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A computational iterative method for solving nonlinear ordinary differential equations

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Abstract

We present a quasi-linear iterative method for solving a system of m-nonlinear coupled differential equations. We provide an error analysis of the method to study its convergence criteria. In order to show the efficiency of the method, we consider some computational examples of this class of problem. These examples validate the accuracy of the method and show that it gives results which are convergent to the exact solutions. We prove that the method is accurate, fast and has a reasonable rate of convergence by computing some local and global error indicators.

1. *Introduction*

We know very well that most real life problems are modeled using nonlinear differential equations and in many cases consist of systems of these equations. Therefore, the solution of coupled systems of nonlinear differential equations is of significant use to researchers in science and engineering. Of course, these types of systems of equations are always difficult to solve and, in some cases, solvers resort to combining the equations into one higher-order equation. When using numerical methods for solving such systems of equations directly, the challenge is to control the errors, because solutions from one equation form the input to another and the errors progress deeper into the problem, which sometimes makes the solution go out of control.

If we consider the problem in detail, it is notable that, in most cases, the difficulty stems from the nonlinearity of the differential equations. We know that most linear ordinary differential equations are solvable, and we are familiar with analyzing differential equations and identifying nonlinearities. An approach which tries to incorporate all of the various tools at a problem-solvers disposal, that is, a combination of analytical, symbolic and numerical computation, has been considered in [**11**]. In essence, the method attempts to linearize the problem and then considers an iterative approach built around analytical and numerical computations.

Several iterative methods have been studied and applied throughout the years to solve nonlinear problems such as nonlinear oscillation equations [**2**, **3**, **5**, **10**], multispecies Lotka–Volterra equations [**9**] and van der Pol equations [**4**]. In fact, one of the most famous iterative methods to solve nonlinear problems is the quasi-linearization technique [**1**].

To solve the aforementioned equations, we propose a method that uses an iterative approach along with analytical computations to provide a solution of a modified reformulated linear problem. It is worth noting that this was inspired by the homotopy analysis method (HAM) [**6**–**8**]. The HAM is a general, approximate analytic approach that is used to obtain convergent series solutions of strongly nonlinear problems. Thanks to the free choice of the initial approximations and auxiliary linear operators, a complicated nonlinear problem can be transformed into an infinite number of simpler linear sub-problems. Our method was also

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inspired by the well-known fixed point iteration method, which we apply in function space. In fact, our method can be described as a quasi-linear iterative method based on the fixed point iteration method, applied in function space.

In addition, we introduce error computation procedures specifically for such problems. In this paper, we present a method for solving nonlinear differential equations that is based on a series of earlier papers. However, here, we formulate a generalized semi-analytical method for solving a reasonably general system of m-nonlinear coupled ordinary differential equations. We develop the error analysis and show that the method is convergent, efficient and easy to use. We also present the associated error control procedures for the method.

The paper is organized as follows. In § 2, we provide a description of the method. In § 3, we provide an error analysis that will be a useful tool for discussing the performance of the method. In § 4, we present some computational examples proving the convergence of the iterative scheme. We conclude with a few remarks in § 5.

2. *Method description*

We consider the system of m-nonlinear coupled differential equations

$$\begin{cases} L_1(u_1(x), u_2(x), \dots, u_m(x)) + N_1(u_1(x), u_2(x), \dots, u_m(x)) + g_1(x) = 0, \\ L_2(u_1(x), u_2(x), \dots, u_m(x)) + N_2(u_1(x), u_2(x), \dots, u_m(x)) + g_2(x) = 0, \\ \vdots \\ L_m(u_1(x), u_2(x), \dots, u_m(x)) + N_m(u_1(x), u_2(x), \dots, u_m(x)) + g_m(x) = 0, \end{cases} \tag{2.1}$$

along with boundary conditions

$$\begin{cases} B_1\left(u_1, \dfrac{du_1}{dx}\right) = 0, \\ B_2\left(u_2, \dfrac{du_2}{dx}\right) = 0, \\ \vdots \\ B_m\left(u_m, \dfrac{du_m}{dx}\right) = 0, \end{cases} \tag{2.2}$$

where x denotes the independent variable, $u_1(x)$, $u_2(x)$, $\dots$, $u_m(x)$ are unknown functions, $g_1(x)$, $g_2(x)$, $\dots$, $g_m(x)$ are known functions, L_1, L_2, $\dots$, L_m are linear operators, N_1, N_2, $\dots$, N_m are nonlinear operators and B_1, B_2, $\dots$, B_m are boundary operators.

This approach is a continuation of a series of papers published by the same authors. Initially, it was used for solving nonlinear differential equations [**11**] and later for nonlinear second-order multi-point boundary value problems [**12**]. The linearization of the method involves splitting the problem into a linear and nonlinear part. Thus the main requirement here is that L_i, $i = 1, 2, \dots, m$ are the linear parts of the system of differential equations. However, it is possible to reconstruct the problem by taking certain parts of the linear terms and adding them to the nonlinear terms N_i, $i = 1, 2, \dots, m$, as needed for smoothness and simplicity of integration. Also, it is acceptable to reformulate this system by using an expansion of the nonlinear parts of the differential equations. The method that we propose works in the following way: we start by assuming that $u_{i,0}(x)$, $i = 1, 2, \dots, m$ are an initial guess for the solutions to the problem (2.1) and satisfy

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$$\begin{cases} L_1(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) + g_1(x) = 0, \\ L_2(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) + g_2(x) = 0, \\ \vdots \\ L_m(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) + g_m(x) = 0, \end{cases} \tag{2.3}$$

along with boundary conditions

$$\begin{cases} B_1\left(u_{1,0}, \dfrac{du_{1,0}}{dx}\right) = 0, \\ B_2\left(u_{2,0}, \dfrac{du_{2,0}}{dx}\right) = 0, \\ \vdots \\ B_m\left(u_{m,0}, \dfrac{du_{m,0}}{dx}\right) = 0. \end{cases} \tag{2.4}$$

The first iterative solutions $u_{i,1}$, $i = 1, 2, \ldots, m$ are defined by solving the problem given by

$$\begin{cases} L_1(u_{1,1}(x), u_{2,1}(x), \ldots, u_{m,1}(x)) = -N_1(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) - g_1(x), \\ L_2(u_{1,1}(x), u_{2,1}(x), \ldots, u_{m,1}(x)) = -N_2(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) - g_2(x), \\ \vdots \\ L_m(u_{1,1}(x), u_{2,1}(x), \ldots, u_{m,1}(x)) = -N_m(u_{1,0}(x), u_{2,0}(x), \ldots, u_{m,0}(x)) - g_m(x), \end{cases} \tag{2.5}$$

along with boundary conditions

$$\begin{cases} B_1\left(u_{1,1}, \dfrac{du_{1,1}}{dx}\right) = 0, \\ B_2\left(u_{2,1}, \dfrac{du_{2,1}}{dx}\right) = 0, \\ \vdots \\ B_m\left(u_{m,1}, \dfrac{du_{m,1}}{dx}\right) = 0. \end{cases} \tag{2.6}$$

Therefore, we establish a simple iterative procedure to find approximate solutions to the nonlinear problem defined by (2.1)–(2.2) based on solving the system of linear equations

$$\begin{cases} L_1(u_{1,n+1}(x), u_{2,n+1}(x), \ldots, u_{m,n+1}(x)) = -N_1(u_{1,n}(x), u_{2,n}(x), \ldots, u_{m,n}(x)) - g_1(x), \\ L_2(u_{1,n+1}(x), u_{2,n+1}(x), \ldots, u_{m,n+1}(x)) = -N_2(u_{1,n}(x), u_{2,n}(x), \ldots, u_{m,n}(x)) - g_2(x), \\ \vdots \\ L_m(u_{1,n+1}(x), u_{2,n+1}(x), \ldots, u_{m,n+1}(x)) = -N_m(u_{1,n}(x), u_{2,n}(x), \ldots, u_{m,n}(x)) - g_m(x), \end{cases} \tag{2.7}$$

along with boundary conditions

$$\begin{cases} B_1\left(u_{1,n+1}, \dfrac{du_{1,n+1}}{dx}\right) = 0, \\ B_2\left(u_{2,n+1}, \dfrac{du_{2,n+1}}{dx}\right) = 0, \\ \vdots \\ B_m\left(u_{m,n+1}, \dfrac{du_{m,n+1}}{dx}\right) = 0. \end{cases} \tag{2.8}$$

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We believe that this iterative procedure will be attractive to implement because of its simplicity, efficiency and convergence. Each set of solutions is an improvement on the previous iterate and, as more and more iterations are taken, the solutions converge to the exact solution of the general problem (2.1)–(2.2).

Of course, as in any approximate method, the error and its control are vital. We suggest that the convergence criteria is monitored using standard error control procedures such as the residual, the $\mathcal{L}^2$-norm and $\mathcal{L}^\infty$-norm.

3. *Error analysis*

In order to provide a reliable error analysis, we need to recall the $\mathcal{L}^2$-norm

$$\|f\| = \left(\int_0^b f^2\,dx\right)^{1/2} \tag{3.1}$$

and we need to introduce the following convergence criteria.

– The pointwise error is given by

$$\mathcal{E}_{k,n}(x) = u_k^{\text{exact}}(x) - u_{k,n}(x), \quad k = 1, 2, \ldots, m. \tag{3.2}$$

– The $\mathcal{L}^2$-norm reference error with respect to the exact solutions is

$$\mathcal{E}_{\mathcal{L}^2,k,n} = \|u_k^{\text{exact}} - u_{k,n}\|, \quad k = 1, 2, \ldots, m \tag{3.3}$$

and the total $\mathcal{L}^2$-norm reference error of the problem is defined by

$$\mathcal{E}_{\mathcal{L}^2,n} = \left(\sum_{k=1}^{m} \mathcal{E}^2_{\mathcal{L}^2,k,n}\right)^{1/2}. \tag{3.4}$$

– The $\mathcal{L}^\infty$-norm reference error with respect to the exact solutions is

$$\mathcal{E}_{\mathcal{L}^\infty,k,n} = \max_{[0,b]} |u_k^{\text{exact}}(x) - u_{k,n}(x)|, \quad k = 1, 2, \ldots, m \tag{3.5}$$

and the total $\mathcal{L}^\infty$-norm reference error of the problem is defined by

$$\mathcal{E}_{\mathcal{L}^\infty,n} = \max_{1\leqslant k\leqslant m} \mathcal{E}_{\mathcal{L}^\infty,k,n}. \tag{3.6}$$

– The residual error of each equation k, $k = 1, 2, \ldots, m$ is given by

$$\mathcal{R}_{k,n} = \left(\int_0^b [L_k(u_{1,n}(x), \ldots, u_{m,n}(x)) + N_k(u_{1,n}(x), \ldots, u_{m,n}(x)) + g_k(x)\,dx]^2\right)^{1/2} \tag{3.7}$$

and the total residual of the problem is

$$\mathcal{R}_n = \left(\sum_{k=1}^{m} \mathcal{R}^2_{k,n}\right)^{1/2}. \tag{3.8}$$

In the following section, we provide an error analysis of a single nonlinear ordinary differential equation, and a generalization of these theorems applied to systems of differential equations will be provided later. Let us recall the nonlinear boundary value problem

$$L(u(x)) + N(u(x)) + g(x) = 0, \tag{3.9}$$

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along with boundary conditions

$$B\left(u, \frac{du}{dx}\right) = 0. \tag{3.10}$$

We can rewrite (3.9) as

$$u'' = f(u'', u', u, x), \tag{3.11}$$

subject to the boundary conditions

$$u(0) = u_a, \quad u(b) = u_b. \tag{3.12}$$

The aim of this section is to show that the sequence of functions u_n, which are solutions of

$$u''_{n+1} = f(u''_n, u'_n, u_n, x), \tag{3.13}$$

subject to the boundary conditions

$$u_n(0) = u_a, \quad u_n(b) = u_b, \tag{3.14}$$

converges to the solution of problem (3.11)–(3.12), where f is a nonlinear analytic function and the initial guess function u_0 can be taken as a solution of the initial problem

$$L(u_0) + g(x) = 0 \tag{3.15}$$

subject to the boundary conditions

$$u_0(0) = u_a, \quad u_0(b) = u_b. \tag{3.16}$$

In order to study the convergence of the above iterative method, we need to recall the Green's function G, which was initially introduced by Bellman and Kabala [**1**] associated with (3.11)–(3.12): that is,

$$G(x,s) = \begin{cases} G_1, & 0 \leqslant x < s \leqslant b, \\ G_2, & 0 \leqslant s < x \leqslant b. \end{cases} \tag{3.17}$$

We let $G_1 = A + Bx$ and $G_2 = C + Dx$. We find G_1 and G_2 such that $G_1(s) = G_2(s)$ and $(dG_2(s))/dx - (dG_1(s))/dx = 1$. Thus

$$G(x,s) = \begin{cases} u_a + \left(\dfrac{u_b - u_a + s}{b} - 1\right)x, & 0 \leqslant x < s \leqslant b, \\ u_a - s + \left(\dfrac{u_b - u_a + s}{b}\right)x, & 0 \leqslant s < x \leqslant b. \end{cases} \tag{3.18}$$

The maximum of this Green's function depends on u_0, u_b and b and will occur when

$$(x,s) = (\tfrac{1}{2}(b + u_a - u_b), \tfrac{1}{2}(b + u_a - u_b))$$

or

$$(x,s) = (0, u_a - u_b + b) \quad \text{or} \quad (x,s) = (b, u_a - u_b).$$

Let

$$K = \max_{x,s} |G(x,s)| = \max\left\{|u_a|, |u_b|, \left|u_a - \frac{1}{4b}(u_b - u_a - b)^2\right|\right\}. \tag{3.19}$$

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If $u_0 = u_b = 0$ and $b = 1$, we obtain $K = \frac{1}{4}$ and we now have the solution of (3.11)–(3.12) in the linear integral form

$$u = \int_0^b G(x,s) f(u'', u', u, s)\, ds \tag{3.20}$$

and the sequence of solutions of problem (3.13)–(3.14) as

$$u_{n+1} = \int_0^b G(x,s) f(u_n'', u_n', u_n, s)\, ds. \tag{3.21}$$

By subtracting (3.20) from (3.21) and applying the mean value theorem, we obtain

$$u_{n+1} - u = \int_0^b G(x,s) f_u(\theta_n)(u_n(s) - u(s))\, ds, \tag{3.22}$$

where $\theta_n \in (u_n, u)$ and $f_u \equiv df/du$. Let

$$M = \max_{u_n < \theta_n < u} |f_u(\theta_n)|. \tag{3.23}$$

In the next theorem, we will prove that the above sequence of functions u_n converges to the exact solution u of problem (3.11)–(3.12).

THEOREM 1. *Let u and u_n, respectively, be the solutions of* (3.11)–(3.12) *and* (3.13)–(3.14). *Assume that f is a nonlinear analytic function. Then, if $MKb < 1$, the sequence of functions u_n converges to the exact solution u in the $\mathcal{L}^2$-norm, where M and K are defined, respectively, by* (3.23) *and* (3.19).

Proof. We start by squaring (3.22) and using the bound for $f_u(\theta_n)$, to obtain

$$(u_{n+1} - u)^2 \leqslant M^2 \left(\int_0^b G(x,s)(u_n(s) - u(s))\, ds \right)^2. \tag{3.24}$$

Using the Cauchy–Schwarz inequality gives

$$(u_{n+1} - u)^2 \leqslant M^2 \int_0^b G(x,s)^2\, ds \int_0^b (u_n(s) - u(s))^2\, ds \tag{3.25}$$

and using the bound for the Green's function leads to

$$(u_{n+1} - u)^2 \leqslant M^2 K^2 b \int_0^b (u_n(s) - u(s))^2\, ds. \tag{3.26}$$

Then

$$\int_0^b (u_{n+1} - u)^2\, dx \leqslant M^2 K^2 b^2 \int_0^b (u_n(s) - u(s))^2\, ds \tag{3.27}$$

$$\|u_{n+1} - u\|^2 \leqslant M^2 K^2 b^2 \|u_n - u\|^2, \tag{3.28}$$

and thus

$$\|u_{n+1} - u\| \leqslant (MKb)^{n+1} \|u_0 - u\|. \tag{3.29}$$

Since $\|u_0 - u\|$ is bounded, if $MKb < 1$, then the sequence of functions u_n converges to the exact solution u of the problem (3.11)–(3.12). □

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In the following theorem, we will prove the convergence of the sequence of functions u_n to the exact solution u of problem (3.11)–(3.12), in terms of the residual error indicator.

THEOREM 2. *Let u and u_n, respectively, be the solutions of (3.11)–(3.12) and (3.13)–(3.14). Assume that f is a nonlinear analytic function. Then, if $MKb < 1$, the residual error defined by (3.7) converges to zero with respect to n and therefore the sequence of functions u_n converges to the exact solution u, where M and K are defined by (3.23) and (3.19), respectively.*

Proof. Let us recall the residual term

$$R_n = u_n'' - f(u_n'', u_n', u_n, x). \tag{3.30}$$

Using (3.13), we obtain

$$R_n = f(u_{n-1}'', u_{n-1}', u_{n-1}, x) - f(u_n'', u_n', u_n, x). \tag{3.31}$$

The mean value theorem gives

$$R_n = f_u(\theta_n)(u_{n-1} - u_n), \tag{3.32}$$

where $\theta_n \in (u_{n-1}, u_n)$. Using the Green's function gives

$$R_n = f_u(\theta_n) \int_0^b G(x,s)(f(u_{n-2}'', u_{n-2}', u_{n-2}, s) - f(u_{n-1}'', u_{n-1}', u_{n-1}, s))\, ds \tag{3.33}$$

then

$$R_n = f_u(\theta_n) \int_0^b G(x,s)(u_{n-1}'' - f(u_{n-1}'', u_{n-1}', u_{n-1}, s))\, ds, \tag{3.34}$$

and thus

$$|R_n| \leqslant |f_u(\theta_n)| \int_0^b |G(x,s)(u_{n-1}'' - f(u_{n-1}'', u_{n-1}', u_{n-1}, s))|\, ds, \tag{3.35}$$

which leads to

$$\int_0^b R_n\, dx \leqslant |f_u(\theta_n)| b \int_0^b |G(x,s) R_{n-1}|\, ds \tag{3.36}$$

then

$$\int_0^b |R_n|\, dx \leqslant MK \int_0^b |R_{n-1}|\, dx. \tag{3.37}$$

Consequently, we obtain

$$\int_0^b |R_n|\, dx \leqslant (MKb)^n \int_0^b |R_0|\, dx. \tag{3.38}$$

Since $\int_0^b |R_0|\, dx$ is bounded by the choice of the initial guess, if $MKb < 1$, then the sequence of functions u_n converges to the exact solution u of the problem (3.11)–(3.12). □

Let us recall the system of m-nonlinear coupled differential equations

$$\begin{cases} L_1(u_1(x), u_2(x), \ldots, u_m(x)) + N_1(u_1(x), u_2(x), \ldots, u_m(x)) + g_1(x) = 0, \\ L_2(u_1(x), u_2(x), \ldots, u_m(x)) + N_2(u_1(x), u_2(x), \ldots, u_m(x)) + g_2(x) = 0, \\ \vdots \\ L_m(u_1(x), u_2(x), \ldots, u_m(x)) + N_m(u_1(x), u_2(x), \ldots, u_m(x)) + g_m(x) = 0, \end{cases} \tag{3.39}$$

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along with boundary conditions

$$\begin{cases} u_1(0) = u_{1a}, & u_1(b) = u_{1b}, \\ u_2(0) = u_{2a}, & u_2(b) = u_{2b}, \\ \vdots \\ u_m(0) = u_{ma}, & u_m(b) = u_{mb}. \end{cases} \tag{3.40}$$

We can easily transform (3.39) to

$$\begin{cases} u_1'' = f_1(u_1'', u_1', u_1, u_2'', u_2', u_2, \ldots, u_m'', u_m', u_m, x), \\ u_2'' = f_2(u_1'', u_1', u_1, u_2'', u_2', u_2, \ldots, u_m'', u_m', u_m, x), \\ \vdots \\ u_m'' = f_m(u_1'', u_1', u_1, u_2'', u_2', u_2, \ldots, u_m'', u_m', u_m, x), \end{cases} \tag{3.41}$$

subject to the boundary conditions

$$\begin{cases} u_1(0) = u_{1a}, & u_1(b) = u_{1b}, \\ u_2(0) = u_{2a}, & u_2(b) = u_{2b}, \\ \vdots \\ u_m(0) = u_{ma}, & u_m(b) = u_{mb}. \end{cases} \tag{3.42}$$

The aim of this section is to show that the sequences of functions $u_{1,n}, u_{2,n}, \ldots, u_{m,n}$ which are solutions of

$$\begin{cases} u_{1,n+1}'' = f_1(u_{1,n}'', u_{1,n}', u_{1,n}, u_{2,n}'', u_{2,n}', u_{2,n}, \ldots, u_{m,n}'', u_{m,n}', u_{m,n}, x), \\ u_{2,n+1}'' = f_2(u_{1,n}'', u_{1,n}', u_{1,n}, u_{2,n}'', u_{2,n}', u_{2,n}, \ldots, u_{m,n}'', u_{m,n}', u_{m,n}, x), \\ \vdots \\ u_{m,n+1}'' = f_m(u_{1,n}'', u_{1,n}', u_{1,n}, u_{2,n}'', u_{2,n}', u_{2,n}, \ldots, u_{m,n}'', u_{m,n}', u_{m,n}, x), \end{cases} \tag{3.43}$$

subject to the boundary conditions

$$\begin{cases} u_{1,n}(0) = u_{1a}, & u_{1,n}(b) = u_{1b}, \\ u_{2,n}(0) = u_{2a}, & u_{2,n}(b) = u_{2b}, \\ \vdots \\ u_{m,n}(0) = u_{ma}, & u_{m,n}(b) = u_{mb}, \end{cases} \tag{3.44}$$

converge to the solutions of problem (3.41)–(3.42), where $f_1, f_2, \ldots, f_m$ are nonlinear analytic functions and the initial guess functions $u_{1,0}, u_{2,0}, \ldots, u_{m,0}$ can be taken as solutions of the initial problem

$$\begin{cases} L_1(u_1(x), u_2(x), \ldots, u_m(x)) + g_1(x) = 0, \\ L_2(u_1(x), u_2(x), \ldots, u_m(x)) + g_2(x) = 0, \\ \vdots \\ L_m(u_1(x), u_2(x), \ldots, u_m(x)) + g_m(x) = 0, \end{cases} \tag{3.45}$$

subject to the boundary conditions

$$\begin{cases} u_{1,0}(0) = u_{1a}, & u_{1,0}(b) = u_{1b}, \\ u_{2,0}(0) = u_{2a}, & u_{2,0}(b) = u_{2b}, \\ \vdots \\ u_{m,0}(0) = u_{ma}, & u_{m,0}(b) = u_{mb}. \end{cases} \tag{3.46}$$

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Following the steps for finding the Green's function associated to problem (3.11)–(3.12), we can write the Green's functions associated with problem (3.41)–(3.42) as

$$\begin{cases} G_1(x,s) = \begin{cases} u_{1a} + \left(\dfrac{u_{1b} - u_{1a} + s}{b} - 1\right)x, & 0 \leqslant x < s \leqslant b, \\ u_{1a} - s + \left(\dfrac{u_{1b} - u_{1a} + s}{b}\right)x, & 0 \leqslant s < x \leqslant b, \end{cases} \\ G_2(x,s) = \begin{cases} u_{2a} + \left(\dfrac{u_{2b} - u_{2a} + s}{b} - 1\right)x, & 0 \leqslant x < s \leqslant b, \\ u_{2a} - s + \left(\dfrac{u_{2b} - u_{2a} + s}{b}\right)x, & 0 \leqslant s < x \leqslant b, \end{cases} \\ \vdots \\ G_m(x,s) = \begin{cases} u_{ma} + \left(\dfrac{u_{mb} - u_{ma} + s}{b} - 1\right)x, & 0 \leqslant x < s \leqslant b, \\ u_{ma} - s + \left(\dfrac{u_{mb} - u_{ma} + s}{b}\right)x, & 0 \leqslant s < x \leqslant b. \end{cases} \end{cases} \tag{3.47}$$

Therefore, for $i = 1, 2, \ldots, m$, we obtain

$$u_i = \int_0^b G_i(x,s) f_i(u_1'', u_1', u_1, u_2'', u_2', u_2, \ldots, u_m'', u_m', u_m, s)\, ds \tag{3.48}$$

and, for $i = 1, 2, \ldots, m$,

$$u_{i,n+1} = \int_0^b G_i(x,s) f_i(u_{1,n}'', u_{1,n}', u_{1,n}, u_{2,n}'', u_{2,n}', u_{2,n}, \ldots, u_{m,n}'', u_{m,n}', u_{m,n}, x)\, ds. \tag{3.49}$$

Let

$$K_i = \max_{x,s} |G_i(x,s)|, \quad i = 1, 2, \ldots, m \tag{3.50}$$

and

$$K = \max_{1 \leqslant i \leqslant m} K_i. \tag{3.51}$$

For the sake of clarity of formulation, we use the notation

$$\tilde{u}_i = (u_i'', u_i', u_i), \quad i = 0, 1, \ldots, m. \tag{3.52}$$

Then, subtracting (3.48) from (3.49) and applying the general mean value theorem for $i = 1, 2, \ldots, m$ leads to

$$u_{i,n+1} - u_i = \int_0^b G_i(x,s) \nabla f_i(\theta_{1,n}).(u_{1,n} - u_1, u_{2,n} - u_2, \ldots, u_{m,n} - u_m)\, ds, \tag{3.53}$$

where $\theta_{i,n} = (\theta_{i,1,n}, \theta_{i,2,n}, \ldots, \theta_{i,m,n})$ and $\theta_{i,k,n} \in (u_{i,n}, u_i)$ for $i, k = 0, 1, \ldots, m$.

Let

$$M_{i,j} = \max_{\|\theta_{i,n} < 1\|} \left| \frac{df_i}{du_j}(\theta_{i,n}) \right|, \quad i, j = 1, 2, \ldots, m \tag{3.54}$$

$$M_i = \max_{1 \leqslant j \leqslant m} M_{i,j}, \quad M = \max_{1 \leqslant i \leqslant m} M_i. \tag{3.55}$$

In the next theorem, we will prove that the sequences of functions $u_{1,n}, u_{2,n}, \ldots, u_{m,n}$ converge to the exact solutions $u_1, u_2, \ldots, u_m$ of problem (3.41)–(3.42).

THEOREM 3. *Let $u_1, u_2, \ldots, u_m$ and $u_{1,n}, u_{2,n}, \ldots, u_{m,n}$, respectively, be the solution of (3.41)–(3.42) and (3.43)–(3.44). Assume that f_i are nonlinear analytic functions for $i = 1, 2, \ldots, m$. Then, if $MKbm < 1$, the sequences of functions $u_{1,n}, u_{2,n}, \ldots, u_{m,n}$ converge to the exact solutions $u_1, u_2, \ldots, u_m$ in the $\mathcal{L}^2$-norm, where M and K are defined, respectively, by (3.55) and (3.51).*

Proof. Squaring (3.53) and applying the Cauchy–Schwarz inequality for $i = 1, 2, \ldots, m$ gives

$$(u_{i,n+1} - u_i)^2 \leqslant \int_0^b G_i^2(x, s)\, ds \int_0^b \left(\sum_{j=1}^m \frac{df_i}{du_j}(\theta_{i,n})(u_{j,n} - u_j) \right)^2 ds. \tag{3.56}$$

Then, for $i = 0, 1, \ldots, m$,

$$(u_{i,n+1} - u_i)^2 \leqslant K_i^2 b \int_0^b \left(\sum_{j=1}^m \left(\frac{df_i}{du_j}(\theta_{i,n}) \right)^2 \sum_{j=1}^m (u_{j,n} - u_j)^2 \right) ds. \tag{3.57}$$

Using (3.54) and (3.55) for $i = 1, 2, \ldots, m$ leads to

$$(u_{i,n+1} - u_i)^2 \leqslant (M_i K_i b)^2 m \sum_{j=1}^m \int_0^b (u_{j,n} - u_j)^2\, ds \tag{3.58}$$

and

$$\int_0^b (u_{i,n+1} - u_i)^2\, ds \leqslant (M_i K_i b)^2 m \left(\sum_{j=1}^m \|u_{j,n} - u_j\|^2 \right). \tag{3.59}$$

Using the notation in (3.3) for $i = 1, 2, \ldots, m$ gives

$$\mathcal{E}^2_{\mathcal{L}^2, i, n+1} \leqslant (M_i K_i b)^2 m \sum_{j=1}^m \mathcal{E}^2_{\mathcal{L}^2, j, n}. \tag{3.60}$$

Using (3.4) for $i = 1, 2, \ldots, m$ leads to

$$\mathcal{E}^2_{\mathcal{L}^2, i, n+1} \leqslant (M_i K_i b)^2 m \mathcal{E}^2_{\mathcal{L}^2, n} \tag{3.61}$$

then

$$\mathcal{E}^2_{\mathcal{L}^2, n+1} \leqslant m b^2 \mathcal{E}^2_{\mathcal{L}^2, n} \sum_{i=1} (M_i K_i)^2. \tag{3.62}$$

Therefore, using (3.55) and (3.51), we obtain

$$\mathcal{E}^2_{\mathcal{L}^2, n+1} \leqslant (MKbm)^2 \mathcal{E}^2_{\mathcal{L}^2, n}, \tag{3.63}$$

which leads to

$$\mathcal{E}^2_{\mathcal{L}^2, n+1} \leqslant (MKbm)^{n+1} \mathcal{E}^2_{\mathcal{L}^2, 0} \tag{3.64}$$

and, since we choose the initial guesses such that $\|u_{i,0} - u_i\|$ are bounded for $i = 1, 2, \ldots, m$, if $MKbm < 1$, then the sequences of functions $u_{1,n}, u_{2,n}, \ldots, u_{m,n}$ converge to the exact solutions $u_1, u_2, \ldots, u_m$ of problem (3.41)–(3.42). □

In the following theorem, we will prove the convergence of the sequences of functions $u_{1,n}$, $u_{2,n}, \ldots, u_{m,n}$ to the exact solutions $u_1, u_2, \ldots, u_m$ of problem (3.41)–(3.42) in terms of the residual error indicator.

THEOREM 4. *Let u_1, $u_2, \ldots, u_m$ and $u_{1,n}$, $u_{2,n}, \ldots, u_{m,n}$, respectively, be the solutions of (3.41)–(3.42) and (3.43)–(3.44). Assume that f_i are nonlinear analytic functions for $i = 1, 2, \ldots, m$. Then, if $MKbm < 1$, the residual error defined by (3.6) converges to zero with respect to n and the sequences of functions $u_{1,n}$, $u_{2,n}, \ldots, u_{m,n}$ converge to the exact solutions u_1, $u_2, \ldots, u_m$, where M and K are defined, respectively, by (3.55) and (3.51).*

Proof. Let us recall the residual term for $i = 1, 2, \ldots, m$ given by

$$R_{i,n} = u''_{i,n} - f_i(u''_{1,n}, u'_{1,n}, u_{1,n}, u''_{2,n}, u'_{2,n}, u_{2,n}, \ldots, u''_{m,n}, u'_{m,n}, u_{m,n}, x). \tag{3.65}$$

Using (3.43) and the notation in (3.52) for $i = 1, 2, \ldots, m$ we obtain

$$R_{i,n} = f_i(\tilde{u}_{1,n-1}, \tilde{u}_{2,n-1}, \ldots, \tilde{u}_{m,n-1}, x) - f_i(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \ldots, \tilde{u}_{m,n}, x). \tag{3.66}$$

Using the mean value theorem for $i = 1, 2, \ldots, m$ gives

$$R_{i,n} = \nabla f_i(\theta_{i,n}).(u_{1,n-1} - u_{1,n}, u_{2,n-1} - u_{2,n}, \ldots, u_{m,n-1} - u_{m,n}), \tag{3.67}$$

where $\theta_{i,n} = (\theta_{i,1,n}, \theta_{i,2,n}, \ldots, \theta_{i,m,n})$ and $\theta_{i,k,n} \in (u_{i,n-1}, u_{i,n})$ for $i, k = 0, 1, \ldots, m$. Using the Green's function (3.47) for $i = 1, 2, \ldots, m$ leads to

$$\begin{aligned} u_{i,n-1} - u_{i,n} = \int_0^b & G_i(x,s)(f_i(\tilde{u}_{1,n-2}, \tilde{u}_{2,n-2}, \ldots, \tilde{u}_{m,n-2}, s) \\ & - f_i(\tilde{u}_{1,n-1}, \tilde{u}_{2,n-1}, \ldots, \tilde{u}_{m,n-1}, s))\, ds \end{aligned} \tag{3.68}$$

and

$$u_{i,n-1} - u_{i,n} = \int_0^b G_i(x,s)(u''_{i,n-1} - f_i(\tilde{u}_{1,n-1}, \tilde{u}_{2,n-1}, \ldots, \tilde{u}_{m,n-1}, s))\, ds \tag{3.69}$$

and then

$$u_{i,n-1} - u_{i,n} = \int_0^b G_i(x,s) R_{i,n-1}\, ds. \tag{3.70}$$

Plugging (3.70) into (3.67) for $i = 1, 2, \ldots, m$ gives

$$R_{i,n} = \nabla f_i(\theta_{i,n}).\bigg(\int_0^b G_1(x,s)R_{1,n-1}\, ds, \int_0^b G_2(x,s)R_{2,n-1}\, ds, \ldots, \int_0^b G_m(x,s)R_{m,n-1}\, ds\bigg). \tag{3.71}$$

Therefore, for $i = 1, 2, \ldots, m$,

$$R_{i,n} = \sum_{j=1}^{m} \frac{df_i}{du_j} \int_0^b G_j(x,s) R_{j,n-1}\, ds. \tag{3.72}$$

Then, for $i = 1, 2, \ldots, m$,

$$|R_{i,n}| \leqslant \sum_{j=1}^{m} M_{i,j} K_j \int_0^b |R_{j,n-1}|\, ds. \tag{3.73}$$

Using the notation in (3.7) gives

$$\mathcal{R}_{i,n} \leqslant M_i K b \int_0^b \sum_{j=1}^{m} |R_{j,n-1}|\, ds. \tag{3.74}$$

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Using (3.8) and some further analysis, for $i = 1, 2, \ldots, m$, we obtain

$$\mathcal{R}_n \leqslant (MKbm)^n \mathcal{R}_0. \tag{3.75}$$

Since $\mathcal{R}_0$ is bounded by the choice of the initial guesses, then, if $MKbm < 1$, the sequences of functions $u_{1,n},\ u_{2,n}, \ldots,\ u_{m,n}$ converge to the exact solutions $u_1,\ u_2, \ldots,\ u_m$ of problem (3.41)–(3.42). □

In the following section, we apply the method that we introduced earlier to three nonlinear coupled systems of differential equations. In the first few iterations, the solutions can easily be analytically derived and presented. However, subsequently, the process gets longer and more complicated. Therefore, at every iteration, we opt to numerically solve the systems of equations using the finite difference method. For the sake of uniformity and without loss of generality, we consider a uniform mesh of size $M = 10^4$ over the unit interval $[0, 1]$, denoted by $\{x_i\}_{i=0\ldots M}$, with step size $h = 10^{-4}$.

The convergence criteria of the proposed method is based on the $\mathcal{L}^\infty$-norm of the error between two consecutive computed solutions. In fact, a desired accuracy δ can be predefined by the user. Actually, our computational scheme converges when the $\mathcal{L}^\infty$-norm of the error between two consecutive computed solutions reaches the desired accuracy δ. Moreover, the $\mathcal{L}^2$-norm of the error between the exact solutions and the numerical solutions and the $\mathcal{L}^2$-norm of the residual will be used as an indicators of the accuracy of the computed solutions. In the presented examples, the desired accuracy is chosen to be $\delta = 10^{-8}$.

4. *Computational examples*

In order to investigate the performance of the method outlined in § 2, we apply it to three examples of nonlinear coupled systems of differential equations.

4.1. *Coupled system of two nonlinear differential equations*

In this problem, we consider the following coupled system of nonlinear second-order differential equations

$$\begin{cases} u''(x) + u(x)v'(x) - v^2(x) - v'(x) + 1 = 0, & 0 < x < 1, \\ v''(x) + u^2(x) - {v'}^2(x) + u'(x) = 0, & 0 < x < 1, \end{cases} \tag{4.1a}$$

subject to the boundary conditions

$$\begin{cases} u(0) = 0, \quad u(1) = \sin(1), \\ v(0) = 1, \quad v(1) = \cos(1). \end{cases} \tag{4.1b}$$

This problem has exact solutions $u(x) = \sin(x)$ and $v(x) = \cos(x)$. We note that the reason for choosing this example which has exact solutions, is to allow us to perform an error analysis to judge the performance of the method. Let us apply the method by first defining the linear and nonlinear parts of the equations. We distribute the system into linear and nonlinear parts given by

$$\begin{cases} L_1(u, v) = u''(x), \\ N_1(u, v) = u(x)v'(x) - v^2(x) - v'(x), \\ g_1(x) = 1, \end{cases} \quad \text{and} \quad \begin{cases} L_2(u, v) = v''(x), \\ N_2(u, v) = u^2(x) - {v'}^2(x) + u'(x), \\ g_2(x) = 0. \end{cases} \tag{4.2}$$

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Thus, we establish the initial problem

$$\begin{cases} u_0''(x) + 1 = 0, \\ v_0''(x) = 0, \end{cases} \tag{4.3a}$$

subject to the boundary conditions

$$\begin{cases} u_0(0) = 0, \quad u_0(1) = \sin(1), \\ v_0(0) = 1, \quad v_0(1) = \cos(1), \end{cases} \tag{4.3b}$$

with solutions given as

$$\begin{cases} u_0(x) = -\dfrac{x^2}{2} + \left(\sin(1) + \dfrac{1}{2}\right)x, \\ v_0(x) = (\cos(1) - 1)x + 1. \end{cases} \tag{4.4}$$

The second iteration can be carried through the system

$$\begin{cases} u_1''(x) = -u_0(x)v_0'(x) + v_0^2(x) + v_0'(x) - 1, \\ v_1''(x) = -u_0^2(x) + v_0'^2(x) - u_0'(x), \end{cases} \tag{4.5a}$$

subject to the boundary conditions

$$\begin{cases} u_1(0) = 0, \quad u_1(1) = \sin(1), \\ v_1(0) = 1, \quad v_1(1) = \cos(1). \end{cases} \tag{4.5b}$$

Solving problem (4.5) provides the first-order pair of solutions $(u_1(x), v_1(x))$

$$\begin{cases} u_1(x) = -1.5439 \times 10^{-3}x^4 - 5.0454 \times 10^{-2}x^3 - 2.2985 \times 10^{-1}x^2 1.1233x, \\ v_1(x) = -8.3333 \times 10^{-3}x^6 + 6.7074 \times 10^{-2}x^5 - 1.4996 \times 10^{-1}x^4 \\ \qquad + 1.6667 \times 10^{-1}x^3 - 5.6507 \times 10^{-1}x^2 + 2.9932 \times 10^{-2}x + 1. \end{cases} \tag{4.6}$$

Similarly, further solutions can be obtained for the problems generated by

$$\begin{cases} u_{n+1}''(x) = -u_n(x)v_n'(x) + v_n^2(x) + v_n'(x) - 1, \\ v_{n+1}''(x) = -u_n^2(x) + v_n'^2(x) - u_n'(x), \end{cases} \tag{4.7a}$$

subject to the boundary conditions

$$\begin{cases} u_{n+1}(0) = 0, \quad u_{n+1}(1) = \sin(1), \\ v_{n+1}(0) = 1, \quad v_{n+1}(1) = \cos(1). \end{cases} \tag{4.7b}$$

We have shown the first two solutions but we have not shown subsequent ones as they become longer and more complicated. In fact, as stated earlier, we use a finite difference scheme to solve the linearized system at each iteration. Based on the chosen accuracy of $\delta = 10^{-8}$, our iterative scheme converges considerably fast in thirteen iterations, which is a good indication of the efficiency and accuracy of the method. We now carry out the error analysis of the method for this example. Figure 1 shows the exact and the first three computed solutions, respectively. These graphs exhibit the convergence of the iterative solutions to the exact ones. In Figure 2, we plot the error functions $\mathcal{E}_{k,n}(x)$, which approach the axis $y = 0$ as the number of iterations increases. These graphs show that the exact errors are getting smaller as the order of the solution increases. These computational results reflect the efficiency of the method outlined

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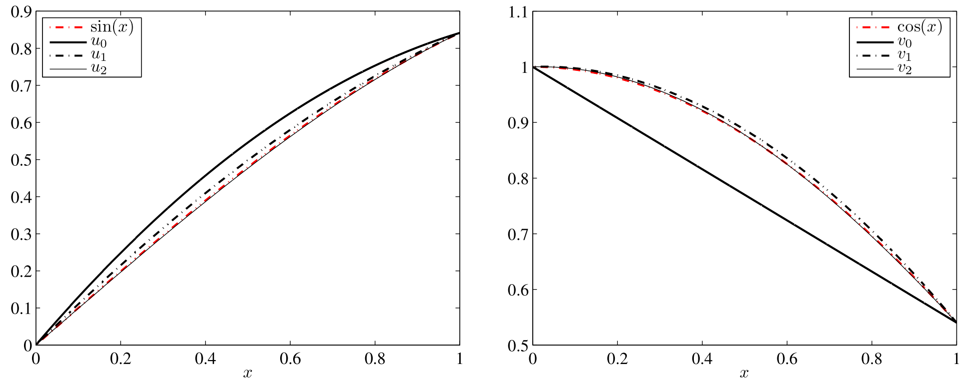

FIGURE 1. *The exact and the first three iterate solutions for problem (4.1).*

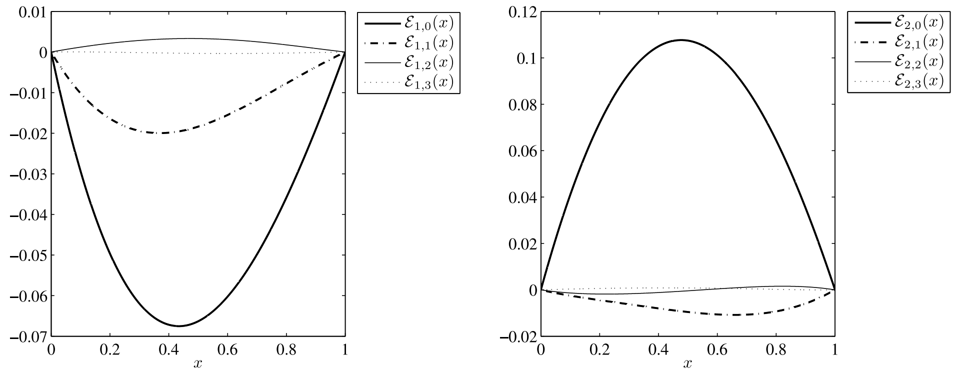

FIGURE 2. *The error functions $\mathcal{E}_{k,n}(x)$ versus x for problem (4.1).*

in § 2. In Tables 1–3, we present the three convergence indicators of the method. These tables clearly show the convergence of the method with respect to the order of iterations.

In Figure 3, we present the residual error and the $\mathcal{L}^2$-norm of the errors between the computed and exact solutions. These error indicators show and confirm the convergence of the method with respect to the order of the solutions.

TABLE 1. *The $\mathcal{L}^\infty$-norm of the error between the exact and computed solutions for the coupled problem (4.1).*

i	$\mathcal{E}_{\mathcal{L}^\infty,1,i}$	$\mathcal{E}_{\mathcal{L}^\infty,2,i}$	$\mathcal{E}_{\mathcal{L}^\infty,i}$
0	6.7517×10^{-2}	1.0765×10^{-1}	1.0765×10^{-1}
1	1.9961×10^{-2}	1.0789×10^{-2}	1.9961×10^{-2}
2	3.3316×10^{-3}	1.8201×10^{-3}	3.3316×10^{-3}
3	3.6159×10^{-4}	8.4440×10^{-4}	8.4440×10^{-4}
4	1.4936×10^{-4}	2.0185×10^{-4}	2.0185×10^{-4}
5	5.2607×10^{-5}	2.8689×10^{-5}	5.2607×10^{-5}
6	7.4988×10^{-6}	9.9179×10^{-6}	9.9179×10^{-6}
7	1.3544×10^{-6}	2.8979×10^{-6}	2.8979×10^{-6}
8	6.9627×10^{-7}	4.8741×10^{-7}	6.9627×10^{-7}

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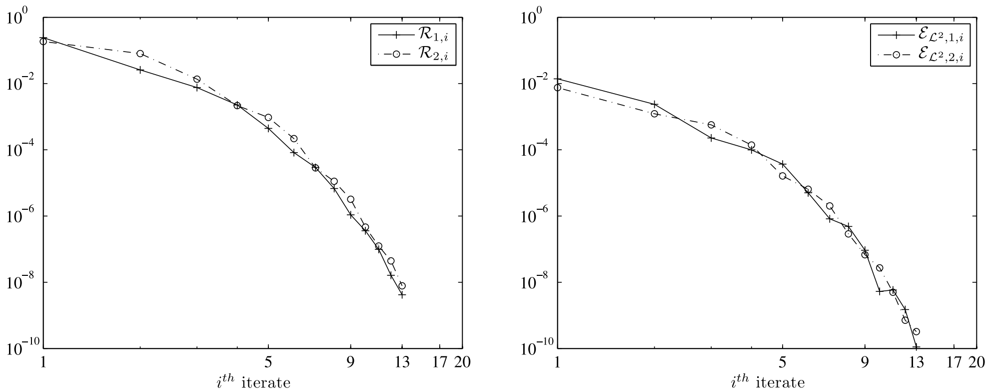

FIGURE 3. *The error indicators with respect to the order of the solution for problem (4.1).*

Based on these computational results, we can derive the following numerical estimates of the error parameters,

$$\begin{cases} \mathcal{R}_i = C_1 e^{-1.4464i}, \\ \mathcal{E}_{\mathcal{L}^2,i} = C_2 e^{-1.4783i}, \\ \mathcal{E}_{\mathcal{L}^\infty,i} = C_3 e^{-1.4927i}, \end{cases} \tag{4.8}$$

where C_1, C_2 and C_3 are constants.

TABLE 2. *The $\mathcal{L}^2$-norm of the error between the exact and computed solutions for the coupled problem (4.1).*

i	$\mathcal{E}_{\mathcal{L}^2,1,i}$	$\mathcal{E}_{\mathcal{L}^2,2,i}$	$\mathcal{E}_{\mathcal{L}^2,i}$
0	4.8808×10^{-2}	7.8268×10^{-2}	9.2240×10^{-2}
1	1.3907×10^{-2}	7.5296×10^{-3}	1.5815×10^{-2}
2	2.3567×10^{-3}	1.2258×10^{-3}	2.6564×10^{-3}
3	2.2723×10^{-4}	5.6636×10^{-4}	6.1024×10^{-4}
4	9.8947×10^{-5}	1.3880×10^{-4}	1.7046×10^{-4}
5	3.6880×10^{-5}	1.6270×10^{-5}	4.0309×10^{-5}
6	5.0279×10^{-6}	6.4335×10^{-6}	8.1651×10^{-6}
7	8.2632×10^{-7}	2.0225×10^{-6}	2.1848×10^{-6}
8	4.8558×10^{-7}	2.8906×10^{-7}	5.6511×10^{-7}

TABLE 3. *The residual errors for the coupled problem (4.1).*

i	$\mathcal{R}_{1,i}$	$\mathcal{R}_{2,i}$	$\mathcal{R}_i$
1	2.4268×10^{-1}	1.8606×10^{-1}	3.0580×10^{-1}
2	2.5848×10^{-2}	8.0363×10^{-2}	8.4417×10^{-2}
3	7.5646×10^{-3}	1.3561×10^{-2}	1.5528×10^{-2}
4	2.2626×10^{-3}	2.1613×10^{-3}	3.1290×10^{-3}
5	4.4374×10^{-4}	9.4492×10^{-4}	1.0439×10^{-3}
6	8.2192×10^{-5}	2.1803×10^{-4}	2.3301×10^{-4}
7	2.9284×10^{-5}	2.8427×10^{-5}	4.0813×10^{-5}
8	6.7743×10^{-6}	1.1188×10^{-5}	1.3079×10^{-5}

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4.2. *Coupled system of three nonlinear integro-differential equations*

In this problem, we consider the coupled system of integro-differential equations given by

$$\begin{cases} u''(x) - v'(x) - \int_0^x [w(t) - u(t)v(t)]\,dt = 0, & 0 < x < 1, \\ v''(x) + u'(x) - w(x) - \int_0^x [u^2(t) - v^2(t)]\,dt = 0, & 0 < x < 1, \\ w''(x) + 4u(x)v(x) + x - \int_0^x [u^2(t) + v^2(t)]\,dt = 0, & 0 < x < 1, \end{cases} \tag{4.9a}$$

subject to the boundary conditions

$$\begin{cases} u(0) = 0, & u(1) = \sin(1), \\ v(0) = 1, & v(1) = \cos(1), \\ w(0) = 0, & w(1) = \sin(1)\cos(1). \end{cases} \tag{4.9b}$$

This problem has exact solutions $u(x) = \sin(x)$, $v(x) = \cos(x)$ and $w(x) = \sin(x)\cos(x)$. An error analysis will be conducted to test the efficacy of the iterative method outlined in § 2. We start by defining the linear and nonlinear parts of the equations as

$$\begin{cases} L_1(u, v, w) = u''(x), \\ N_1(u, v, w) = -v'(x) - \int_0^x [w(t) - u(t)v(t)]\,dt, \\ g_1(x) = 0, \end{cases} \tag{4.10}$$

$$\begin{cases} L_2(u, v, w) = v''(x), \\ N_2(u, v, w) = u'(x) - w(x) - \int_0^x [u^2(t) - v^2 v]\,dt, \\ g_2(x) = 0, \end{cases} \tag{4.11}$$

and

$$\begin{cases} L_3(u, v, w) = v''(x), \\ N_3(u, v, w) = 4u(x)v(x) - \int_0^x [u^2(t) + v^2(t)]\,dt, \\ g_3(x) = -x. \end{cases} \tag{4.12}$$

Thus, we establish the initial problem as

$$\begin{cases} u_0''(x) = 0, \\ v_0''(x) = 0, \\ w_0''(x) = -x, \end{cases} \tag{4.13a}$$

subject to the boundary conditions

$$\begin{cases} u_0(0) = 0, & u_0(1) = \sin(1), \\ v_0(0) = 1, & v_0(1) = \cos(1), \\ w_0(0) = 0, & w_0(1) = \sin(1)\cos(1), \end{cases} \tag{4.13b}$$

to obtain the initial solutions

$$\begin{cases} u_0(x) = \sin(1)x, \\ v_0(x) = (\cos(1) - 1)x + 1, \\ w_0(x) = -\dfrac{x^3}{6} + \left(\sin(1)\cos(1) + \dfrac{1}{6}\right)x. \end{cases} \tag{4.14}$$

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The second iteration can be carried through the system

$$\begin{cases} u_1(x)'' = v_0'(x) + \int_0^x [w_0(t) - u_0(t)v_0(t)]\, dt, \\ v_1(x)'' = -u_0'(x) + w_0(x) + \int_0^x [u_0^2(t) - v_0^2(t)]\, dt, \\ w_1(x)'' = -4u_0(x)v_0(x) - x + \int_0^x [u_0^2(t) + v_0^2(t)]\, dt, \end{cases} \tag{4.15a}$$

subject to the boundary conditions

$$\begin{cases} u_1(0) = 0, \quad u_1(1) = \sin(1), \\ v_1(0) = 1, \quad v_1(1) = \cos(1), \\ w_1(0) = 0, \quad w_1(1) = \sin(1)\cos(1), \end{cases} \tag{4.15b}$$

to provide the first-order approximate solutions

$$\begin{cases} u_1(x) = -1.3889 \times 10^{-3}x^6 + 6.4470 \times 10^{-3}x^5 - 9.1732 \times 10^{-3}x^4 \\ \qquad\qquad - 2.2985 \times 10^{-1}x^2 + 1.0754x, \\ v_1(x) = -5.4143 \times 10^{-3}x^5 + 3.8308 \times 10^{-2}x^4 - 6.3114 \times 10^{-2}x^3 \\ \qquad\qquad - 4.2074 \times 10^{-1}x^2 - 1.4102 \times 10^{-2}x + 1, \\ w_1(x) = 1.5323 \times 10^{-2}x^5 + 9.0633 \times 10^{-2}x^4 - 5.6098 \times 10^{-1}x^3 \\ \qquad\qquad + 9.0967 \times 10^{-1}x. \end{cases} \tag{4.16}$$

Consecutively, the iterative algorithm

$$\begin{cases} u_{n+1}''(x) = v_n'(x) + \int_0^x [w_n(t) - u_n(t)v_n(t)]\, dt, \\ v_{n+1}''(x) = -u_n'(x) + w_n(x) + \int_0^x [u_n^2(t) - v_n^2(t)]\, dt, \\ w_{n+1}''(x) = -4u_n(x)v_n(x) - x + \int_0^x [u_n^2(t) + v_n^2(t)]\, dt \end{cases} \tag{4.17a}$$

can provide an improved set of solutions at each iteration subject to the boundary conditions

$$\begin{cases} u_{n+1}(0) = 0, \quad u_{n+1}(1) = \sin(1), \\ v_{n+1}(0) = 1, \quad v_{n+1}(1) = \cos(1), \\ w_{n+1}(0) = 0, \quad w_{n+1}(1) = \sin(1)\cos(1). \end{cases} \tag{4.17b}$$

We stop showing the computed solutions at this level because the remaining ones are too long to display here. However, in the computational results provided later, we continue deriving solutions for problem (4.9) until we achieve convergence to the earlier predefined accuracy. In fact, our iterative scheme converges in only ten iterations, which is a good indication of the efficiency and reliability of the method. Now, we carry out the error analysis of the method for this example. Figure 4 shows the exact and the three first iterative solutions, respectively. These graphs exhibit the convergence of the computed solutions to the exact ones.

In Figure 5, we plot the error functions $\mathcal{E}_{k,n}(x)$, which attenuate as the number of iterations increases. These graphs show that the exact errors are getting smaller as the order of the solution is increasing.

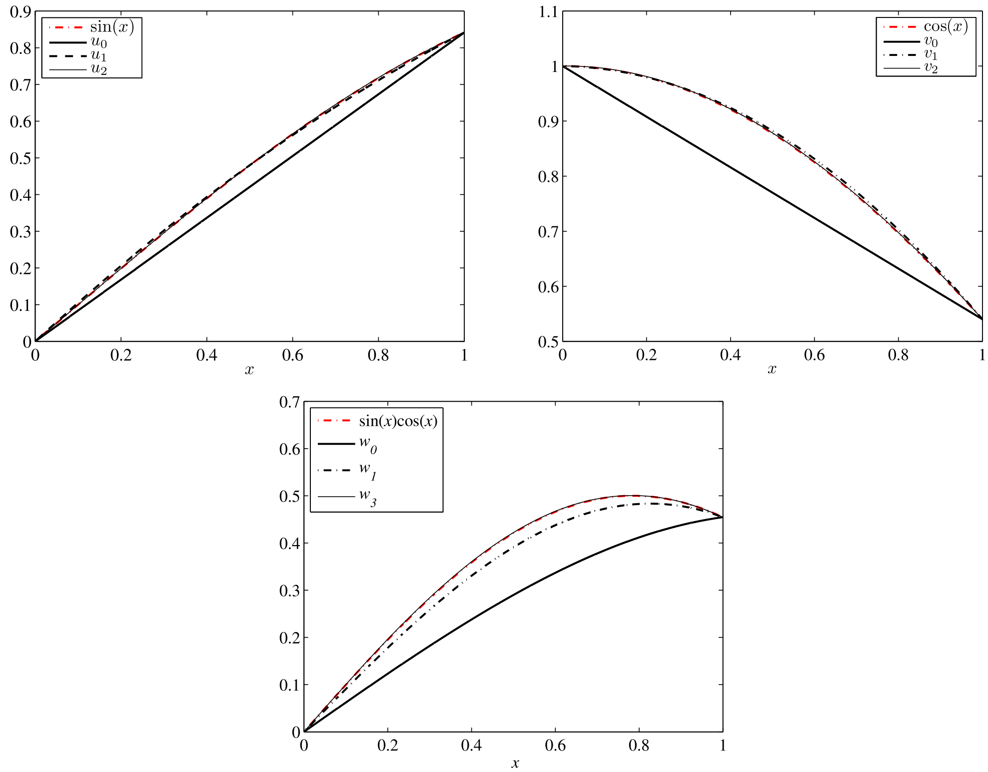

FIGURE 4. *The exact and the first three iterative solutions for problem (4.9).*

In Tables 4–6, we provide the error measures to show the convergence and efficiency of the method.

Figure 6 exhibits the residual errors and the $\mathcal{L}_2$-norm of the errors between the exact and the computed solutions. These error indicators confirm the rapid convergence of the iterative method.

These numerical results endorse the convergence of the method with respect to the order of the solution. In addition, we can establish the numerical estimates of the error indicators

TABLE 4. *The $\mathcal{L}^\infty$-norm of the error between the exact and computed solutions for the coupled problem (4.9).*

i	$\mathcal{E}_{\mathcal{L}^\infty,1,i}$	$\mathcal{E}_{\mathcal{L}^\infty,2,i}$	$\mathcal{E}_{\mathcal{L}^\infty,3,i}$	$\mathcal{E}_{\mathcal{L}^\infty,i}$
0	5.9994×10^{-2}	1.0765×10^{-1}	1.3172×10^{-1}	1.3172×10^{-1}
1	7.2304×10^{-3}	6.6692×10^{-3}	2.9880×10^{-2}	2.9880×10^{-2}
2	1.0395×10^{-3}	1.0458×10^{-3}	1.4766×10^{-3}	1.4766×10^{-3}
3	6.6283×10^{-5}	1.2292×10^{-4}	3.1353×10^{-4}	3.1353×10^{-4}
4	1.8103×10^{-5}	1.1999×10^{-5}	7.7916×10^{-6}	1.8103×10^{-5}
5	2.1736×10^{-6}	2.6022×10^{-6}	5.3907×10^{-6}	5.3907×10^{-6}
6	2.5623×10^{-7}	5.1204×10^{-7}	2.0937×10^{-7}	5.1204×10^{-7}
7	5.1335×10^{-8}	1.2075×10^{-7}	8.0342×10^{-8}	1.2075×10^{-7}
8	1.1874×10^{-8}	1.2436×10^{-8}	5.8714×10^{-9}	1.2436×10^{-8}

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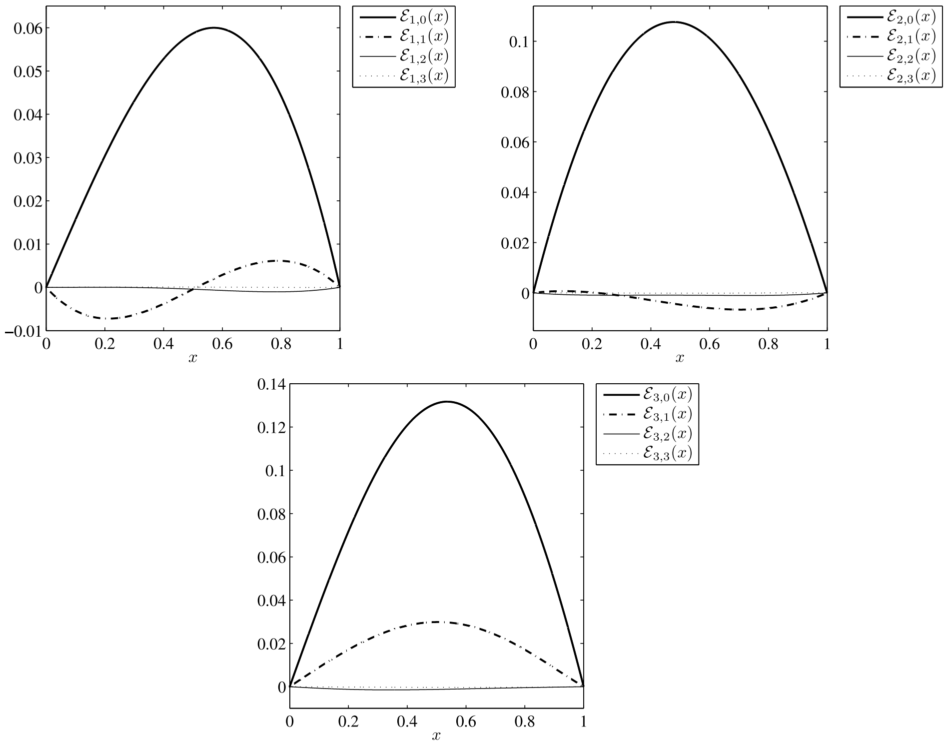

FIGURE 5. *The error functions* $\mathcal{E}_{k,n}(x)$ *versus* x *for problem* (4.9).

TABLE 5. *The* $\mathcal{L}^2$*-norm of the error between the exact and computed solutions for the coupled problem* (4.9).

i	$\mathcal{E}_{\mathcal{L}^2,1,i}$	$\mathcal{E}_{\mathcal{L}^2,2,i}$	$\mathcal{E}_{\mathcal{L}^2,3,i}$	$\mathcal{E}_{\mathcal{L}^2,i}$
0	4.3020×10^{-2}	7.8268×10^{-2}	9.4060×10^{-2}	1.2971×10^{-1}
1	4.8146×10^{-3}	4.0850×10^{-3}	2.0937×10^{-2}	2.1868×10^{-2}
2	5.7845×10^{-4}	8.6425×10^{-4}	9.7312×10^{-4}	1.4242×10^{-3}
3	4.6405×10^{-5}	7.4503×10^{-5}	2.1758×10^{-4}	2.3462×10^{-4}
4	1.0754×10^{-5}	7.7847×10^{-6}	4.4157×10^{-6}	1.3991×10^{-5}
5	1.3489×10^{-6}	1.5552×10^{-6}	3.6108×10^{-6}	4.1565×10^{-6}
6	1.6263×10^{-7}	3.3661×10^{-7}	1.4372×10^{-7}	4.0051×10^{-7}
7	3.2046×10^{-8}	7.8062×10^{-8}	4.9373×10^{-8}	9.7766×10^{-8}
8	7.0352×10^{-9}	7.6549×10^{-9}	3.3133×10^{-9}	1.0912×10^{-8}

TABLE 6. *The residual errors for the coupled problem* (4.9).

i	$\mathcal{R}_{1,i}$	$\mathcal{R}_{2,i}$	$\mathcal{R}_{3,i}$	$\mathcal{R}_i$
1	2.5744×10^{-1}	1.1523×10^{-1}	2.1949×10^{-1}	3.5740×10^{-1}
2	2.3979×10^{-2}	3.2394×10^{-2}	1.4945×10^{-2}	4.2985×10^{-2}
3	3.4288×10^{-3}	2.7070×10^{-3}	2.5206×10^{-3}	5.0436×10^{-3}
4	2.7688×10^{-4}	2.3011×10^{-4}	1.0329×10^{-4}	3.7454×10^{-4}
5	3.5794×10^{-5}	4.2921×10^{-5}	4.4196×10^{-5}	7.1251×10^{-5}
6	6.3440×10^{-6}	6.6369×10^{-6}	2.0504×10^{-6}	9.4073×10^{-6}
7	1.1636×10^{-6}	1.0516×10^{-6}	8.2801×10^{-7}	1.7736×10^{-6}
8	2.2604×10^{-7}	1.3407×10^{-7}	7.5631×10^{-8}	2.7347×10^{-7}

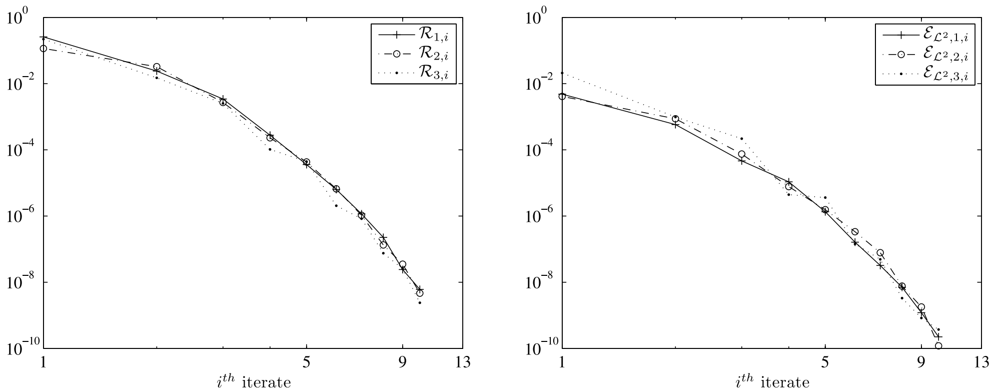

FIGURE 6. *The error indicators with respect to the order of the solution for problem (4.9).*

given by

$$\begin{cases} \mathcal{R}_i = C_1 e^{-1.9565 i}, \\ \mathcal{E}_{\mathcal{L}^2,i} = C_2 e^{-1.9472 i}, \\ \mathcal{E}_{\mathcal{L}^\infty,i} = C_3 e^{-1.9250 i}, \end{cases} \tag{4.18}$$

where C_1, C_2 and C_3 are constants.

In the next example, we apply our method to a system of differential equations in which the nonlinearity appears mainly in the higher-order derivative terms. In fact, a direct application of the method as defined earlier would fail. Therefore, a reformulation of the problem is required in order to guarantee the convergence to the exact solutions.

4.3. *Coupled system of differential equations with nonlinearity on highest order derivative terms*

In this problem, we consider the coupled system of nonlinear second order differential equations given by

$$\begin{cases} u''(x)u(x) + v'(x) + v(x) = 0, & 0 < x < 1, \\ v''(x)u'(x) - 2u(x)v'(x) + u'(x) + u(x) = 0, & 0 < x < 1, \end{cases} \tag{4.19a}$$

subject to the boundary conditions

$$\begin{cases} u(0) = 1, \quad u(1) = \exp(-1), \\ v(0) = 1, \quad v(1) = \exp(-2). \end{cases} \tag{4.19b}$$

This problem has exact solutions $u(x) = \exp(-x)$ and $v(x) = \exp(-2x)$.

We notice that this example is quite challenging since the highest order derivative terms do not appear in the linear parts of equations (4.19a). Thus, in order to apply our method and guarantee its convergence, we have to reformulate the problem (4.19) to linearize the highest order derivative terms (4.19a).

Therefore, problem (4.19) becomes

$$\begin{cases} u''(x) + \dfrac{v'(x) + v(x)}{u(x)} = 0, & 0 < x < 1, \\ v''(x) - \dfrac{2u(x)v'(x) + u'(x) + u(x)}{u'(x)} = 0, & 0 < x < 1, \end{cases} \tag{4.20a}$$

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subject to the boundary conditions

$$\begin{cases} u(0) = 1, & u(1) = \exp(-1), \\ v(0) = 1, & v(1) = \exp(-2). \end{cases} \tag{4.20b}$$

First, let us define the linear and nonlinear parts of the reformulated equations (4.20a) as

$$\begin{cases} L_1(u,v) = u''(x), \\ N_1(u,v) = \dfrac{v'(x)+v(x)}{u(x)}, \\ g_1(x) = 0, \end{cases}$$
$$\begin{cases} L_2(u,v) = v''(x), \\ N_2(u,v) = \dfrac{-2u(x)v'(x)+u'(x)+u(x)}{u'(x)}, \\ g_2(x) = 0. \end{cases} \tag{4.21}$$

We generate the sequence of solutions $u_n(x)$ by solving the following recurrence relations

$$\begin{cases} u''_{n+1}(x) = \dfrac{-v'_n(x) - v_n(x)}{u_n(x)}, & 0 < x < 1, \\ v''_{n+1}(x) = \dfrac{2u_n(x)v'_n(x) - u'_n(x) - u_n(x)}{u'_n(x)}, & 0 < x < 1, \end{cases} \tag{4.22a}$$

subject to the boundary conditions

$$\begin{cases} u_{n+1}(0) = 1, & u_{n+1}(1) = \exp(-1), \\ v_{n+1}(0) = 1, & v_{n+1}(1) = \exp(-2). \end{cases} \tag{4.22b}$$

Thus, we establish the initial problem as

$$\begin{cases} u''_0(x) = 0, & 0 < x < 1, \\ v''_0(x) = 0, & 0 < x < 1, \end{cases} \tag{4.23a}$$

subject to the boundary conditions

$$\begin{cases} u_0(0) = 1, & u_0(1) = \exp(-1), \\ v_0(0) = 1, & v_0(1) = \exp(-2). \end{cases} \tag{4.23b}$$

We obtain the initial approximate solutions as

$$\begin{cases} u_0(x) = (\exp(-1) - 1)x + 1, \\ v_0(x) = (\exp(-2) - 1)x + 1. \end{cases} \tag{4.24}$$

In fact, since a direct double integration is not possible to solve problem (4.20), we have used, from the first iteration, a finite difference method with step size $h = 10^{-5}$.

Figure 7 shows the exact and the first three iterative solutions. These graphs exhibit the convergence of the approximate solutions to the exact ones with respect to the order of recurrence.

In Figure 8, we plot the error functions $\mathcal{E}_{k,n}(x)$, which approach the x-axis as the number of iterations increases. These graphs show that the exact errors are getting smaller as the order of the solution is increasing.

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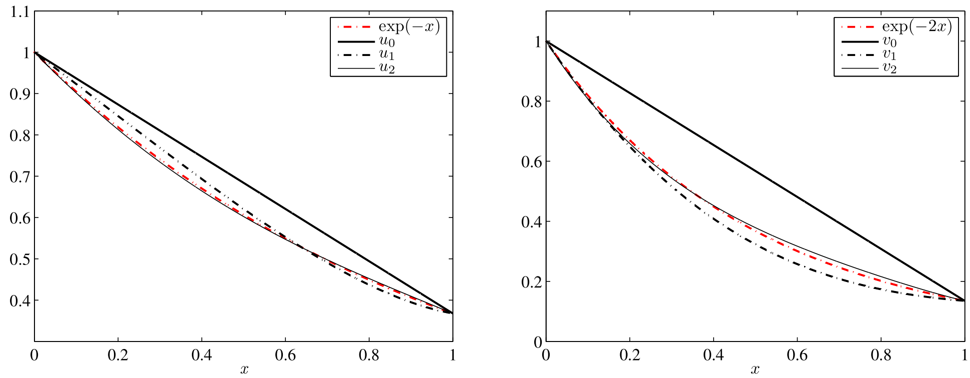

FIGURE 7. *The exact and the first three iterative solutions for problem (4.19).*

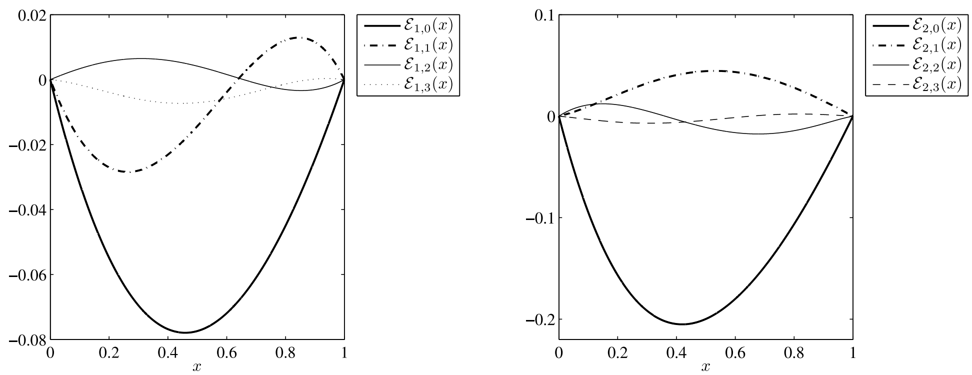

FIGURE 8. *The error functions $\mathcal{E}_{k,n}(x)$ versus x for problem (4.19).*

In Tables 7–9, we provide the error measures to show the convergence and efficiency of the method.

Figure 9 exhibits the residual errors and the $\mathcal{L}_2$-norm of the errors between the exact and the computed solutions. These error indicators confirm the convergence of the iterative method.

TABLE 7. *The $\mathcal{L}^\infty$-norm of the error between the exact and computed solutions for the coupled problem (4.19).*

i	$\mathcal{E}_{\mathcal{L}^\infty,1,i}$	$\mathcal{E}_{\mathcal{L}^\infty,2,i}$	$\mathcal{E}_{\mathcal{L}^\infty,i}$
0	7.7941×10^{-2}	2.0513×10^{-1}	2.0513×10^{-1}
1	2.8419×10^{-2}	4.4584×10^{-2}	4.4584×10^{-2}
2	6.4839×10^{-3}	1.7555×10^{-2}	1.7555×10^{-2}
3	7.3113×10^{-3}	6.9774×10^{-3}	7.3113×10^{-3}
4	1.5210×10^{-3}	5.0248×10^{-3}	5.0248×10^{-3}
5	8.7565×10^{-4}	2.4809×10^{-3}	2.4809×10^{-3}
6	4.9766×10^{-4}	8.1834×10^{-4}	8.1834×10^{-4}
7	1.7956×10^{-4}	2.8008×10^{-4}	2.8008×10^{-4}
8	1.2764×10^{-4}	1.6527×10^{-4}	1.6527×10^{-4}

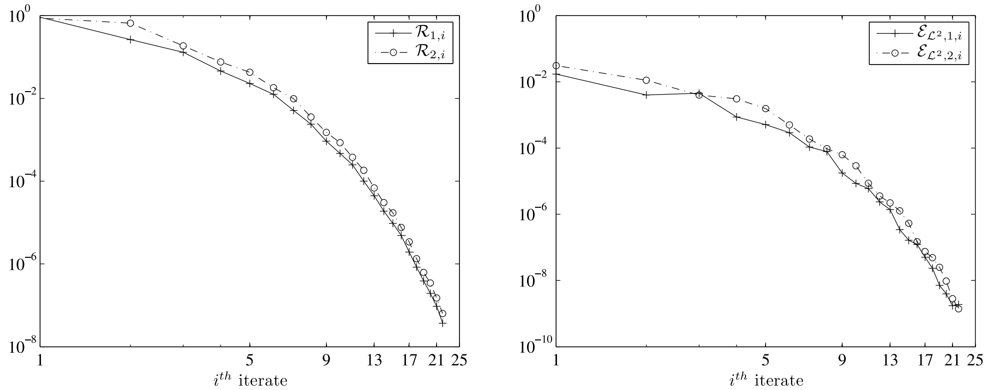

FIGURE 9. *The error indicators with respect to the order of the solution for problem* (4.19).

Based on the these results, we can establish the numerical estimates of the error indicators to be

$$\begin{cases} \mathcal{R}_i = C_1 e^{-0.7931i}, \\ \mathcal{E}_{\mathcal{L}^2,i} = C_2 e^{-0.8198i}, \\ \mathcal{E}_{\mathcal{L}^\infty,i} = C_3 e^{-0.8215i}, \end{cases} \tag{4.25}$$

where C_1, C_2 and C_3 are constants.

Once again, these computational experiments prove the convergence of the sequences of approximating solutions to the exact solutions, as stated in Theorems 3 and 4. Therefore, the computational results are in full agreement with the theory.

TABLE 8. *The $\mathcal{L}^2$-norm of the error between the exact and computed solutions for the coupled problem* (4.19).

i	$\mathcal{E}_{\mathcal{L}^2,1,i}$	$\mathcal{E}_{\mathcal{L}^2,2,i}$	$\mathcal{E}_{\mathcal{L}^2,i}$
0	5.6809×10^{-2}	1.4869×10^{-1}	1.5918×10^{-1}
1	1.7093×10^{-2}	3.0868×10^{-2}	3.5285×10^{-2}
2	3.9983×10^{-3}	1.1143×10^{-2}	1.1839×10^{-2}
3	4.5323×10^{-3}	3.9774×10^{-3}	6.0300×10^{-3}
4	8.6313×10^{-4}	3.0612×10^{-3}	3.1806×10^{-3}
5	5.0816×10^{-4}	1.5494×10^{-3}	1.6306×10^{-3}
6	2.9024×10^{-4}	5.0009×10^{-4}	5.7821×10^{-4}
7	1.0695×10^{-4}	1.8790×10^{-4}	2.1620×10^{-4}
8	7.7795×10^{-5}	9.5637×10^{-5}	1.2328×10^{-4}

TABLE 9. *The residual errors for the coupled problem* (4.19).

i	$\mathcal{R}_{1,i}$	$\mathcal{R}_{2,i}$	$\mathcal{R}_i$
1	9.0892×10^{-1}	8.6715×10^{-1}	1.2562
2	2.6224×10^{-1}	6.5774×10^{-1}	7.0809×10^{-1}
3	1.2952×10^{-1}	1.8586×10^{-1}	2.2654×10^{-1}
4	4.5990×10^{-2}	7.5953×10^{-2}	8.8792×10^{-2}
5	2.2942×10^{-2}	4.2775×10^{-2}	4.8539×10^{-2}
6	1.2531×10^{-2}	1.8194×10^{-2}	2.2091×10^{-2}
7	5.1178×10^{-3}	9.8038×10^{-3}	1.1059×10^{-2}
8	2.3755×10^{-3}	3.5509×10^{-3}	4.2722×10^{-3}

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5. *Conclusion*

In this paper, we have introduced an iterative quasi-linear method for solving systems of nonlinear ordinary differential equations. We have demonstrated the accuracy and rapid convergence of the method. We have also analyzed the method from a theoretical perspective and our resulting theorems show the convergence and accuracy of the method. By means of several computational examples, we have demonstrated that the method is accurate and gives results which are convergent to the exact solutions of the problem. Thus, our computational results are in full agreement with our theoretical analysis. In addition, adopting a finite difference scheme to our iterative method significantly improves the convergence and accuracy of the method.

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