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ON THE WEAK-HASH METRIC FOR BOUNDEDLY FINITE INTEGER-VALUED MEASURES

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Abstract

It is known that the space of boundedly finite integer-valued measures on a complete separable metric space becomes a complete separable metric space when endowed with the weak-hash metric. It is also known that convergence under this topology can be characterised in a way that is similar to the weak convergence of totally finite measures. However, the original proofs of these two fundamental results assume that a certain term is monotonic, which is not the case as we show by a counterexample. We clarify these original proofs by addressing the parts that rely on this assumption and finding alternative arguments.

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1. Introduction

Let $\mathcal{X}$ be a complete separable metric space and $x_0 \in \mathcal{X}$ be a fixed origin. Let $B_r(x)$ denote the open ball with radius $r \in \mathbb{R}_{\geq 0}$ and centre $x \in \mathcal{X}$; write $B_r := B_r(x_0)$ for the open balls centred at x_0. For any subset $A \subset \mathcal{X}$ and $\varepsilon \in \mathbb{R}_{>0}$, the ε-neighbourhood of A is defined by $A^\varepsilon := \bigcup_{a\in A} B_\varepsilon(a)$, the boundary of A is denoted by ∂A and the closure of A is denoted by $\overline{A}$. For any Borel measure ξ on $\mathcal{X}$ and any $r \in \mathbb{R}_{\geq 0}$, we use the notation $\xi^{(r)}$ to refer to the restriction of ξ to the open ball B_r, that is, $\xi^{(r)}(A) = \xi(A \cap B_r)$ for all $A \in \mathcal{B}(\mathcal{X})$. A Borel measure ξ on $\mathcal{X}$ is called totally finite if $\xi(\mathcal{X}) < \infty$. Let $\mathcal{M}_\mathcal{X}$ denote the space of totally finite measures on $\mathcal{X}$ and define the Prohorov distance d on $\mathcal{M}_\mathcal{X}$ by

$$\begin{aligned} d\ &: \mathcal{M}_\mathcal{X} \times \mathcal{M}_\mathcal{X} \to \mathbb{R}_{\geq 0} \\ &(\mu,\nu) \mapsto d(\mu,\nu) := \inf\{\varepsilon \in \mathbb{R}_{\geq 0} : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \\ &\qquad\qquad\qquad\qquad \text{for all closed } A \subset \mathcal{X}\}. \end{aligned}$$

It is known that d makes $\mathcal{M}_\mathcal{X}$ a complete separable metric space (see, for example, [1, Section A2.5, pages 398–402].

In this paper, we are interested in boundedly finite integer-valued measures. A Borel measure ξ on $\mathcal{X}$ is called boundedly finite if $\xi(A) < \infty$ for all bounded Borel sets $A \in \mathcal{B}(\mathcal{X})$. We denote by $\mathcal{N}^{\#}_{\mathcal{X}}$ the space of boundedly finite measures on $\mathcal{X}$ with values in $\mathbb{N} \cup \{\infty\}$. Note that such measures are always atomic, that is, a superposition of Dirac measures (see, for example, [2, Proposition 9.1.III(ii), page 4]). One might ask if the Prohorov distance d on the space $\mathcal{M}_{\mathcal{X}}$ has a counterpart on the space $\mathcal{N}^{\#}_{\mathcal{X}}$. Daley and Vere-Jones [1, page 403] tackled this question by considering the distance function

$$\begin{aligned} d^{\#} &: \mathcal{N}^{\#}_{\mathcal{X}} \times \mathcal{N}^{\#}_{\mathcal{X}} \to \mathbb{R}_{\geq 0} \\ &(\mu, \nu) \mapsto d^{\#}(\mu, \nu) := \int_0^{\infty} e^{-r} \frac{d(\mu^{(r)}, \nu^{(r)})}{1 + d(\mu^{(r)}, \nu^{(r)})}\, dr. \end{aligned} \tag{1.1}$$

The core idea is to use the Prohorov metric on the restrictions to the open balls and compute a weighted average. Daley and Vere-Jones call the corresponding topology the $w^{\#}$-topology ('weak-hash') and refer to $d^{\#}$ as the $w^{\#}$-distance. They then obtained the following two fundamental results. The first is a characterisation of convergence under this metric.

Theorem 1.1 (Characterisation of convergence). *Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{N}^{\#}_{\mathcal{X}}$ and $\mu \in \mathcal{N}^{\#}_{\mathcal{X}}$. Then the following statements are equivalent:*

(i) *$d^{\#}(\mu_k, \mu) \to 0$ as $k \to \infty$;*
(ii) *$\int_{\mathcal{X}} f(x)\mu_k(dx) \to \int_{\mathcal{X}} f(x)\mu(dx)$ as $k \to \infty$ for all bounded continuous functions f on $\mathcal{X}$ vanishing outside a bounded set;*
(iii) *there exists an increasing sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n \to \infty$ as $n \to \infty$ such that $d(\mu_k^{(r_n)}, \mu^{(r_n)}) \to 0$ as $k \to \infty$ for all $n \in \mathbb{N}$;*
(iv) *$\mu_k(A) \to \mu(A)$ as $k \to \infty$ for all bounded sets $A \in \mathcal{B}(\mathcal{X})$ such that $\mu(\partial A) = 0$.*

The second result confirms that $d^{\#}$ is indeed the counterpart of d, that is, $\mathcal{N}^{\#}_{\mathcal{X}}$ inherits the completeness and separability properties of $\mathcal{X}$ under the metric $d^{\#}$. This second result also provides us with a characterisation of the Borel σ-algebra $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$.

Theorem 1.2 (Metric properties of $\mathcal{N}^{\#}_{\mathcal{X}}$).

(i) *The space $\mathcal{N}^{\#}_{\mathcal{X}}$ is a complete separable metric space when it is equipped with the distance function $d^{\#}$.*
(ii) *The corresponding Borel σ-algebra $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$ is the smallest σ-algebra that makes all mappings $\Phi_A : \mathcal{N}^{\#}_{\mathcal{X}} \to \mathbb{N} \cup \{\infty\}$, $A \in \mathcal{B}(\mathcal{X})$, measurable, where $\Phi_A(\xi) = \xi(A)$.*

Theorems 1.1 and 1.2 are Proposition A2.6.II and Theorem A2.6.III in [1, pages 403–405], respectively.

Regarding the motivation of this paper, the metric space $(\mathcal{N}^{\#}_{\mathcal{X}}, d^{\#})$ is a stepping stone to the theory of point processes as presented by Daley and Vere-Jones [2], who define a point process as a random element in $\mathcal{N}^{\#}_{\mathcal{X}}$. The present research was in fact triggered by the work [6] on the existence and uniqueness of marked point processes defined via their intensity. Since the above theorems are crucial in the framework and proofs, the present author examined them carefully, which resulted in this paper.

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We now turn to the precise purpose of this paper. To argue that the integrand in (1.1) is measurable and prove the above properties of the metric $d^{\#}$, Daley and Vere-Jones [1, pages 403–405] assumed that $d(\mu^{(r)}, \nu^{(r)})$ is nondecreasing as a function of $r \in \mathbb{R}_{\geq 0}$. Example 1.3 below shows that this need not necessarily be true.

Example 1.3. Set $\mathcal{X} = \mathbb{R}$, $x_0 = 0$, $\mu = \delta_0$ and $\nu = \delta_{0.5}$, where, for any $x \in \mathcal{X}$, δ_x denotes the Dirac measure at x. Then, as long as $r < 0.5$, $d(\mu^{(r)}, \nu^{(r)}) = 1$. However, as soon as $r > 0.5$, $d(\mu^{(r)}, \nu^{(r)}) = 0.5$.

Consequently, our goal is to clarify the original proofs of Theorems 1.1 and 1.2 given in [1] by addressing specifically the parts that rely on the assumed monotonicity of $d(\mu^{(r)}, \nu^{(r)})$. Note that Daley and Vere-Jones [1] considered the larger space $\mathcal{M}^{\#}_{\mathcal{X}}$ of boundedly finite measures, that is, not necessarily integer-valued. The proofs we develop here (except in Section 3) are specialised to the subspace $\mathcal{N}^{\#}_{\mathcal{X}}$ and take advantage of the discrete nature of its elements. Besides, we should add that an alternative metrisation of $\mathcal{M}^{\#}_{\mathcal{X}}$, leading to the same properties, is presented in [4, Section 4.1, pages 111–117]. According to the historical and bibliographical notes in [4, page 638], the extension from totally finite measures to boundedly finite measures under this alternative metric was first developed in [5].

The paper is organised as follows. Section 2 gives some preliminary results on the Prohorov metric. Section 3 shows that the distance function in (1.1) is well defined. Section 4 deals with the proof of Theorem 1.1. Sections 5 and 6 address the proof of Theorem 1.2.

Remark 1.4. We would like to stress that this paper focuses on the parts of the original proofs that assume that $r \mapsto d(\mu^{(r)}, \nu^{(r)})$ is nondecreasing (with the exception of Section 6). Our main objective is to find alternative arguments for these parts specifically. To understand the proofs of Theorems 1.1 and 1.2 in their entirety and the details of the other parts that are not treated here, we refer the reader to the original text [1, pages 403–405].

2. Preliminaries on the Prohorov metric

As the Prohorov metric d is the main building block of the $w^{\#}$-distance $d^{\#}$, it is not surprising that we need to study its behaviour. In particular, we will apply the following lemmas.

Lemma 2.1. *Let $\mu \in \mathcal{M}^{\#}_{\mathcal{X}}$ and $p, r \in \mathbb{R}_{\geq 0}$ with $p \leq r$. Then $d(\mu^{(p)}, \mu^{(r)}) \leq \mu(S_r \backslash S_p)$.*

Proof. Let $\varepsilon > \mu(S_r \backslash S_p)$. Let $F \in \mathcal{B}(\mathcal{X})$ be a closed set. Clearly,

$$\mu^{(p)}(F) = \mu(F \cap S_p) \leq \mu(F^{\varepsilon} \cap S_r) + \varepsilon = \mu^{(r)}(F^{\varepsilon}) + \varepsilon.$$

Moreover,

$$\begin{aligned}\mu^{(r)}(F) &= \mu(F \cap S_p) + \mu(F \cap S_r \backslash S_p)\\ &\leq \mu^{(p)}(F) + \mu(S_r \backslash S_p) \leq \mu^{(p)}(F^{\varepsilon}) + \varepsilon.\end{aligned}$$

This means that $d(\mu^{(p)}, \mu^{(r)}) \leq \mu(S_r \backslash S_p)$ by definition of the Prohorov distance d. □

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Lemma 2.2. *Let $\mu, \nu \in \mathcal{N}^{\#}_{\mathcal{X}}$ be such that $\mu(\mathcal{X}) < \infty$, $\nu(\mathcal{X}) < \infty$. Let $\underline{r}, \bar{r}, \varepsilon \in \mathbb{R}_{>0}$ be such that $\underline{r} < \bar{r}$ and $\varepsilon < (\bar{r} - \underline{r})/2 < 1$. If $\mu(B_{\bar{r}} \backslash B_{\underline{r}}) = 0$ and $\nu(B_{\bar{r}-\varepsilon} \backslash B_{\underline{r}+\varepsilon}) > 0$, then $d(\mu, \nu) \geq \varepsilon$.*

Proof. Let $0 \leq \delta < \varepsilon$ and $u \in B_{\bar{r}-\varepsilon} \backslash B_{\underline{r}+\varepsilon}$ be such that $\nu(\{u\}) \geq 1$. Then

$$\nu(\{u\}) \geq 1 > \delta = \mu(\{u\}^{\delta}) + \delta,$$

which implies that $d(\mu, \nu) \geq \delta$ by definition of the Prohorov distance. Consequently, $d(\mu, \nu) \geq \varepsilon$. □

Lemma 2.3. *Let $r \in \mathbb{R}_{\geq 0}$ and $\mu, \nu \in \mathcal{N}^{\#}_{\mathcal{X}}$. Then $d(\mu^{(r)}, \nu^{(r)}) \geq |\mu(B_r) - \nu(B_r)|$.*

Proof. Without loss of generality, we can assume that $\mu(B_r) > \nu(B_r)$. Choose ε, δ with $\varepsilon \in [0, \mu(B_r) - \nu(B_r))$ and $\delta \in [0, \mu(B_r) - \nu(B_r) - \varepsilon)$. By [1, Proposition A2.2.II, page 386], there exists a closed set $F \subset B_r$ such that $\mu^{(r)}(B_r \backslash F) < \delta$. Then

$$\begin{aligned} \mu^{(r)}(F) &= \mu^{(r)}(B_r) - \mu^{(r)}(B_r \backslash F) > \mu^{(r)}(B_r) - \delta \\ &> \mu^{(r)}(B_r) + \varepsilon + \nu(B_r) - \mu(B_r) \geq \nu^{(r)}(F^{\varepsilon}) + \varepsilon. \end{aligned}$$

Again, this implies that $d(\mu^{(r)}, \nu^{(r)}) \geq |\mu(B_r) - \nu(B_r)|$ by definition of the Prohorov distance. □

3. The metric $d^{\#}$ is well defined

In this section, we address the proof in [1, page 403] that shows that $d^{\#}$ is indeed a well-defined metric. We have to check that the integral in (1.1) is well defined and, in particular, that $r \mapsto d(\mu^{(r)}, \nu^{(r)})$ is measurable. To achieve this, it suffices to notice that this function is actually piecewise constant since μ and ν are atomic with finitely many atoms in any bounded set. In fact, for any $R \in \mathbb{R}_{>0}$, as r goes from 0 to R, the restricted measures $\mu^{(r)}$ and $\nu^{(r)}$ change only a finite number of times and so does $d(\mu^{(r)}, \nu^{(r)})$. The other arguments in [1, page 403] are then enough to ensure that $d^{\#}$ satisfies all the conditions of a distance function.

As a side note, for the general case where $\mu, \nu \in \mathcal{M}^{\#}_{\mathcal{X}}$, we can prove that $r \mapsto d(\mu^{(r)}, \nu^{(r)})$ is measurable by showing that it is of finite variation.

Proposition 3.1. *Let $\mu, \nu \in \mathcal{M}^{\#}_{\mathcal{X}}$ and $R \in \mathbb{R}_{\geq 0}$. Then, as a function of $r \in \mathbb{R}_{\geq 0}$, the variation of $d(\mu^{(r)}, \nu^{(r)})$ over $[0, R]$ is bounded by $\mu(S_R) + \nu(S_R)$. In particular, $r \mapsto d(\mu^{(r)}, \nu^{(r)})$ is of bounded variation and, thus, measurable.*

Proof. Let $r \in \mathbb{R}_{\geq 0}$ and $\delta > 0$. Applying the triangle inequality to the Prohorov distance, we obtain the following two inequalities:

$$\begin{aligned} d(\mu^{(r+\delta)}, \nu^{(r+\delta)}) &\leq d(\mu^{(r+\delta)}, \mu^{(r)}) + d(\mu^{(r)}, \nu^{(r)}) + d(\nu^{(r)}, \nu^{(r+\delta)}), \\ d(\mu^{(r)}, \nu^{(r)}) &\leq d(\mu^{(r)}, \mu^{(r+\delta)}) + d(\mu^{(r+\delta)}, \nu^{(r+\delta)}) + d(\nu^{(r+\delta)}, \nu^{(r)}). \end{aligned}$$

This implies that

$$|d(\mu^{(r+\delta)}, \nu^{(r+\delta)}) - d(\mu^{(r)}, \nu^{(r)})| \leq d(\mu^{(r)}, \mu^{(r+\delta)}) + d(\nu^{(r)}, \nu^{(r+\delta)}).$$

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Using Lemma 2.1, we can go further and conclude that

$$|d(\mu^{(r+\delta)}, \nu^{(r+\delta)}) - d(\mu^{(r)}, \nu^{(r)})| \leq \mu(S_{r+\delta}) - \mu(S_r) + \nu(S_{r+\delta}) - \nu(S_r).$$

Since $\mu(S_r)$ and $\nu(S_r)$ are nondecreasing in r and always finite (because μ and ν are boundedly finite), they are of bounded variation, which concludes the proof. □

4. Characterisation of convergence in the $w^\#$-topology

In this section, we address the proof of Theorem 1.1, which characterises the convergence of boundedly finite integer-valued measures.

Proof of Theorem 1.1. We need to show only the implication (i) $\implies$ (iii) because this is the only part in [1, page 403] which relies on the assumption that $d(\mu^{(r)}, \nu^{(r)})$ is nondecreasing in $r \in \mathbb{R}_{\geq 0}$. The rest of the proof of Proposition A2.6.II in [1, pages 403–404] can be used to show that (iii) $\implies$ (ii) $\implies$ (iv) $\implies$ (i).

Let $n \in \mathbb{N}$ and $\underline{r}_n, \bar{r}_n \in \mathbb{R}_{\geq 0}$ be such that $n < \underline{r}_n < \bar{r}_n < n+1$ and $\mu(B_{\bar{r}_n} \backslash B_{\underline{r}_n}) = 0$. Let $0 < \varepsilon < (\bar{r}_n - \underline{r}_n)/2$. By contradiction, assume that for any $K \in \mathbb{N}$, there exists $k > K$ such that $\mu_k(B_{\bar{r}_n - \varepsilon} \backslash B_{\underline{r}_n + \varepsilon}) \geq 1$. Then, by Lemma 2.2, there exists a subsequence $(k_p)_{p \in \mathbb{N}}$ such that $d(\mu_{k_p}^{(r)}, \mu^{(r)}) \geq \varepsilon$ for all $r \geq n+1$, $p \in \mathbb{N}$. Along this subsequence,

$$d^\#(\mu_{k_p}, \mu) = \int_0^\infty e^{-r} \frac{d(\mu_{k_p}^{(r)}, \mu^{(r)})}{1 + d(\mu_{k_p}^{(r)}, \mu^{(r)})}\, dr \geq \int_{n+1}^\infty e^{-r} \frac{\varepsilon}{1+\varepsilon}\, dr = \frac{\varepsilon}{1+\varepsilon} e^{-n-1} > 0,$$

which contradicts the assumption that $d^\#(\mu_k, \mu) \to 0$ as $k \to \infty$. Consequently, there exists a $K \in \mathbb{N}$ such that, for all $k \geq K$, $\mu_k(S_{\bar{r}_n - \varepsilon} \backslash S_{\underline{r}_n + \varepsilon}) = 0$. This means that, for all $k \geq K$, neither μ_k nor μ can have any atom in $S_{\bar{r}_n - \varepsilon} \backslash S_{\underline{r}_n + \varepsilon}$, whence there is some constant $d_k \in \mathbb{R}_{\geq 0}$ such that $d(\mu_k^{(r)}, \mu^{(r)}) = d_k$ for all $r \in (\underline{r}_n + \varepsilon, \bar{r}_n - \varepsilon)$. This implies that, for all $k \geq K$,

$$\begin{aligned} d^\#(\mu_k, \mu) &= \int_0^\infty e^{-r} \frac{d(\mu_k^{(r)}, \mu^{(r)})}{1 + d(\mu_k^{(r)}, \mu^{(r)})}\, dr \\ &\geq \int_{\underline{r}_n + \varepsilon}^{\bar{r}_n - \varepsilon} e^{-r} \frac{d_k}{1 + d_k}\, dr \geq \frac{d_k}{1 + d_k} e^{-\underline{r}_n - \varepsilon} (1 - e^{-(\bar{r}_n - \underline{r}_n - 2\varepsilon)}) \end{aligned}$$

and, thus, $d_k \to 0$ as $k \to \infty$. If we set $r_n = (\underline{r}_n + \bar{r}_n)/2$, we obtain $d(\mu_k^{(r_n)}, \mu^{(r_n)}) \to 0$ as $k \to \infty$.

5. Completeness and separability of $\mathcal{N}_\mathcal{X}^\#$

In this section, we address the proof of the first part of Theorem 1.2, which states that $\mathcal{N}_\mathcal{X}^\#$ is complete and separable when it is endowed with the $w^\#$-metric $d^\#$.

5.1. Completeness. To begin with, we show that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $(\mathcal{N}^{\#}_{\mathcal{X}}, d^{\#})$ is Cauchy, then the restrictions along an increasing sequence of balls are also Cauchy for the Prohorov metric d.

PROPOSITION 5.1. *Let $(\mu_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{N}^{\#}_{\mathcal{X}}$ for the $w^{\#}$-metric $d^{\#}$. Then there exists an increasing sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ with $r_n \to \infty$ as $n \to \infty$ such that, for each $n \in \mathbb{N}$, $(\mu_k^{(r_n)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_{\mathcal{X}}$ for the Prohorov metric d.*

PROOF.

Step 1. We show that $\mu_k(B_r)$ is bounded in $k \in \mathbb{N}$ for all $r \in \mathbb{R}_{\geq 0}$. By contradiction, assume this is not the case. Then there exists a subsequence such that $\mu_{k_p}(B_r) \to \infty$. Along this subsequence, for p large enough and any fixed $q \in \mathbb{N}$,

$$\begin{aligned}\int_r^{r+1} e^{-s} \frac{d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}{1 + d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}\, ds &\geq \int_r^{r+1} e^{-s} \frac{|\mu_{k_p}(B_s) - \mu_{k_q}(B_s)|}{1 + |\mu_{k_p}(B_s) - \mu_{k_q}(B_s)|}\, ds \\ &\geq \int_r^{r+1} e^{-s} \frac{\mu_{k_p}(B_r) - \mu_{k_q}(B_{r+1})}{1 + \mu_{k_p}(B_r) - \mu_{k_q}(B_{r+1})}\, ds \to e^{-r}(1 - e^{-1})\end{aligned}$$

as $p \to \infty$, where we used Lemma 2.3 and the fact that $\mu_{k_p}(B_s)$ and $\mu_{k_q}(B_s)$ are nondecreasing in s. But this is incompatible with the Cauchy assumption on $(\mu_k)_{k\in\mathbb{N}}$. Indeed, let $\varepsilon < e^{-r}(1 - e^{-1})$. Then the Cauchy assumption implies that there exists $K \in \mathbb{N}$ such that, for all $k, k' \geq K$,

$$d^{\#}(\mu_k, \mu_{k'}) = \int_0^{\infty} e^{-s} \frac{d(\mu_k^{(s)}, \mu_{k'}^{(s)})}{1 + d(\mu_k^{(s)}, \mu_{k'}^{(s)})}\, ds \leq \varepsilon.$$

But, then, for $p, q \in \mathbb{N}$ large enough, we must have

$$\varepsilon \geq \int_0^{\infty} e^{-s} \frac{d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}{1 + d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}\, ds \geq \int_r^{r+1} e^{-s} \frac{d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}{1 + d(\mu_{k_p}^{(s)}, \mu_{k_q}^{(s)})}\, ds > \varepsilon.$$

Step 2. Let $n \in \mathbb{N}$. We show that for $k, p \in \mathbb{N}$ large enough, there is a subinterval of $[n, n + 1]$ on which the functions $r \mapsto d(\mu_k^{(r)}, \mu_p^{(r)})$ are constant. Define $M := \sup_{k\in\mathbb{N}} \mu_k(B_{n+1})$, which is finite by the first step and can be understood as a bound on the number of points in the ball B_{n+1} among all measures μ_k. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$ be such that $\varepsilon_1 < \varepsilon_2 < 1/2(M + 1)$ and $\varepsilon_1 < \varepsilon_2 e^{-n-1}/(1 + \varepsilon_2)$. Let $K \in \mathbb{N}$ be such that, for all $k, k' \geq K$, $d^{\#}(\mu_k, \mu_{k'}) \leq \varepsilon_1$ (Cauchy assumption). Since $\mu_K(B_{n+1} \backslash B_n) \leq M$, we can find $\underline{r}_n, \overline{r}_n \in (n, n + 1)$ such that $\mu_K(B_{\overline{r}_n} \backslash B_{\underline{r}_n}) = 0$ and $\overline{r}_n - \underline{r}_n \geq 1/(M + 1)$. Now, by contradiction, assume that for some $p > K$, we have $\mu_p(B_{\overline{r}_n - \varepsilon_2} \backslash B_{\underline{r}_n + \varepsilon_2}) \geq 1$. Then, using Lemma 2.2,

$$\begin{aligned}\varepsilon_1 \geq d^{\#}(\mu_K, \mu_p) &= \int_0^{\infty} e^{-r} \frac{d(\mu_K^{(r)}, \mu_p^{(r)})}{1 + d(\mu_K^{(r)}, \mu_p^{(r)})}\, dr \\ &\geq \int_{n+1}^{\infty} e^{-r} \frac{d(\mu_K^{(r)}, \mu_p^{(r)})}{1 + d(\mu_K^{(r)}, \mu_p^{(r)})}\, dr \geq \frac{\varepsilon_2}{1 + \varepsilon_2} e^{-n-1},\end{aligned}$$

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which contradicts the original assumption on ε_1 and ε_2. As a consequence, for all $k \geq K$, $\mu_k(B_{\overline{r}_n-\varepsilon_2} \backslash B_{\underline{r}_n+\varepsilon_2}) = 0$, which implies that $r \mapsto d(\mu_p^{(r)}, \mu_q^{(r)})$ is constant on $(\underline{r}_n + \varepsilon_2, \overline{r}_n - \varepsilon_2)$ for all $p, q \geq K$.

Step 3. We finally show that $(\mu_k^{(r_n)})_{k\in\mathbb{N}}$ is a Cauchy sequence for the Prohorov metric d when $r_n =: (\underline{r}_n + \overline{r}_n)/2$. Let $\varepsilon > 0$ and set $\delta := (\overline{r}_n - \underline{r}_n - 2\varepsilon_2)e^{-n-1}\varepsilon/(1 + \varepsilon)$. Let $J \in \mathbb{N}$ be such that $d^{\#}(\mu_k, \mu_{k'}) \leq \delta$ for all $p, q \geq J$ (Cauchy assumption). Then, for all $p, q \geq K \vee J$,

$$\delta \geq \int_{\underline{r}_n+\varepsilon_2}^{\overline{r}_n-\varepsilon_2} e^{-r} \frac{d_{pq}}{1 + d_{pq}}\, dr \geq \frac{d_{pq}}{1 + d_{pq}}(\overline{r}_n - \underline{r}_n - 2\varepsilon_2)e^{-n-1},$$

where $d_{pq} := d(\mu_p^{(r_n)}, \mu_q^{(r_n)})$, which implies that

$$d_{pq} \leq \frac{\delta}{(\overline{r}_n - \underline{r}_n - 2\varepsilon_2)e^{-n-1} - \delta} = \frac{1}{(1 + \varepsilon)/\varepsilon - 1} = \varepsilon. \qquad \square$$

Reusing a part of the proof of Theorem A2.6.III in [1, page 404], the above proposition implies that $\mathcal{N}_{\mathcal{X}}^{\#}$ is complete. Still, we would like to mention some points that deserve a bit more detail. First, one needs to ensure that the limit of each Cauchy sequence $(\mu_k^{(r_n)})_{k\in\mathbb{N}}$ in Proposition 5.1 is still integer-valued. This can be done by adapting the proof of [2, Lemma 9.1.V, page 6]. Second, if we denote the limit of $(\mu_k^{(r_n)})_{k\in\mathbb{N}}$ by ν_n, we can show that $\nu_m^{(r_n)} = \nu_n$ when $n < m$ (that is, the sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ is consistent) by using Theorem A2.3.II(iv) in [1, page 391] and the fact that $\nu_m(\partial B_{r_n}) = 0$. Third, to show that $\mu(\cdot) := \lim_{n\to\infty} \nu_n(\cdot)$ is continuous from below, one can use the fact that $\lim_{i\to\infty} \lim_{j\to\infty} a_{ij} = \lim_{j\to\infty} \lim_{i\to\infty} a_{ij}$ for any double sequence (a_{ij}) that is nondecreasing in both i and j.

5.2. Separability. Next we prove that the space of boundedly finite integer-valued measures $\mathcal{N}_{\mathcal{X}}^{\#}$ is separable. We wish to show that there exists a countable set in $\mathcal{N}_{\mathcal{X}}^{\#}$ that can approximate well enough any element of $\mathcal{N}_{\mathcal{X}}^{\#}$. Let $\mathcal{D}_{\mathcal{X}}$ be the separability set of $\mathcal{X}$. It seems natural to expect that the set of totally finite (hence with a finite number of atoms) integer-valued measures with atoms only in $\mathcal{D}_{\mathcal{X}}$ is a good candidate. We denote this set by $\mathcal{D}_{\mathcal{N}}$. Notice that one can define an injection between $\mathcal{D}_{\mathcal{N}}$ and the finite subsets of $\mathbb{N}^2$. For example, the Dirac measure with mass $n \in \mathbb{N}$ at the mth element of $\mathcal{D}_{\mathcal{X}}$ can be represented by the set $\{(m, n)\}$. Since the finite subsets of a countable set form a countable set, $\mathcal{D}_{\mathcal{N}}$ is countable. The following proposition coupled with a part of the proof of Theorem A2.6.III in [1, page 404] allows us to conclude the argument.

Proposition 5.2. *Let $\mu \in \mathcal{N}_{\mathcal{X}}^{\#}$ and $R, \varepsilon \in \mathbb{R}_{>0}$. Then there exists $\tilde{\mu} \in \mathcal{D}_{\mathcal{N}}$ such that*

$$\int_0^R e^{-r} \frac{d(\mu^{(r)}, \tilde{\mu}^{(r)})}{1 + d(\mu^{(r)}, \tilde{\mu}^{(r)})}\, dr \leq \varepsilon.$$

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Proof. Let $(u_n)_{n\in\{1,\dots,N\}}$ be the atoms of μ in B_R, where $N \in \mathbb{N}$ is their total number, and let $(w_n)_{n\in\{1,\dots,N\}}$ be their corresponding weights. Let $\varepsilon_1 > 0$ be such that $B_{\varepsilon_1}(u_n) \subset B_R$ for all $n = 1, \dots, N$. Let $0 \le r_1 < \cdots < r_{N'} < R$ be the radii at which the atoms are located, where $N' \in \mathbb{N}$, $N' \le N$ ($r_1 = 0$ means that $x_0 \in (u_n)_{n\in\{1,\dots,N\}}$). Define $\varepsilon_2 := \frac{1}{2}\min_{n<N'}(r_{n+1} - r_n)$ and $\varepsilon_3 := \varepsilon/4N'$. Define $\varepsilon_4 := \varepsilon/(2c - \varepsilon)$, where $c = 1 - e^{-R}$, and assume that $\varepsilon < 2c$ (if this is not the case, then the desired inequality already holds no matter what $\tilde{\mu}$ is). Finally, set $\delta := \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and let $(\tilde{u}_n)_{n\in\{1,\dots,N\}}$ be such that $\tilde{u}_n \in \mathcal{D}_{\mathcal{X}}$, $\tilde{u}_n \in B_\delta(u_n)$, $n = 1, \dots, N$. We will show that $\tilde{\mu} := \sum_{n=1}^{N} w_n \delta_{\tilde{u}_n}$ satisfies the desired inequality.

Let $n = 1, \dots, N' - 1$ and $r \in (r_n + \delta, r_{n+1} - \delta)$. We can check that $d(\mu^{(r)}, \tilde{\mu}^{(r)}) \le \delta$. Indeed, since $\delta \le \varepsilon_1$ and $\delta \le \varepsilon_2$, we have $u_i \in B_r$ if and only if $\tilde{u}_i \in B_r$. Consequently, for any closed set $A \in \mathcal{B}(\mathcal{X}) \cap B_r$, using the fact that $\tilde{u}_i \in B_\delta(u_i)$,

$$\mu^{(r)}(A) = \mu(A) \le \tilde{\mu}(A^\delta \cap B_r) = \tilde{\mu}^{(r)}(A^\delta) \ \text{ and } \ \tilde{\mu}^{(r)}(A) = \tilde{\mu}(A) \le \mu(A^\delta \cap B_r) = \mu^{(r)}(A^\delta),$$

which means that $d(\mu^{(r)}, \tilde{\mu}^{(r)}) \le \delta$. Similarly, $d(\mu^{(r)}, \tilde{\mu}^{(r)}) \le \delta$ for all $r \in [0, 0 \vee (r_1 - \delta))$ and all $r \in (r_{N'} + \delta, R]$. Using this bound on the Prohorov distance between the restrictions,

$$\begin{aligned}
&\int_0^R \frac{e^{-r} d(\mu^{(r)}, \tilde{\mu}^{(r)})}{1 + d(\mu^{(r)}, \tilde{\mu}^{(r)})}\, dr \\
&\quad = \Big(\int_0^{0\vee(r_1-\delta)} + \sum_{n=1}^{N'} \int_{(r_n-\delta)\vee 0}^{r_n+\delta} + \sum_{n=1}^{N'-1} \int_{r_n+\delta}^{r_{n+1}-\delta} + \int_{r_{N'}+\delta}^{R} \Big) \frac{e^{-r} d(\mu^{(r)}, \tilde{\mu}^{(r)})}{1 + d(\mu^{(r)}, \tilde{\mu}^{(r)})}\, dr \\
&\quad \le \int_0^R e^{-r} \frac{\delta}{1+\delta}\, dr + \sum_{n=1}^{N'} \int_{(r_n-\delta)\vee 0}^{r_n+\delta} e^{-r} \frac{d(\mu^{(r)}, \tilde{\mu}^{(r)})}{1 + d(\mu^{(r)}, \tilde{\mu}^{(r)})}\, dr \\
&\quad \le (1 - e^{-R}) \frac{\delta}{1+\delta} + 2\delta N' \le (1 - e^{-R}) \frac{\varepsilon_4}{1+\varepsilon_4} + 2\varepsilon_3 N' = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square
\end{aligned}$$

6. Characterisation of the σ-algebra $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}})$

This section proves the second part of Theorem 1.2. We show that all mappings $\Phi_A : \xi \mapsto \xi(A)$, $\xi \in \mathcal{N}^\#_{\mathcal{X}}$, $A \in \mathcal{B}(\mathcal{X})$, are measurable with respect to the Borel σ-algebra $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}})$ and that $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}})$ is actually generated by all these mappings. This property is very useful to check the measurability of functionals on $\mathcal{N}^\#_{\mathcal{X}}$, such as Hawkes functionals, as demonstrated in [6]. Our proof is different from the original one in [1, page 405] as we identify a convenient basis for the $w^\#$-hash topology (Proposition 6.1). Note however that this last result is directly inspired by [1, page 398, Proposition A2.5.I], where three different bases for the weak topology on $\mathcal{M}_{\mathcal{X}}$ are given. Besides, our proof of Theorem 1.2(ii) shows explicitly why the mapping Φ_A is $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}})$-measurable when A is a bounded closed set.

Proposition 6.1. *Consider the family of sets*

$$\begin{aligned}
\{\xi \in \mathcal{N}^\#_{\mathcal{X}} : {} & \xi(F_i) < \mu(F_i) + \varepsilon \text{ for } i = 1, \dots, m, \\
& |\xi(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})| < \varepsilon \text{ and } \xi(\partial B_{r_j}) = 0 \text{ for } j = 1, \dots, n\}, \qquad (6.1)
\end{aligned}$$

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where $\mu \in \mathcal{N}^{\#}_{\mathcal{X}}$, $\varepsilon \in \mathbb{R}_{>0}$, $m, n \in \mathbb{N}$, F_i, for $i = 1, \ldots, m$, is a bounded closed set of $\mathcal{X}$ and $r_j \in \mathbb{R}_{>0}$, for $j = 1, \ldots, n$, is such that $\mu(\partial B_{r_j}) = 0$. This family forms a basis that generates the $w^{\#}$-topology.

Proof.

Step 1. We check that this family is a basis. Let $\mu, \mu' \in \mathcal{N}^{\#}_{\mathcal{X}}$, $\varepsilon, \varepsilon' \in \mathbb{R}_{>0}$, let $F_1, \ldots, F_m$ and $F'_1, \ldots, F'_{m'}$ be bounded closed sets and let $r_1, \ldots, r_n, r'_1, \ldots, r'_{n'} > 0$ be such that $\mu(\partial B_{r_j}) = 0$ and $\mu'(\partial B_{r'_j}) = 0$. Consider the sets A and B of the form (6.1) generated by these two collections, respectively, and let $\mu'' \in A \cap B$. We will now find a set C, again of the form (6.1), such that $\mu'' \in C$ and $C \subset A \cap B$. Set the following parameters:

$$\delta := \min_{i=1,\ldots,m} \mu(F_i) + \varepsilon - \mu''(F_i), \quad \delta' := \min_{i=1,\ldots,m'} \mu'(F'_i) + \varepsilon' - \mu''(F'_i),$$
$$\gamma := \min_{j=1,\ldots,n} \varepsilon - |\mu''(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})|, \quad \gamma' := \min_{j=1,\ldots,n'} \varepsilon' - |\mu''(\overline{B}_{r'_j}) - \mu'(\overline{B}_{r'_j})|$$

and let $\varepsilon'' := \min(\delta, \delta', \gamma, \gamma')$. Now consider the set

$$\begin{aligned} C := \{\xi \in \mathcal{N}^{\#}_{\mathcal{X}} : \; & \xi(F_i) < \mu''(F_i) + \varepsilon'' \text{ for } i = 1, \ldots, m, \\ & \xi(F'_i) < \mu''(F'_i) + \varepsilon'' \text{ for } i = 1, \ldots, m', \\ & |\xi(\overline{B}_{r_j}) - \mu''(\overline{B}_{r_j})| < \varepsilon'' \text{ and } \xi(\partial B_{r_j}) = 0 \text{ for } j = 1, \ldots, n, \\ & |\xi(\overline{B}_{r'_j}) - \mu''(\overline{B}_{r'_j})| < \varepsilon'' \text{ and } \xi(\partial B_{r'_j}) = 0 \text{ for } j = 1, \ldots, n'\}. \end{aligned}$$

Clearly, the set C is of the form (6.1). We check that $C \subset A \cap B$. Let $\xi \in C$. For all $i = 1, \ldots, m$,

$$\xi(F_i) < \mu''(F_i) + \varepsilon'' \leq \mu(F_i) + \varepsilon,$$

because $\varepsilon'' \leq \mu(F_i) + \varepsilon - \mu''(F_i)$. For all $j = 1, \ldots, n$,

$$|\xi(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})| \leq |\xi(\overline{B}_{r_j}) - \mu''(\overline{B}_{r_j})| + |\mu''(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})| < \varepsilon,$$

because $\varepsilon'' \leq \varepsilon - |\mu''(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})|$. Thus, $\xi \in A$. A similar argument yields $\xi \in B$ and so $C \subset A \cap B$.

Step 2. We check that every element of this basis contains an open ball. Consider first any set A of the form (6.1) but for which $n = 1$ (only one ball). Let $\delta \in (0, 1)$ be such that $2\delta < \varepsilon$, $\mu(F^{\delta}_i) = \mu(F_i)$ for all $i = 1, \ldots, m$ and

$$\mu\Big(\overline{\overline{B}^{\delta}_{r_1} \backslash \overline{B}_{r_1}}^{\delta}\Big) = 0,$$

which means that δ is chosen small enough such that there are no atoms within a distance δ of the boundary ∂B_{r_1}. Let $R \in \mathbb{R}_{>0}$ be such that $F^{\delta}_i \subset B_R$ for all $i = 1, \ldots, m$ and such that $r_1 + 2\delta < R$. Consider now the ball $B := \{\xi \in \mathcal{N}^{\#}_{\mathcal{X}} : d^{\#}(\mu, \xi) < \gamma\}$, where $\gamma := e^{-R}\delta/(1 + \delta)$. Take any $\xi \in B$ and, by contradiction, assume that $\xi(F_i) > \mu(F^{\delta}_i) + \delta$

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for some $i = 1, \ldots, m$. This implies that $d(\xi^{(r)}, \mu^{(r)}) \geq \delta$ for all $r \geq R$, which in turn implies that

$$d^{\#}(\xi, \mu) \geq \int_R^\infty e^{-r} \frac{\delta}{1+\delta}\, dr = \gamma.$$

This contradicts the fact that $\xi \in B$ and, thus, we must have

$$\xi(F_i) \leq \mu(F_i^\delta) + \delta = \mu(F_i) + \delta < \mu(F_i) + \varepsilon, \quad i = 1, \ldots, m.$$

The same reasoning holds for the closed sets $\overline{B}_{r_1}$ and ∂B_{r_1}, finally implying that

$$\xi(\partial B_{r_1}) = \mu(\partial B_{r_1}) = 0 \quad \text{and} \quad \xi(\overline{B}_{r_1}) - \mu(\overline{B}_{r_1}) \leq \delta < \varepsilon.$$

To obtain $\xi \in A$, it remains only to show that $\mu(\overline{B}_{r_1}) - \xi(\overline{B}_{r_1}) < \varepsilon$. Using again the previous reasoning,

$$\xi(\overline{B}_{r_1}^\delta) - \xi(\overline{B}_{r_1}) = \xi(\overline{B}_{r_1}^\delta \backslash \overline{B}_{r_1}) \leq \xi\Big(\overline{\overline{B}_{r_1}^\delta \backslash \overline{B}_{r_1}}\Big) \leq \mu\Big(\overline{\overline{B}_{r_1}^\delta \backslash \overline{B}_{r_1}}^{\,\delta}\Big) + \delta = \delta$$

and also $\mu(\overline{B}_{r_1}) \leq \xi(\overline{B}_{r_1}^\delta) + \delta$. This implies the desired inequality,

$$\mu(\overline{B}_{r_1}) - \xi(\overline{B}_{r_1}) = \mu(\overline{B}_{r_1}) - \xi(\overline{B}_{r_1}^\delta) + \xi(\overline{B}_{r_1}^\delta) - \xi(\overline{B}_{r_1}) \leq \delta + \delta < \varepsilon,$$

and allows us to conclude that the ball B is included in the neighbourhood A. For the general case when the set A is defined by multiple balls (that is, $n > 1$), we simply view it as an intersection of sets A_j, where each A_j is defined by one ball (that is, $m = 1$). As shown above, for each A_j, we can find an adequate ball with centre μ and radius γ_j. Then the ball with radius $\gamma = \min \gamma_i$ must be included in A.

Step 3. We check that every open ball contains an element of this basis. Let $\mu \in \mathcal{N}_{\mathcal{X}}^{\#}$, $\varepsilon \in \mathbb{R}_{>0}$ and consider the ball $B := \{\xi \in \mathcal{N}_{\mathcal{X}}^{\#} : d^{\#}(\mu, \xi) < \varepsilon\}$. Let $R > 0$ be such that $e^{-R} < \frac{1}{2}\varepsilon$. Let $\rho_1 < \cdots < \rho_N$ be all the radii in $(0, R)$ such that $\mu(\partial B_{\rho_j}) > 0$, $j = 1, \ldots, N$. Set also $\rho_0 := 0$ and $\rho_{N+1} := R$. Define $\rho := \frac{1}{2} \min_{j=1,\ldots,N+1}(\rho_j - \rho_{j-1})$, let $\gamma < \varepsilon/8(N+2)$ and set $\delta := \min(\rho, \gamma)$. Define the bounded closed sets G_j by $G_j := \overline{B}_{\rho_j - \delta} \backslash B_{\rho_{j-1}+\delta}$ for $j = 1, \ldots, N+1$ and notice that $\mu(G_j) = 0$. Also, define the radii $r_j := (\rho_{j-1} + \rho_j)/2$, $j = 1, \ldots, N+1$. For all r_j, reusing the last part of the proof of Proposition A2.5.I in [1, page 399], we know that we can find $\tilde{\varepsilon}_j \in (0, 1)$ and a finite family of closed bounded sets $F_{1,j}, \ldots, F_{m_j,j}$ such that

$$\begin{aligned} A_j &:= \{\xi \in \mathcal{N}_{\mathcal{X}}^{\#} : \xi(F_{i,j}) < \mu(F_{i,j}) + \tilde{\varepsilon}_j \text{ for } i = 1, \ldots, m_j, |\xi(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})| < \tilde{\varepsilon}_j\} \\ &\subset \{\xi \in \mathcal{N}_{\mathcal{X}}^{\#} : d(\mu^{(r_j)}, \xi^{(r_j)}) < c\}, \end{aligned}$$

where we choose c such that $(1 - e^{-R})c/(1 + c) < \varepsilon/4$. Finally, set $\tilde{\varepsilon} = \min \tilde{\varepsilon}_j$ and consider the set

$$\begin{aligned} A := \{\xi \in \mathcal{N}_{\mathcal{X}}^{\#} : &\xi(F_{i,j}) < \mu(F_{i,j}) + \tilde{\varepsilon} \text{ for } i = 1, \ldots, m_j,\ \xi(G_j) < \mu(G_j) + \tilde{\varepsilon}, \\ &|\xi(\overline{B}_{r_j}) - \mu(\overline{B}_{r_j})| < \tilde{\varepsilon} \text{ and } \xi(\partial B_{r_j}) = 0 \text{ for } j = 1, \ldots, N+1\}, \end{aligned}$$

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which is of the form (6.1) and is such that $A \subset A_j$, $j = 1, \dots, N+1$. For all $\xi \in A$, this implies that $d(\mu^{(r_j)}, \xi^{(r_j)}) < c$, $j = 1, \dots, N+1$. This also implies that $\xi(G_j) = 0$ and thus $r \mapsto d(\mu^{(r)}, \xi^{(r)})$ is constant on each interval $(\rho_{j-1} + \delta, \rho_j - \delta)$, $j = 1, \dots, N+1$. Noting that $r_j \in (\rho_{j-1} + \delta, \rho_j - \delta)$, it remains to check that

$$\begin{aligned} d^{\#}(\mu, \xi) &< \int_0^R e^{-r} \frac{d(\mu^{(r)}, \xi^{(r)})}{1 + d(\mu^{(r)}, \xi^{(r)})}\, dr + \frac{1}{2}\varepsilon \\ &< 2\delta(N+2) + (1 - e^{-R}) \frac{c}{1+c} + \frac{1}{2}\varepsilon < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Consequently, we have indeed that $A \subset B$, which concludes the proof. □

Proof of Theorem 1.2(ii).

Step 1. We first show that Φ_A is $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$-measurable for all bounded closed sets A. Let $n \in \mathbb{N}$. We prove that $I := \{\xi \in \mathcal{N}^{\#}_{\mathcal{X}} : \xi(A) \leq n\}$ is an open set of $\mathcal{N}^{\#}_{\mathcal{X}}$, implying that Φ_A is indeed $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$-measurable. If $A = \emptyset$, then $I = \mathcal{N}^{\#}_{\mathcal{X}}$, which is open. From now on, we assume that $A \neq \emptyset$. Let $\mu \in I$ (I is clearly not empty). Let $\delta \in (0,1)$ be such that $\mu(A) = \mu(A^{\delta})$ (this is always possible since μ has a finite number of atoms in $A^{\gamma} \backslash A$, with $\gamma = 1$, say). Let $R > 0$ be such that $A^{\delta} \subset B_R$. Consider the open ball $J := \{\nu \in \mathcal{N}^{\#}_{\mathcal{X}} : d^{\#}(\mu, \nu) < \varepsilon\}$ with $\varepsilon = e^{-R}\delta/(1+\delta)$. Then $J \subset I$, which implies that I is open. Indeed, let $\nu \in J$ and, by contradiction, assume that $\nu(A) > \mu(A^{\delta}) + \delta$. Then, for all $r \geq R$,

$$\nu^{(r)}(A) = \nu(A) > \mu(A^{\delta}) + \delta = \mu^{(r)}(A^{\delta}) + \delta,$$

which means that $d(\mu^{(r)}, \nu^{(r)}) \geq \delta$. Hence,

$$d^{\#}(\mu, \nu) \geq \int_R^{\infty} e^{-r} \frac{\delta}{1+\delta}\, dr = \varepsilon,$$

which contradicts the assumption that $\nu \in J$. Thus, $\nu(A) \leq \mu(A^{\delta}) + \delta = \mu(A) + \delta$. Since $\nu(A) \in \mathbb{N}$, $\mu(A) \in \mathbb{N}$ and $\delta < 1$, this implies that $\nu(A) \leq \mu(A) \leq n$ and thus $\nu \in I$.

Step 2. Consider the class C of sets $C := \{A \in \mathcal{B}(\mathcal{X}) : \Phi_A \text{ is } \mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})\text{-measurable}\}$. By the continuity of measures [3, Lemma 1.14, page 8], we have $\Phi_{A_n} \uparrow \Phi_A$ for any sequence $A_n \uparrow A$ and, since the limit of measurable functions is measurable [3, Lemma 1.9, page 6], C is closed under increasing limits. In other words, C forms a monotone class. Moreover, consider the class $\mathcal{R}$ of sets of the form $\bigcup_{i=1}^n A_i \backslash B_i$, where $n \in \mathbb{N}$ and $A_i, B_i \in \mathcal{B}(\mathcal{X})$ are bounded closed sets such that $(A_i \backslash B_i) \cap (A_j \backslash B_j) = \emptyset$ as soon as $i \neq j$ (that is, we consider finite disjoint unions of differences of bounded closed sets). One can check that $\mathcal{R}$ is stable by finite intersections and symmetric differences (perhaps the most difficult part is to see that, for any bounded closed sets A_1, A_2, B_1, B_2, the difference $(A_1 \backslash B_1) \backslash (A_2 \backslash B_2)$ can be written as a disjoint union of differences of bounded closed sets). This means that $\mathcal{R}$ forms a ring. Besides, for any bounded closed sets $A, B \in \mathcal{X}$, since $\xi(A) < \infty$ for all $\xi \in \mathcal{N}^{\#}_{\mathcal{X}}$, we have $\Phi_{A \backslash B} = \Phi_{A \backslash (A \cap B)} = \Phi_A - \Phi_{A \cap B}$. As $A \cap B$ is still a bounded closed set, by applying the first part of the proof, we see that

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$\Phi_{A\setminus B}$ is measurable. By the countable additivity of measures, this implies that Φ_A is measurable for any set $A \in \mathcal{R}$ and thus $\mathcal{R} \subset \mathcal{C}$. By the monotone class theorem [1, page 369], we then have $\sigma(\mathcal{R}) \subset \mathcal{C}$. But $\mathcal{R}$ contains all the bounded closed balls and any open set in $\mathcal{X}$ is a countable union of those since $\mathcal{X}$ is separable. Consequently, $\mathcal{B}(\mathcal{X}) = \sigma(\mathcal{R}) \subset \mathcal{C}$, meaning that Φ_A is measurable for all $A \in \mathcal{B}(\mathcal{X})$.

Step 3. To show that $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}})$ is actually generated by all mappings Φ_A, $A \in \mathcal{B}(\mathcal{X})$, consider any σ-algebra $\mathcal{R}$ on $\mathcal{N}^\#_{\mathcal{X}}$ such that all mappings Φ_A are measurable. Then all the sets of the form (6.1) belong to $\mathcal{R}$ and, by Proposition 6.1, these sets form a basis for the $w^\#$-topology. Since $\mathcal{N}^\#_{\mathcal{X}}$ is separable, any open set of the $w^\#$-topology can be represented as a countable union of these sets and, thus, $\mathcal{B}(\mathcal{N}^\#_{\mathcal{X}}) \subset \mathcal{R}$. □

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