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Homoclinic points and moduli

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Abstract. In this paper we study some conjugacy invariants (moduli) for discrete two dimensional dynamical systems, with a homoclinic tangency. We show that the modulus obtained by Palis in the heteroclinic case also turns up in the case considered here. We also present two new conjugacy invariants.

1. *Introduction*

We start our introduction by recalling some notions from the theory of differentiable dynamical systems and bifurcation theory. See [7, 9].

Let $\text{Diff}^r(M)$ be the set of C^r-diffeomorphisms $(2 \leq r \leq \infty)$ on a compact two dimensional manifold M endowed with the C^r-topology. Two diffeomorphisms f, g are called *conjugate* when there exists a homeomorphism h (a conjugacy) such that $fh = hg$. This equivalence relation defines the *conjugacy classes.* If there is a neighbourhood of a diffeomorphism f contained in its equivalence class, then we say that f is *structurally stable.*

Let p be a fixed point of f. Then p is called *hyperbolic* if the eigenvalues of $Df(p)$ have absolute values different from one. If one eigenvalue has an absolute value less than one and the other larger than one, then we say that p is a *saddle point* (recall that we only consider dynamical systems on two-manifolds).

The stable and unstable manifolds of a hyperbolic fixed point p are defined by:

$$W^s(p) = \{x \in M \mid f^n(x) \to p,\ n \to \infty\},$$
$$W^u(p) = \{x \in M \mid f^n(x) \to p,\ n \to -\infty\}.$$

Invariant manifold theory [2] gives us that $W^s(p)$ and $W^u(p)$ are immersed submanifolds of M, as differentiable as f and transversal to each other in p, i.e. $T_pM = T_p(W^s(p)) \oplus T_p(W^u(p))$.

When p is a hyperbolic fixed point of f, a point $q \in M$ is called *homoclinic to p* if $p \neq q \in W^s(p) \cap W^u(p)$, i.e. $p \neq q$ and $\lim_{i \to \pm\infty} f^i(q) = p$, q is called a *transversal homoclinic point* if $W^s(p)$ and $W^u(p)$ intersect transversally at q. If this intersection is non-transversal, then q is a point of a *homoclinic tangency.* If p and q are two distinct hyperbolic fixed points of a diffeormorphism f then a point $r \in M$ is called *heteroclinic to p, q* if $r \in W^s(p) \cap W^u(q)$, i.e. $\lim_{i \to \infty} f^i(r) = p$ and $\lim_{i \to -\infty} f^i(r) = q$. As above we define the notions of a *transversal heteroclinic point* and a *heteroclinic tangency.*

There are corresponding definitions for periodic points instead of fixed points.

Remark. Generically all homoclinic points are transverse (see [7]). When we consider a one parameter family of diffeomorphisms however, we can expect tangencies at isolated values of the parameter.

Now we come to the basic question considered in this article. Given two diffeomorphisms f, f' with a hyperbolic fixed point p (resp. p') of saddle-type and a point r (resp. r') of a homoclinic tangency, when are f and f' conjugated?

In [6] Palis studied the analogous question for heteroclinic points: given two diffeomorphisms f and f' (at least C^2) with hyperbolic fixed points p, q and p', q' resp. Suppose that $W^s(p)$ and $W^u(q)$ (resp. $W^s(p')$ and $W^u(q')$) have a point of tangency r (resp. r'). Then under some conditions on this tangency he has shown: if f and f' are conjugated then

$$\frac{\log|\lambda|}{\log|\mu|}=\frac{\log|\lambda'|}{\log|\mu'|}$$

where λ (resp. λ') denotes the contracting eigenvalue of $Df(q)$ (resp. $Df'(q')$) and μ (resp. μ') denotes the expanding eigenvalue of $Df(p)$ (resp. $Df'(p')$). Thus the ratio $\log|\lambda|/\log|\mu|$ is an invariant under topological conjugacy. We call such an invariant a *modulus.*

As mentioned before we study the corresponding question for homoclinic tangencies. In particular we will show, that the same modulus as above turns up, when we have a homoclinic tangency (λ (resp. λ') now denotes the contracting eigenvalue of $Df(p)$ (resp. $Df'(p')$). But we shall show the existence of more moduli.

This gives us other reasons for the fact that one-parameter families of diffeomorphisms, with a homoclinic tangency are *not* structurally stable, (with the usual definition of structural stability for one-parameter families of diffeomorphisms), because a little perturbation of our original diffeomorphism leads to different values of the moduli.

2. *Preliminaries*

In order to prove the existence of moduli in the next section we have to compare different metrics on our manifold M. These metrics are induced by C^1-coordinate systems. In this section we state some properties of C^r-metrics and introduce some notation, to be used further on. For the proof of these properties we refer to [4], from which we have taken these properties verbatim.

Definition. A C^r-metric $d: M\times M\to\mathbb{R}$ on $M(0\le r\le\infty)$ is a metric induced by a C^r-Riemannian structure g on M such that:

$$d(x,y) = \inf\{l_g(\gamma)\mid \gamma:[0,1]\to M \text{ is a piecewise } C^1 \text{ curve with } \gamma(0)=x \quad\text{and}\quad \gamma(1)=y\}$$

where $l_g(\gamma)=\int_0^1 \sqrt{g(\gamma'(t),\gamma'(t))}\,dt.$

The distance from a point x to a set S will be denoted $d(x,S)$ where $d(x,S)=\inf\{d(x,y)\mid y\in S\}$. Furthermore it will be convenient to introduce the following notation for (real) sequences $\{\alpha_i\}$, $\{\beta_i\}$:

$\alpha_i \sim \beta_i$ iff $|\alpha_i/\beta_i|$ is bounded and bounded away from zero.

$\alpha_i \simeq \beta_i$ iff α_i/β_i converges to one.

LEMMA 2.1. *Let $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be a C^0 metric induced by the Riemannian structure g. Let d_0 denote the metric induced by the constant Riemannian structure g_0 which coincides with g at O. If $S\subset\mathbb{R}^n$ contains O and $x_i\in\mathbb{R}^n-S$ converges to O then one has $d(x_i, S)\simeq d_0(x_i, S)$.*

LEMMA 2.2. *Let $S\subset\mathbb{R}^n$ be a codimension one C^1-manifold, containing O and d_j $(j=1,2)$ be C^0-metrics on $\mathbb{R}^n$. Then there exists a positive real number A such that $d_1(x_i, S)\simeq Ad_2(x_i, S)$, for any sequence $x_i\in\mathbb{R}^n-S$, converging to O.*

Remark. If $\tilde{S}$ is another codimension one submanifold, tangent to S at O and $x_i\in\mathbb{R}^n-\tilde{S}$ converges to O then $d_1(x_i,\tilde{S})\simeq Ad_2(x_i, S)$ where A is the same constant as for S.

By the use of C^0-metrics we can introduce the notion of contact of manifolds:

Definition. Let x be a point of tangency of two C^1-manifolds $S_1, S_2\subset M$. We say that S_1 has *contact of order n* with S_2 at x if the following limit exists and is positive.

$$\lim_{\substack{w\to x\\ w\in S_1}}\frac{d(w,S_2)}{[d(w,x]^n}.$$

If this limit is infinite then we say that *the order of contact is at most n.* If this limit is zero for every n, then we say that *the order of contact is infinite.* Otherwise the order of contact is not defined.

Remark. From the lemmas above it follows that the definition of the order of contact is independent of the chosen metric. Notice also that the order of contact may not exist; however if there is a C^1-coordinate system ϕ on a neighbourhood of x such that $\phi(S_1)$ and $\phi(S_2)$ are both C^∞-submanifolds of $\mathbb{R}^2$ then the order of contact is defined or it is infinite.

Definition. Let p be a hyperbolic fixed point of saddle type of a C^2-diffeormorphism $f:M\to M$. Then a *linearising metric at p* is a C^0-metric d on a neighbourhood U on $W^u(p)\cup W^s(p)$ such that d coincides with the Euclidian metric in a C^1-coordinate system in U linearising f.

Remark. These linearising metrics always exist in dimension two for saddle points like p (this follows from a theorem of Hartman [**1**]). They are not unique. However, with the above lemmas it is easy to see that if $\tilde{d}$ is another linearising metric then the restrictions of d and $\tilde{d}$ to each connected component of $(W^u(p)\cup W^s(p))-\{p\}$ differ only by a multiplicative factor.

In the sequel we shall make extensive use of the next two lemmas.

LEMMA 2.3. *Let p be a hyperbolic fixed point of saddle type of a C^2 diffeomorphism $f:M\to M$. Let $x\in W^u(p)-\{p\}$, d a C^0-metric on M and μ the contracting eigenvalue of $Df(p)$. Then for any sequence $x_i\to x$ we have:*

(i) *If there exists a sequence* $n_i \to \infty$ *such that* $f^{-n_i}(x_i) \to z \in W^s(p)$ *and* $f^j(x_i)$ *is for* $0 \le j \le n_i$ *in a linearising neighbourhood, then*

$$d(x_i, W^u(p)) \simeq cd(z, p)|\mu|^{n_i},$$

for some constant c which depends on x, z and d but not on the sequence. If d is a linearising metric then c is independent of x and z.

(ii) *If* $d(x_i, W^u(p)) \simeq c|\mu|^{n_i}$ *for some constant c and some sequence* $n_i \to \infty$ *then the sequence* $f^{-n_i}(x_i)$ *has at least one and at most two limit points which are contained in* $W^s(p)$.

3. *Moduli in the homoclinic case*

The first thing we want to deal with is the modulus introduced by Palis in [**6**] for the heteroclinic case but now in the homoclinic case.

THEOREM 3.1. *Let p (resp. p') be a hyperbolic fixed point of saddle type of a diffeomorphism f (resp. f') of a compact two-dimensional* C^∞ *manifold M. Let r (resp. r') be a quadratic tangency between* $W^u(p)$ *and* $W^s(p)$ *(resp. between* $W^u(p')$ *and* $W^s(p')$*), i.e. a second order contact as defined in* § 2. *Let* μ *(resp.* μ'*) denote the contracting eigenvalue of* $Df(p)$ *(resp.* $Df'(p')$*) and* λ *(resp.* λ'*) denote the expanding eigenvalue of* $Df(p)$ *(resp.* $Df'(p')$*). Let h be a conjugacy between f and f', such that* $h(p) = p'$ *and* $h(r) = r'$. *Then we have*:

$$\frac{\log|\lambda|}{\log|\mu|} = \frac{\log|\lambda'|}{\log|\mu'|}$$

Proof. (See also [**5**].) Replacing f, f' by a power we may assume $\mu, \lambda, \mu', \lambda' > 0$. We have for example the situation shown in figure 1.

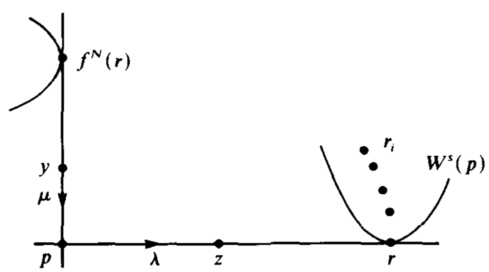

FIGURE 1

We consider a sequence of points r_i converging to r with $r_i \notin W^u(p) \cup W^s(p)$. By choosing subsequences $n_i \to \infty$, $m_i \to \infty$ we can arrange that $f^{-n_i}(r_i)$, resp. $f^{m_i}(r_i)$ has a limit y in $W^s(p) - \{p\}$, resp. z in $W^u(p) - \{p\}$. We can C^1 linearise f on $W^s(p)$ and $W^u(p)$. If $f^{-j}(r_i)$, $0 \ge -j \ge -n_i$ and $f^j(r_i)$, $N \le j \le m_j$ are in a linearizing neighbourhood then we have:

$$d(r_i, W^u(p)) \simeq c_1 \mu^{n_i} \quad \text{and} \quad d(r_i, W^s(p)) \simeq c_2 \lambda^{-m_i}.$$

Where c_1 and c_2 are constants. It is clear from the picture that we can choose the sequence r_i so that

$$d(r_i, W^u(p)) \simeq d(r_i, W^s(p)).$$

In that case we have

$$\frac{\log|\lambda|}{\log|\mu|} = -\lim \frac{m_i}{n_i}.$$

Denote by r_i' the images under h of r_i. From the topology of the intersection of $W^u(p)$ and $W^s(p)$ and the positions of the r_i's we have:

$$d(r_i', W^s(p')) \leq d(r_i', W^u(p')). \tag{$*$}$$

Furthermore since $(f')^{-n_i}(r_i')$ and $(f')^{m_i}(r_i')$ must have a limit in $W^s(p') - \{p'\}$ resp. $W^u(p') - \{p'\}$ we have:

$$d(r_i', W^n(p')) \cong c_1'(\mu')^{-1} \quad \text{and} \quad d(r_i', W^s(p')) \simeq c_2'(\lambda')^{-m_i}.$$

Where c_1' and c_2' are constants. This together with $(*)$ implies:

$$\frac{\log|\lambda'|}{\log|\mu'|} \leq -\lim \frac{m_i}{n_i} = \frac{\log|\lambda|}{\log|\mu|}.$$

Using a sequence r_i on the other side of $W^u(p)$ we find:

$$\frac{\log|\lambda'|}{\log|\mu'|} \geq \frac{\log|\lambda|}{\log|\mu|}.$$

So we have

$$\frac{\log|\lambda|}{\log|\mu|} = \frac{\log|\lambda'|}{\log|\mu'|}.$$ □

The next theorem shows the rigidity of the conjugacy h in case $\log|\lambda|/\log|\mu| \in \mathbb{R} - \mathbb{Q}$.

THEOREM 3.2. *Take the situation as described in Theorem* 3.1. *Let d_p be a linearising metric at p. If* $\log|\lambda|/\log|\mu|$ *is irrational then we have*: $d'_{p'}(h(z), p')/[d_p(z, p)]^\delta$ *is constant in each connected component of* $W^s(p) - \{p\}$ *and*: $d'_{p'}(h(w), p')/[d_p(w, p)]^\delta$ *is constant in each connected component of* $W^u(p) - \{p\}$, *where*

$$\delta = \frac{\log|\mu'|}{\log|\mu|}\left(= \frac{\log|\lambda'|}{\log|\lambda|}\right).$$

Proof. The proof follows from arguments similar to those in [3]. □

To be more precise: if $h: M \to M$ be a conjugacy between f and f', $h(p) = p'$, $h(r) = r'$ then there exists constants a_-, a_+, b_-, b_+ such that:

$$h((x, 0)) = (a_+(x)^\delta, 0);\ (x, 0) \in U \cap W^s(p);\ x \geq 0,$$
$$h((x, 0)) = (a_-|x|^\delta, 0);\ (x, 0) \in U \cap W^s(p);\ x < 0,$$
$$h((0, y)) = (0, b_+(y)^\delta);\ (0, y) \in U \cap W^u(p);\ y \geq 0,$$
$$h((0, y)) = (0, b_-|y|^\delta);\ (0, y) \in U \cap W^u(p);\ y < 0,$$

where U denotes a neighbourhood of p such that there is a C^1-coordinate system $\phi: U \to \mathbb{R}^2$ linearising f i.e.: $\phi \circ f \circ \phi^{-1}(x, y) = (\lambda x, \mu y)$ and $r \in U$.

Remark 1. From the formulas above for h it follows that the restriction of h to $W^s(p) - \{p\}$ and to $W^u(p) - \{p\}$ is a C^1-diffeomorphism.

Remark 2. If there are no further restrictions on h due to global configurations, then the restriction of h to each component of $W^s(p)-\{p\}$ and $W^u(p)-\{p\}$ is determined by the image of one point. This is the rigidity of the conjugacy mentioned before.

COROLLARY 3.3. *Each extra orbit of tangency between stable and unstable manifolds gives rise to at least two more moduli, because of the rigidity of h.*

Next we prove that in the case of homoclinic tangency we have both $\mu=\mu'$ and $\lambda=\lambda'$ instead of the weaker result

$$\frac{\log|\lambda|}{\log|\mu|}=\frac{\log|\lambda'|}{\log|\mu'|}.$$

So now both λ and μ are moduli.

THEOREM 3.4. *Let f, f' be two C^∞ diffeomorphisms of a two dimensional manifold M; p (resp. p') a hyperbolic fixed point of saddle type of f (resp. f'). Let r (resp. r') be a point of quadratic tangency between $W^s(p)$ and $W^u(p)$ (resp. $W^s(p')$ and $W^u(p')$). Let μ (resp. μ') denote the contracting eigenvalue of $Df(p)$ (resp. $Df'(p')$), and λ (resp. λ') the expanding eigenvalue of $Df(p)$ (resp. $Df'(p')$). If h is a conjugacy between f and f' with $h(p)=p'$; $h(r)=r'$ and $\log|\lambda|/\log|\mu|$ is irrational then we have: $\mu=\mu'$ and $\lambda=\lambda'$.*

Proof. Because $\log|\lambda|/\log|\mu|$ is irrational we know that $h|_{W^s(p)-\{p\}}$ is a C^1-map. Take a sequence $r_i\in W^s(p)$ with $r_i\to r$ and $f^{-n_i}(r_i)\to q\in W^s(p)$, when $n_i\to\infty$ then we have:

$$d(r_i,\, W^u(p))\simeq c|\mu|^{n_i}d(p,q), \tag{1}$$

where c is a positive constant independent of the sequence. Now $W^u(p)$ and $w^s(p)$ have a quadratic tangency at r. For a C^∞ metric $\tilde d$ induced by a C^∞ coordinate system in which $W^u(p)$ is a straight line, and $W^s(p)$ is the graph of a homogeneous polynomial of degree two, we have:

$$\frac{\tilde d(r_i,\, W^u(p))}{[\tilde d(r_i,r)]^2}\to\tilde s(r), \tag{2}$$

where $\tilde s(r)$ is a positive number. But since d is a C^0-metric we have by Lemma 2.2: $d(r_i,\, W^u(p))/\tilde d(r_i,\, W^u(p))$ converges to a positive constant.

Because $r\in W^s(p)$ we have that the sequence $d(r_i,r)/\tilde d(r_i,r)$ also converges to a positive constant. This together with (2) implies that

$$d(r_i,\, W^u(p))/[d(r_i,r)]^2\to s(r);\quad s(r)>0. \tag{3}$$

Because $h|_{W^s(p)-\{p\}}$ is C^1 we have

$$d(r_i,r)\sim d'(h(r_i),r') \tag{4}$$

Equations (1), (3), (4) imply that $|\mu|=|\mu'|$. Because a conjugacy also preserves the sign of μ, μ' we have $\mu=\mu'$. From

$$\frac{\log|\lambda|}{\log|\mu|}=\frac{\log|\lambda'|}{\log|\mu'|}$$

we finally get $\lambda=\lambda'$. □

Before we can define our last modulus, we have to make some estimates on the iterates of points near the homoclinic point. Suppose we have the situation indicated in figure 2:

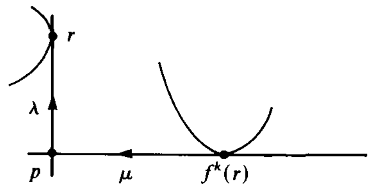

FIGURE 2

So we have a diffeomorphism f with a homoclinic point r. We assume that the eigenvalues of $Df(p)$ are such that $0<\mu<1<\lambda$. Furthermore we assume that the tangencies are quadratic. Next we choose linearising coordinates (V, ϕ) so that we have: $\phi(r)=(0, r_2)$ where r is a point of tangency of $W^s(p)$ and $W^u(p)$, in our coordinate neighbourhood. Then we can find an integer k such that $f^k(r)\in V$ lies on the local stable-manifold of p. We assume $\phi(f^k(r))=(r_1, 0)$. Now we follow further iterates of r. Without loss of generality we may assume that our diffeomorphism is linear in V.

$$f(x_1, x_2)=Df(x_1, x_2)=(\mu x_1, \lambda x_2) \quad \text{when } (x_1, x_2)\in V.$$

Because all our tangencies are quadratic we have that f^k is a quadratic mapping at r in the following sense:

$$f^k(x_1, x_2)=(r_1-\alpha(x_2-r_2), \beta x_1+\gamma(x_2-r_2)^2)+\text{h.o.t.},$$

where α, β, γ are positive constants, h.o.t. stands for higher order terms.

Note. The curvature of $W^u(p)$ at $f^k(r)$ in this coordinate system is in fact: $\gamma\alpha^{-2}$.

For all $n>0$, f^{n+k} is a quadratic mapping provided $f^{i+k}(x_1, x_2)$ is in the coordinate neighbourhood V, when $i<n$.

Furthermore we have the following formula for f^{n+k} (restricted to $x_1=0$):

$$f^{n+k}(0, x_2)=(\mu^n(r_1-\alpha(x_2-r_2)), \lambda^n(\gamma(x_2-r_2)^2)).$$

Next we want to know how the coordinates of the point $f^{n+k}(0, x_2)$ behave, when $n\to\infty$.

It is clear that the x_1-coordinate goes to zero. For the x_2-coordinate we have: The x_2-coordinate is approximately r_2 when

$$|x_2-r_2|\le\sqrt{(r_2\lambda^{-n}\gamma^{-1})}.$$

So by choosing an appropriate sequence $\{r_i\}$ of points converging to r we can achieve that $f^{i+k}(r_i)$ converges to r. This argument shows that we can expect another modulus. This is related with the fact that our homeomorphism h is completely determined on components of $W^s(p)-\{p\}$ and $W^u(p)-\{p\}$. Because $W^u(p)$ accumulates on itself, we can come in conflict if we want to have that h is continuous.

We will now start to derive the modulus mentioned above.

We assume that for our diffeomorphism f we have $\log|\lambda|/\log|\mu|$ irrational. (See figure 3.) Let (V, ϕ) be a C^1-coordinate system which linearizes f. Take $r \in V$ and pick an integer k such that $\tilde{r} = f^k(r)$ lies on the local stable manifold of p. Also assume $\tilde{r} \in V$. We may assume $\phi(p) = (0, 0)$; $\phi(r) = (0, 1)$ and $\phi(\tilde{r}) = (1, 0)$. This fixes ϕ completely on $W^u(p)$ and $W^s(p)$. The corresponding points for a similar diffeomorphism f' are denoted by p', r, $\tilde{r}$. Let d be a linearising metric at p. Assume there is a conjugacy h between f and f', with $h(p) = p'$; $h(r) = r'$; $h(\tilde{r}) = \tilde{r}$. Then we have a modulus of the following form:

Take linearising coordinates z on $W^u(p)$ with $z(r) = 1$, such that the mapping f^k restricted to $W^u(p)$ is given by $f^{k+n}(z) = (x_{k+n}(z), y_{k+n}(z))$ and $y_{k+n}(z)$ is given by a homogeneous quadratic polynomial + h.o.t. i.e.:

$$y_{k+n}(1+z) = c_{k+n}^2 z^2 + \text{h.o.t.}, \quad c_{k+n}^2 = \lambda^n c_0^2.$$

Take a sequence of points $\{r_i\}$, $r_i \in W^u(p)$, $r_i \to r$. Then define the sequence $\tilde{r}_i$ by $\tilde{r}_i = f^k(r_i)$. This gives us $\tilde{r}_i \to \tilde{r}$. By choosing the sequence r_i in a right way we can achieve that $\bar{r}_i$ defined by $\bar{r}_i = f^i(\tilde{r}_i)$ converges to r. More explicitly: We have $\bar{r}_i \to r$ if and only if $d(r_i, r)/(\sqrt{\lambda})^{-i} \to c_0$. Going back to our sequences r_i, $\tilde{r}_i$, $\bar{r}_i$ we have: if $d(r_i, r)/\sqrt{\lambda}^{-i} \to c_0$ then $\bar{r}_i \to r$. Note that $\bar{r}_i$ can go to r in the following way (see figure 4). Because $h|_{W^s(p)-\{p\}}$ is a C^1 map and d is a linearising metric we have $d(h(r_i), r') = d(r_i, r)$. With the same reasoning as above we conclude that there exists a constant c_0' such that $\bar{r}_i' = h(\bar{r}_i) \to r'$ if and only if $d(h(r_i), r')/\sqrt{\lambda}^{-i} \to c_0'$. So we get $c_0(\sqrt{\lambda})^{-i} \simeq d(r_i, r) = d(h(r_i), r') \simeq c_0'(\sqrt{\lambda'})^{-i}$. Because $\log|\lambda|/\log|\mu|$ is irrational we have from Theorem 5.4: $\lambda = \lambda'$ and so we must have $c_0 = c_0'$.

So we have proven:

THEOREM 3.5. *Let f, f' be two C^∞-diffeomorphisms of a two dimensional manifold M; p (resp. p') a hyperbolic fixed point of saddle type of f (resp. f').*

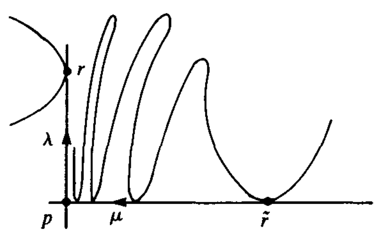

FIGURE 3

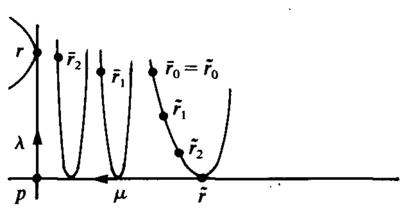

FIGURE 4

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Choose coordinates (V, ϕ) (resp. (V', ϕ')) which linearize f (resp. f'). Let $r \in V$ (resp. $r' \in V'$) be a point of tangency between $W^s(p)$ and $W^u(p)$ (resp. $W^s(p')$ and $W^u(p')$). Let k be an integer such that $\tilde{r} = f^k(r)$ is an element of V. Let μ (resp. μ') denote the contracting eigenvalue of $Df(p)$ (resp. $Df'(p')$) and λ (resp. λ') the expanding eigenvalue of $f(p)$ (resp. $Df'(p')$). Let h be a conjugacy between f and f' with $h(p) = p$, $h(r) = r'$ and $\log|\lambda|/\log|\mu|$ irrational.

If we take linearising coordinates z, with $z(r) = 1$, on $W^u(p)$ such that the mapping f^k restricted to $W^u(p)$ is given by $f^k(z) = (x(z), y(z))$ where $y(z)$ is a quadratic mapping – $y(1) = 0$, $y'(1) = 0$, $y''(1) \neq 0$ – and similarly for f'.

Then there exist constants c, c' such that:

(i) *For a sequence $\{r_i\}$, $r_i \in W^u(p)$, $r_i \to r$, let the sequence $\{\overline{r_i}\}$ be defined by $\overline{r_i} = f^{k+i}(r_i)$ then we have: $\overline{r_i} \to r$ if and only if $d(r_i, r)/\sqrt{\lambda}^{-i} \to c$ (and an analogous condition for c').*

(ii) *If f and f' are conjugated then we have $c = c'$.* □

4. *Some remarks about the higher dimensional case*

The moduli obtained in the previous section also turn up in the higher dimensional case. One has to assume that there are C^2-linearizing coordinates in a neighborhood of the hyperbolic fixed point p and that there is an orbit of regular quasi-transversal tangency between $W^s(p)$ and $W^u(p)$. See [**5**] for this notion. As a consequence we have that the weakest expanding and weakest contracting eigenvalues of $Df(p)$ exist. Denote these eigenvalues by λ, μ respectively. Then we have the same moduli as in the preceding section. But there are more moduli. We intend to come back to this in another article.

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Added in proof

After submitting this paper the author found the following paper:
S. V. Gonchenko & L. P. Shilnikov. Arithmetic properties of topological invariants of systems with non-structurally stable homoclinic trajectories. *Ukr. Math. J.* **39** (1987), 15–21. Topological invariants related with homoclinic tangencies are also considered in this paper.

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