mathbf{DT}_\gamma \neq 0$). Then z_γ has degree $r+n$ and the leading term $(-1)^{\phi(\gamma)}\mathbb{L}^{n/2}x_r$ (if $\mathbf{DT}_\gamma \neq 0$).

We conclude that, if $\mathbf{DT}_\gamma = 0$, then the right hand side has degree $< n$, hence $\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}} = 0$ and $\mathcal{M}^{\mathsf{s}}_\gamma = \varnothing$. If $\mathbf{DT}_\gamma \neq 0$, then the right hand side has degree $r+n$ and the leading term $(-1)^{\phi(\gamma)}\mathbb{L}^{n/2}x_r$. Comparing it to the left hand side, we conclude that $r=0$ and $x_0 = [\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}]$. As $\mathbf{DT}_\gamma$ is self-dual and $x_i = 0$ for $i > 0$, we conclude that $\mathbf{DT}_\gamma = x_0 = [\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}]$. □

The above result translates verbatim to moduli spaces in other hereditary categories under the assumption that the restriction of the Euler form to $\mathcal{A}^\mu$ is symmetric (so that we can apply Theorem 4.11).

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Corollary 5.8 (cf. Theorem 1.3). *We have*

$$\mathbf{DT}_\gamma = \begin{cases} [\mathbf{IC}_{\mathcal{M}_\gamma}] & \mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing, \\ 0 & \mathcal{M}^{\mathsf{s}}_\gamma = \varnothing. \end{cases} \tag{66}$$

$$\mathrm{DT}_\gamma = \begin{cases} [H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma})] & \mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing, \\ 0 & \mathcal{M}^{\mathsf{s}}_\gamma = \varnothing. \end{cases} \tag{67}$$

Proof. The moduli space $\mathcal{M}_\gamma$ is irreducible and $\mathcal{M}^{\mathsf{s}}_\gamma \subseteq \mathcal{M}_\gamma$ is open. If $\mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing$, then $\overline{\mathcal{M}^{\mathsf{s}}_\gamma} = \mathcal{M}_\gamma$, hence $\mathbf{DT}_\gamma = [\mathbf{IC}_{\overline{\mathcal{M}^{\mathsf{s}}_\gamma}}] = [\mathbf{IC}_{\mathcal{M}_\gamma}]$ and we obtain (66). To prove (67), we apply $a_! : \mathbf{K}(\mathrm{MHM}(\mathcal{M}_\gamma)) \to \mathbf{K}(\mathrm{MHM}(\mathbf{pt}))$ for the projection $a : \mathcal{M}_\gamma \to \mathbf{pt}$. □

Example 5.9. If C is an elliptic curve, then $\mathcal{M}_\gamma \simeq S^k(C)$ for $k = \gcd(r, d)$, $\gamma = (r, d)$. Moreover, $\mathcal{M}^{\mathsf{s}}_\gamma = \mathcal{M}_\gamma$ if $k = 1$ and $\mathcal{M}^{\mathsf{s}}_\gamma = \varnothing$ otherwise. Therefore,

$$\mathbf{DT}_\gamma = \begin{cases} [\mathbf{IC}_{\mathcal{M}_\gamma}] = \mathbb{L}^{-1/2}[\mathbb{Q}_{\mathcal{M}_\gamma}] & \gcd(r,d) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

5.6 Hodge–Deligne polynomials

Given a mixed Hodge structure V, we define its Hodge–Deligne polynomial (also called Hodge–Euler polynomial or E-polynomial)

$$E(V;\ u, v) = \sum_{p,q} h^{p,q}(V) u^p v^q, \quad h^{p,q}(V) = \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q}(V_\mathbb{C}). \tag{68}$$

This induces a λ-ring homomorphism

$$E : K(\mathrm{MHM}(\mathbf{pt})) = K(\mathrm{MHS}^p(\mathbb{Q})) \to \mathbb{Z}[u^{\pm1}, v^{\pm1}], \tag{69}$$

where the λ-ring structure on the right is given by the Adams operations $\psi^n(f(u,v)) = f(u^n, v^n)$. The fact that the above map preserves λ-ring structures follows from the fact that it is induced by an exact monoidal functor $\mathrm{MHS}^p(\mathbb{Q}) \to \mathrm{Vec}^{\mathbb{Z}^2}_\mathbb{C}$, $V \mapsto V_\mathbb{C}$ (where $\mathrm{Vec}^{\mathbb{Z}^2}_\mathbb{C}$ denotes the category of $\mathbb{Z}^2$-graded objects in $\mathrm{Vec}_\mathbb{C}$) and that $K(\mathrm{Vec}^{\mathbb{Z}^2}_\mathbb{C}) \simeq \mathbb{Z}[u^{\pm1}, v^{\pm1}]$. For an algebraic variety X, we have the class $\chi_c[X] = [H^*_c(X, \mathbb{Q})] = \sum_i (-1)^i [H^i_c(X, \mathbb{Q})] \in K(\mathrm{MHM}(\mathbf{pt}))$ and we define

$$E(X) = E(\chi_c[X]) = \sum_i (-1)^i E(H^i_c(X, \mathbb{Q})) \in \mathbb{Z}[u^{\pm1}, v^{\pm1}]. \tag{70}$$

In particular, $\mathbb{L} = H^*_c(\mathbb{A}^1, \mathbb{Q}) = \mathbb{Q}(-1)[-2]$ and $E(\mathbb{L}) = E(\mathbb{Q}(-1)) = uv$. We extend E to a λ-ring homomorphism

$$E : \mathbf{K}(\mathrm{MHM}(\mathbf{pt})) \to \mathbb{Q}(u^{1/2}, v^{1/2})$$

with $E(\mathbb{L}^{1/2}) = -(uv)^{1/2}$ (the minus sign corresponds to the fact that $\mathbb{L}^{1/2} = \mathbb{Q}(-1/2)[-1]$ has odd homological degree). In what follows we will denote $E(\mathbb{L}^{1/2})$ by $\mathbb{L}^{1/2}$. For a finite type Artin stack X over $\mathbb{C}$ with affine stabilizers, we have the classes $[X] \in \mathbf{K}(\mathrm{Sch}/\mathbf{pt})$, $\chi_c[X] \in \mathbf{K}(\mathrm{MHM}(\mathbf{pt}))$ and we define $E(X) = E(\chi_c[X])$.

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For $\mu \in \mathbb{Q}$, we define the series (cf. (1))

$$Q_\mu = 1 + \sum_{d/r=\mu} Q_{r,d} t^r = 1 + \sum_{d/r=\mu} \mathbb{L}^{(1-g)r^2/2} E(\mathfrak{M}_{r,d}) t^r \in \mathbb{Q}(u^{1/2}, v^{1/2})[\![t]\!]. \tag{71}$$

Note that $\dim \mathfrak{M}_\gamma = -\chi(\gamma, \gamma) = (g-1)r^2$ for $\gamma = (r, d)$. As in the introduction, we define the DT invariants of the curve by the formula (2),

$$Q_\mu = \operatorname{Exp}\left(\frac{\sum_{d/r=\mu} \mathrm{DT}^E_{r,d}\, t^r}{\mathbb{L}^{\frac12} - \mathbb{L}^{-\frac12}} \right). \tag{72}$$

Theorem 5.10 (cf. Theorem 1.1). *If $\mathcal{M}^{\mathsf{s}}_\gamma = \varnothing$, then $\mathrm{DT}^E_\gamma = 0$. If $\mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing$, then*

$$\mathrm{DT}^E_\gamma = E(H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma})) = \mathbb{L}^{-\dim \mathcal{M}_\gamma/2} E(\mathrm{IH}^*(\mathcal{M}_\gamma, \mathbb{Q})), \tag{73}$$

$$\mathrm{DT}^E_\gamma(y, y) = (-y)^{-\dim \mathcal{M}_\gamma} \sum\nolimits_k \dim \mathrm{IH}^k(\mathcal{M}_\gamma, \mathbb{Q})(-y)^k, \tag{74}$$

where $\dim \mathcal{M}_\gamma = \dim \mathcal{M}^{\mathsf{s}}_\gamma = (g-1)r^2 + 1$ *for* $\gamma = (r, d) \in \mathbb{Z}^2$.

Proof. Comparing (72) and (62), we conclude that $\mathrm{DT}^E_\gamma = E(\mathrm{DT}_\gamma)$. Taking the E-polynomials on both sides of (67), we obtain $\mathrm{DT}^E_\gamma = \begin{cases} E(H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma})) & \mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing, \\ 0 & \mathcal{M}^{\mathsf{s}}_\gamma = \varnothing. \end{cases}$

We have (cf. (12))

$$\mathrm{IH}^*(\mathcal{M}_\gamma, \mathbb{Q}) = H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma} \langle -\dim \mathcal{M}_\gamma \rangle) = \mathbb{L}^{\dim \mathcal{M}_\gamma/2} H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma}),$$

hence $E(H^*(\mathcal{M}_\gamma, \mathbf{IC}_{\mathcal{M}_\gamma})) = \mathbb{L}^{-\dim \mathcal{M}_\gamma/2} E(\mathrm{IH}^*(\mathcal{M}_\gamma, \mathbb{Q}))$. The second formula follows by substitution $u = v = y$, $\mathbb{L}^{1/2} = -y$ and the fact that $\mathrm{IH}^k(\mathcal{M}_\gamma, \mathbb{Q})$ is pure of weight k (note that $\langle 1 \rangle$ does not change the weight). □

If $g \geqslant 2$, then $\mathcal{M}^{\mathsf{s}}_\gamma \neq \varnothing$, hence $\dim \mathcal{M}_\gamma = (g-1)r^2 + 1$ for $\gamma = (r, d)$. Therefore,

$$\sum_{d/r=\mu} \mathbb{L}^{(1-g)r^2/2} E(\mathrm{IH}^*(\mathcal{M}_{r,d})) t^r = \sum_{d/r=\mu} \mathbb{L}^{1/2}\, \mathrm{DT}^E_{r,d}\, t^r = (\mathbb{L} - 1) \operatorname{Log}(Q_\mu). \tag{75}$$

The next result is an explicit formula for $Q_{r,d}$ mentioned in the introduction; see [Zag96, LR96, MR14].

Theorem 5.11. *For any r, d, we have*

$$Q_{r,d} = \sum_{\substack{r_1, \dots, r_k > 0 \\ r_1 + \dots + r_k = r}} \prod_{i=1}^{k-1} \frac{\mathbb{L}^{(r_i + r_{i+1})\{(r_1 + \dots + r_i)d/r\}}}{1 - \mathbb{L}^{r_i + r_{i+1}}} Q_{r_1} \dots Q_{r_k},$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x and

$$Q_r = \mathbb{L}^{(1-g)r^2/2} \operatorname{Res}_{t=1} \prod_{i=0}^{r-1} Z_C(\mathbb{L}^i t), \quad Z_C(t) = \sum_{n \geqslant 0} E(S^n C) t^n = \frac{(1-ut)^g (1-vt)^g}{(1-t)(1-uvt)}.$$

Remark 5.12. With $r_{\leqslant i} = r_1 + \dots + r_i$, we have $\sum_{i=1}^{k-1} (r_i + r_{i+1}) r_{\leqslant i} = (r - r_k) r$, hence

$$\sum_{i=1}^{k-1} (r_i + r_{i+1}) \{r_{\leqslant i} d/r\} = (r - r_k)d - \sum_{i=1}^{k-1} (r_i + r_{i+1}) \lfloor r_{\leqslant i} d/r \rfloor \in \mathbb{Z}.$$

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Appendix A. Relative hard Lefschetz theorem

Let $\pi : X \to Y$ be a projective morphism between smooth (connected) algebraic varieties and $\ell \in H^2(X, \mathbb{Z}(1))$ be the first Chern class of a relatively ample line bundle on X. This induces the map $\ell : M \to M(1)[2]$ for any $M \in D^b\mathrm{MHM}(X)$.

THEOREM A.1 (Relative hard Lefschetz theorem [BBD82, Sai88]). *For a pure $M \in \mathrm{MHM}(X)$, the map*

$$\ell^i : H^{-i}\pi_*M \to H^i\pi_*M(i), \quad i \geqslant 0,$$

is an isomorphism.

We define the primitive parts of π_*M

$$P_i = \mathrm{Ker}(\ell^{i+1} : H^{-i}\pi_*M \to H^{i+2}\pi_*M(i+1)), \qquad i \geqslant 0. \tag{A1}$$

Applying the isomorphism ℓ^i, we obtain

$$P_i \simeq \mathrm{Ker}(\ell : H^i\pi_*M \to H^{i+2}\pi_*M(1))(i).$$

By the above theorem, there are decompositions

$$M = \bigoplus_{i \geqslant 0} P_i[i] \otimes H^*(\mathbb{P}^i, \mathbb{Q}), \tag{A2}$$

$$H^kM = \bigoplus_{\substack{0 \leqslant j \leqslant i \\ -i+2j=k}} P_i(-j) = \bigoplus_{j \geqslant 0, k} P_{2j-k}(-j), \quad k \in \mathbb{Z}. \tag{A3}$$

LEMMA A.2. *Let $\pi : X \to Y$ be a smooth projective morphism with smooth Y and connected fibres of dimension n. Then the primitive parts of $\pi_*\mathbf{IC}_X$ satisfy $P_n = H^{-n}\pi_*\mathbf{IC}_X = \mathbf{IC}_Y(n/2)$ and $P_i = 0$ for $i > n$. We also have $H^n\pi_*\mathbf{IC}_X \simeq \mathbf{IC}_Y(-n/2)$.*

Proof. The functor π_* has amplitude $[-n, n]$, hence $H^{-i}\pi_*\mathbf{IC}_X = H^i\pi_*\mathbf{IC}_X = 0$ for $i > n$. Therefore, $P_n = H^{-n}\pi_*\mathbf{IC}_X$. The map $\mathbb{Q}_Y \to \pi_*\pi^*\mathbb{Q}_Y = \pi_*\mathbb{Q}_X$ induces the map $\mathbf{IC}_Y \to \pi_*\mathbf{IC}_X(-n/2)[-n]$, hence $\mathbf{IC}_Y \to H^{-n}\pi_*\mathbf{IC}_X(-n/2)$. To show that this is an isomorphism, we consider the corresponding map between perverse sheaves $\mathrm{IC}_Y \to {}^{\mathrm{p}}H^{-n}\pi_*\mathrm{IC}_X$. As π is smooth, the object $M = \pi_*\mathrm{IC}_X \in D^b_c(\mathbb{Q}_Y)$ is smooth. Therefore, ${}^{\mathrm{p}}H^iM = (\mathsf{H}^{i-d_Y}M)[d_Y]$ for all $i \in \mathbb{Z}$, where $\mathsf{H}^iM \in \mathrm{Sh}(\mathbb{Q}_Y)$ denotes the ith cohomology sheaf. In particular, ${}^{\mathrm{p}}H^{-n}\pi_*\mathrm{IC}_X = (\mathsf{H}^{-d_X}\pi_*\mathrm{IC}_X)[d_Y] = (\mathsf{H}^0\pi_*\mathbb{Q}_X)[d_Y]$. The map $\mathrm{IC}_Y = \mathbb{Q}_Y[d_Y] \to (\mathsf{H}^0\pi_*\mathbb{Q}_X)[d_Y]$ is an isomorphism as the fibres are connected. □

Remark A.3. If $\pi : X \to Y$ is a $\mathbb{P}^n$-fibration, then the primitive part P_n of $\pi_*\mathbf{IC}_X$ is the only nonzero component (we can check this on the fibres). Therefore,

$$\pi_*\mathbf{IC}_X = \mathbf{IC}_Y(n/2)[n] \otimes H^*(\mathbb{P}^n, \mathbb{Q}) = \mathbf{IC}_Y \otimes H^*(\mathbb{P}^n, \mathbb{Q}) \langle n \rangle . \tag{A4}$$

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