THE LINEARISATIONS OF CYCLIC PERMUTATIONS HAVE RATIONAL ZETA FUNCTIONS

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Let \( \mathbb{Z} \) be an integer. Let \( \mathbb{P} \) be the set of all integers in \([1, n+1]\) and let \( \sigma \) be a cyclic permutation on \( \mathbb{P} \). Assume that \( f \) is the linearisation of \( \sigma \) on \( \mathbb{P} \). Then we show that \( f \) has rational Artin-Mazur zeta function which is closely related to the characteristic polynomial of some \( n \times n \) matrix with entries either zero or one. Some examples of non-conjugate maps with the same Artin-Mazur zeta function are also given.

Let \( [a, b] \) be a nondegenerate compact interval on the real line and let \( f \) be a continuous map from \([a, b]\) into itself. For every positive integer \( k \), let \( f^k \), the \( k \)'th iterate of \( f \), be defined by: \( f^1 = f \) and \( f^k = f \circ f^{k-1} \) if \( k > 1 \). For \( x_0 \in [a, b] \), we call \( x_0 \) a periodic point of \( f \) if \( f^m(x_0) = x_0 \) for some positive integer \( m \) and call the smallest such positive integer \( m \) the least period of \( x_0 \) (under \( f \)). We call the set \( \{ f^k(x_0) \mid k \text{ is any nonnegative integer} \} \) the periodic orbit of \( x_0 \) (under \( f \)). It is easy to see that, if \( f^m(x_0) = x_0 \) for some positive integer \( m \), then the least period of \( x_0 \) must divide \( m \). We shall need this fact later. A periodic point of least period 1 is also called a fixed point. In discrete dynamical systems theory, one problem related to the numbers of periodic points is: For every positive integer \( k \), let \( P_k = \{ x \in [a, b] \mid f^k(x) = x \} \). Let \( N(f^k) \) be the number of points in \( \{ x \in P_k \mid x \text{ is isolated in } P_m \text{ for some positive integer } m \text{ dividing } k \} \). Assume that, for every positive integer \( k \), the number \( N(f^k) \) is finite. (Note that this definition of \( N(f^k) \) is a slight generalisation of that of Artin and Mazur [1].) Find the reduced Artin-Mazur zeta function [5] \( \bar{\zeta}_f(z) = \sum_{k=1}^{\infty} N(f^k) z^k \) of \( f \) or find the Artin-Mazur zeta function [1, 5] \( \zeta_f(z) = \exp \left( \sum_{k=1}^{\infty} \left( N(f^k)/k \right) z^k \right) \) of \( f \), where \( z \) is the complex variable.

When we actually compute [2, 3] the reduced Artin-Mazur zeta functions of some special types of continuous piecewise linear maps \( f \) on \([a, b]\), we find some \( n \times n \) matrices \( A_f \) (depending on \( f \)) such that \( N(f^k) = \text{tr} \left( A_f^k \right) \), the trace of \( A_f^k \), for every positive integer \( k \) (see also [6]) and the reduced Artin-Mazur zeta functions of \( f \) are closely
related to the characteristic polynomials of the matrices $A_f$. In this note, we extend this result to a class called the linearisations of cyclic permutations. To this end, we shall need the following result from matrix theory.

**Theorem 1.** Let $A$ be an $n \times n$ complex matrix and let $\det (xE_n - A) = z^n + \sum_{j=0}^{n-1} \beta_j z^j$ be the characteristic polynomial of $A$, where $E_n$ is the $n \times n$ identity matrix. Then

$$
\sum_{k=1}^{\infty} \frac{\text{tr} (A^k) z^k}{k} = -z \frac{d}{dz} \left( 1 + \sum_{j=1}^{n} \beta_{n-j} z^j \right) \quad \text{or} \quad \exp \left( \sum_{k=1}^{\infty} \frac{\text{tr} (A^k) z^k}{k} \right) = \frac{1}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j}.
$$

**Proof:** Write $z^n + \sum_{j=0}^{n-1} \beta_j z^j = \prod_{j=1}^{n} (z - \lambda_j)$. Then, by replacing $z$ by $1/z$ and simplifying, we obtain that $1 + \sum_{j=1}^{n} \beta_{n-j} z^j = \prod_{j=1}^{n} (1 - \lambda_j z)$. Since $A$ is similar to an upper triangular matrix with main diagonal entries the eigenvalues of $A$, we easily obtain that $\text{tr} (A^k) = \sum_{j=1}^{n} \lambda_j^k$ for all positive integers $k$. This fact will also be used later. So,

$$
\sum_{k=1}^{\infty} \text{tr} (A^k) z^k = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{n} \lambda_j^k \right) z^k = \sum_{j=1}^{n} \left( \sum_{k=1}^{\infty} (\lambda_j z)^k \right) = \sum_{j=1}^{n} \frac{\lambda_j z}{1 - \lambda_j z} = -z \frac{d}{dz} \left( \frac{\prod_{j=1}^{n} (1 - \lambda_j z)}{\prod_{j=1}^{n} (1 - \lambda_j z)} \right) = -z \frac{d}{dz} \left( 1 + \sum_{j=1}^{n} \beta_{n-j} z^j \right).
$$

Or, by formal integration,

$$
\exp \left( \sum_{k=1}^{\infty} \frac{\text{tr} (A^k) z^k}{k} \right) = \frac{1}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j}.
$$

We now return to the discrete dynamical systems theory on the interval. Let $n \geq 2$ be an integer. Let $P$ be the set of all integers in $[1, n+1]$ and let $\sigma$ be a map from $P$ into itself. For every integer $1 \leq k \leq n$, let $J_k = [k, k+1]$. Assume that $f$ is the continuous map from $[1, n+1]$ into itself such that $f(i) = \sigma(i)$ for every integer $1 \leq i \leq n+1$ and $f$ is linear on $J_k$ (and so, the absolute value of the slope of $f$ on $J_k$ is $\geq 1$) for every integer $1 \leq k \leq n$. This continuous map $f$ is called the linearisation.
of σ on P. Let \( A_f = (\alpha_{ij}) \) be the \( n \times n \) matrix defined by \( \alpha_{ij} = 1 \) if \( f(J_i) \supset J_j \) and \( \alpha_{ij} = 0 \) otherwise. This matrix \( A_f \) is called the Markov matrix of \( f \). Note that, for every positive integer \( k \), since \( f \) is piecewise linear, so is \( f^k \) and the slope of every linear piece of \( f^k \) is the product of the slopes of \( k \) linear pieces of \( f \). Now, for every positive integer \( k \), if \( \alpha_{i_1i_2} \alpha_{i_2i_3} \cdots \alpha_{i_{k+1}i_{k+2}} \) is nonzero (in this case, this product is 1) for some integers \( i_1, i_2, \cdots, i_{k+1} \) in \([1, n]\) then, by definition of \( A_f \), \( f(J_{i_s}) \supset J_{i_{s+1}} \) for every integer \( 1 \leq s \leq k \). If \( I \) and \( J \) are closed subintervals of \([1, n+1]\) such that \( f(J) \supset J \), then it is well-known that there is a closed interval \( L \subset I \) such that \( f(L) = J \). So, since \( f(J_{ik}) \supset J_{ik+1} \), there is a closed interval \( L_1 \subset J_{ik} \) such that \( f(L_1) = J_{ik+1} \). Since \( L_1 \subset J_{ik} \subset f(J_{ik-1}) \), there is a closed interval \( L_2 \subset J_{ik-1} \) such that \( f(L_2) = L_1 \). Inductively, there are closed intervals \( L_1, L_2, \cdots, L_k \) such that \( L_s \subset J_{ik+1-s} \) and \( f(L_s) = L_{s-1} \) for every integer \( 1 \leq s \leq k \), where we define \( L_0 = J_{ik+1} \). Consequently, \( f^k \) is linear on \( L_k \) and \( f^k(L_k) = J_{ik+1} \). Note that, since \( f^k \) is linear on \( L_k \), the slope of \( f^k \) on \( L_k \) is nonzero.

Conversely, let \( k \) be a fixed positive integer and let \( T_k \) be a maximum closed interval on which \( f^k \) is linear with nonzero slope. Since the \( y \)-coordinates of the turning points and the boundary points, \((1, f(1))\) and \((n+1, f(n+1))\), of \( f \) are contained in \( P \), \( f^k(T_k) \) is a compact interval whose endpoints are distinct and contained in \( P \). So, for some integers \( 1 \leq u_k \leq v_k \leq n \), \( f^k(T_k) = \bigcup_{i=u_k}^{v_k} J_i \). Let \( i_{k+1} \) be any integer such that \( u_k \leq i_{k+1} \leq v_k \). Then \( f^k(T_k) \supset J_{i_{k+1}} \) and there is a closed subinterval \( L_k \) of \( T_k \) such that \( f^k(L_k) = J_{i_{k+1}} \). Since \( f^k \) is linear on \( L_k \), \( f \) is linear on the interval \( f^k(L_k) \) for every integer \( 0 \leq s \leq k - 1 \). In particular, \( f \) is linear on the interval \( f^{k-1}(L_k) \) and since the interior of \( f(f^{k-1}(L_k)) = J_{i_{k+1}} \) contains no point of \( P \), neither does the interior of \( f^{k-1}(L_k) \). This implies that \( f^{k-1}(L_k) \subset J_{i_k} \) for some integer \( 1 \leq i_k \leq n \). Similarly, since \( f \) is linear on the interval \( f^{k-2}(L_k) \) and the interior of \( f(f^{k-2}(L_k)) = f^{k-1}(L_k) \subset J_{i_k} \) contains no point of \( P \), neither does the interior of \( f^{k-2}(L_k) \). Therefore, \( f^{k-2}(L_k) \subset J_{i_{k-1}} \) for some integer \( 1 \leq i_{k-1} \leq n \). Similar arguments imply that there are integers \( i_k, i_{k-1}, \cdots, i_3, i_2, i_1 \) in \([1, n]\) such that \( f^t(L_k) \subset J_{i_{t+1}} \) for every integer \( 0 \leq t \leq k - 1 \) and \( f^k(L_k) = J_{i_{k+1}} \). Note that, when \( t = 0 \), \( L_k \subset J_{i_1} \). Thus, \( f^s(L_k) \subset f(J_{i_s}) \) for every integer \( 1 \leq s \leq k \). Consequently, \( f^s(L_k) \subset J_{i_{s+1}} \cap f(J_{i_s}) \) for every integer \( 1 \leq s \leq k - 1 \). So, \( f(J_{i_s}) \) contains some interior points of \( J_{i_{s+1}} \) for every integer \( 1 \leq s \leq k - 1 \). Since both endpoints of \( f(J_{i_s}) \) are distinct points in \( P \), this implies that \( f(J_{i_s}) \supset J_{i_{s+1}} \), and so, \( \alpha_{i_s,i_{s+1}} = 1 \) for every integer \( 1 \leq s \leq k-1 \). Furthermore, since \( J_{i_k} \supset f^{k-1}(L_k) \), we have \( f(J_{i_k}) \supset f(f^{k-1}(L_k)) = J_{i_{k+1}} \). Thus, \( \alpha_{i_k,i_{k+1}} = 1 \). Hence \( \alpha_{i_1i_2} \alpha_{i_2i_3} \cdots \alpha_{i_ki_{k+1}} = 1 \). Therefore, for every positive integer \( k \), there is a one-to-one correspondence between the collection of nonzero products \( \alpha_{i_1i_2} \alpha_{i_2i_3} \cdots \alpha_{i_ki_{k+1}} \) and the collection of closed
intervals $L_k \subset J_i$ such that $f^k$ is linear on $L_k$ and $f^k(L_k) = J_{i_{k+1}}$. Consequently, if $A_f^k = (\alpha_{ij}^{(k)})$; then every entry $\alpha_{ij}^{(k)}$ represents the number of closed intervals $L_k \subset J_i$ with disjoint interiors such that $f^k$ is linear on $L_k$ and $f^k(L_k) = J_{j}$. In the following, we shall show how to relate this number $\sum_{j=1}^n \alpha_{jj}^{(k)}$ to the number $N(f^k)$. We have two cases to consider:

**Case 1.** Assume that, for some positive integer $t$, the absolute value of the slope of every linear piece of $f^t$ is $> 1$. Then, it is easy to see that, for every integer $j > t$, the absolute value of the slope of every linear piece of $f^j$ is $> 1$. So, for any integers $1 \leq i \leq n$, $k \geq 1$ (we do not require $k \geq t$) and any closed interval $L \subset J_i$, if $f^k$ is linear on $L$ and $f^k(L) = J_i \supset L$, then the slope of $f^k$ on $L$ cannot be equal to 1 and hence the equation $f^k(x) = x$ has exactly one solution in $L$. So, we can associate this unique solution to the interval $L$. This implies that, for every positive integer $k$, $N(f^k) = \sum_{j=1}^n \alpha_{jj}^{(k)} = \tr(A_f^k) = \sum_{j=1}^n \lambda_j^k$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the Markov matrix $A_f$ of $f$.

**Case 2.** Assume that the absolute value of the slope of some linear piece of $f^n$ is 1. Let $T_n$ be a maximum closed interval on which $f^n$ is linear and the absolute value of the slope of $f^n$ on $T_n$ is 1. As was just shown above, there exist integers $i_1, i_2, \ldots, i_{n+1}$ in $[1, n]$ and a closed interval $L_n \subset T_n$ such that $J_{i_{n+1}} \subset f^n(T_n), f^n(L_n) = J_{i_{n+1}}$ and $f^s(L_n) \subset J_{i_{s+1}}$ for every integer $0 \leq s \leq n - 1$. Since the absolute value of the slope of $f^n$ on $L_n$ (which is 1) equals the product of the absolute values of the slopes of $f$ on $f^s(L_n)$ (which are $\geq 1$) for all integers $0 \leq s \leq n - 1$, the absolute value of the slope of $f$ on $f^s(L_n)$ must be 1 for every integer $0 \leq s \leq n - 1$. So, the length of $f^s(L_n)$ is equal to that of $f^{s+1}(L_n)$ for every integer $0 \leq s \leq n - 1$. Since the length of $f^n(L_n)(= J_{i_{n+1}})$ is 1, we obtain that the length of $f^s(L_n)(\subset J_{i_{s+1}})$ is also 1 for every integer $0 \leq s \leq n - 1$. This, together with the fact that $f^n(L_n) = J_{i_{n+1}}$, implies that $f^s(L_n) = J_{i_{s+1}}$ for every integer $0 \leq s \leq n$. Note that, when $s = 0$, $L_n = J_{i_1}$. Thus, $f(J_{i_1}) = f(f^{s-1}(L_n)) = f^s(L_n) = J_{i_{s+1}}$ for every integer $1 \leq s \leq n$. Since there are $n + 1$ closed intervals $J_{i_{s+1}}(= f^s(L_n))$, $0 \leq s \leq n$, taken from the $n$ distinct $J_i$'s, some interval $J_{i_s}$ appears at least twice. Without loss of generality, we may assume that $J_{i_1} = J_{i_{m+1}}$ for some integer $1 \leq m \leq n$ and the $J_{i_s}$'s are distinct for all integers $1 \leq s \leq m$. Since $f(J_{i_s}) = J_{i_{s+1}}$ and $f$ is linear on $J_{i_s}$ for all integers $1 \leq s \leq m$, we obtain that, for every integer $1 \leq s \leq m$, $f^m(J_{i_s}) = J_{i_s}$ and $f^m$ is linear on $J_{i_s}$, and hence, we have either $f^m(x) = x$ for all $x \in J_{i_s}$ and all integers $1 \leq s \leq m$ or $f^m(x) = -x + a_s + b_s$, where $J_{i_s} = [a_s, b_s]$, for all $x \in J_{i_s}$ and all integers $1 \leq s \leq m$. In the following, we assume, for simplicity, that $\sigma$ is a cyclic permutation on $P$. If, for some integer $1 \leq j \leq m$, $f^m(x) = x$ for all $x \in J_{i_j}$, then, in particular,
f^m(a_j) = a_j$. Since $a_j$ is a periodic point of $f$ with least period $n + 1$, we must have $n + 1 \leq m$ which contradicts the assumption that $1 \leq m \leq n$. This contradiction implies that $f^m(x) = -x + a_s + b_s$ and hence $f^{2m}(x) = x$ for all $x \in J_i$, and all integers $1 \leq s \leq m$. In particular, $f^{2m}(a_s) = a_s$ for all integers $1 \leq s \leq m$. Since the least period of $a_s$ is $n + 1$, we have $n + 1 \mid m$ for some positive integer $r$. But, since $1 \leq m \leq n$, we must have $r = 1$, and so, $n + 1 = 2m$. Furthermore, since $f^m$ maps every endpoint of $J_i$ to the other for every integer $1 \leq s \leq m$, the $m$ closed intervals $J_i$'s are pairwise disjoint. Since there are exactly $n = 2m - 1 = m + (m - 1)$ distinct closed intervals $J_i$'s, we obtain that $\{J_i, 1 \leq s \leq m\} = \{J_{2j-1}, 1 \leq j \leq m\}$. Consequently, $f^m(x) = -x + 4s - 1$ and $f^{2m}(x) = x$ for all $x \in J_{2j-1}$ and all integers $1 \leq s \leq m$. This also implies that $T_n = J_{2j-1}$ for some integer $1 \leq j \leq m$ and the absolute value of the slope of the linear piece of $f^n$ and hence of $f^k$ for every integer $k \geq n$ on any closed interval contained in $J_{2j}$ for any integer $1 \leq i \leq m - 1$ is $> 1$. Thus, for any positive integer $k$ (we do not require $k \geq n$), any integer $1 \leq i \leq m - 1$ and any closed interval $L \subset J_{2i}$, if $f^k$ is linear on $L$ and $f^k(L) = L_{2i} \supset L$, then the slope of $f^k$ on $L$ cannot be equal to 1 and hence the equation $f^k(x) = x$ has a unique solution in $L$. So, we can associate this unique solution to the interval $L$. Furthermore, since $f$ permutes cyclically the intervals $J_{2j-1}$, $1 \leq j \leq m$, and since $f^m$ maps every endpoint of any $J_{2j-1}$, $1 \leq j \leq m$, to the other, we obtain that there exists a cyclic permutation $\rho$ on the set of all integers in $[1, m]$ such that $\sigma^{j}(1) \in \{2\rho^{j}(1) - 1, 2\rho^{j}(1)\}$ and $\{\sigma^{j}(1), \sigma^{m+j}(1)\} = \{2\rho^{j}(1) - 1, 2\rho^{j}(1)\}$ for all integers $1 \leq j \leq m$. On the other hand, since $f$ permutes cyclically the intervals $J_{2j-1}$, $1 \leq j \leq m$, and since $f^m(x) = -x + 4s - 1$ and $f^{2m}(x) = x$ for all $x \in J_{2j-1}$ and all integers $1 \leq j \leq m$, we see that, for any positive integer $k$, (i) if $k$ is not a multiple of $m$, then the equation $f^k(x) = x$ has no solution in $J_{2j-1}$ for any integer $1 \leq j \leq m$; (ii) if $k$ is an odd multiple of $m$, then each midpoint of $J_{2j-1}$ is the unique (and isolated) solution of the equation $f^k(x) = x$ in $J_{2j-1}$ for every integer $1 \leq j \leq m$; and (iii) if $k$ is an even multiple of $m$, then, for every integer $1 \leq j \leq m$, every point of $J_{2j-1}$ is a (non-isolated) solution of the equation $f^k(x) = x$, but, the midpoint of $J_{2j-1}$ is an isolated solution of the equation $f^m(x) = x$. Therefore, we can associate, for every integer $1 \leq j \leq m$, the midpoint of $J_{2j-1}$ to the interval $J_{2j-1}$ when $k$ is an (odd or even) multiple of $m$ and nothing otherwise. This implies that, for every positive integer $k$, $N(f^k) = \sum_{j=1}^{n} \alpha_{jj}^{(k)} = \text{tr} \left(A_f^k\right) = \sum_{j=1}^{n} \lambda_j^k$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the Markov matrix $A_f$ of $f$. Consequently, by Theorem 1, we have proved the following two results:

**Theorem 2.** Let $n \geq 2$ be an integer. Let $P$ be the set of all integers in $[1, n+1]$ and let $\sigma$ be a map from $P$ into itself. Assume that $f$ is the linearisation of $\sigma$ on $P$ such that, for some positive integer $t$, the absolute value of the slope of every linear
piece of $f^t$ is $> 1$. Then the following hold:

(a) For every positive integer $k$, $N(f^k) = \text{tr} \left( A_f^k \right) = \sum_{j=1}^{n} \lambda_j^k$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the Markov matrix $A_f$ of $f$.

(b) The Artin-Mazur zeta function $\zeta_f(z)$ of $f$ is $\zeta_f(z) = 1/\left(1 + \sum_{k=1}^{n} \beta_{n-k}z^k\right)$, where $x^n + \sum_{k=0}^{n-1} \beta_k x^k$ is the characteristic polynomial of the Markov matrix of $f$.

**Theorem 3.** Let $n \geq 2$ be an integer. Let $P$ be the set of all integers in $[1, n+1]$ and let $\sigma$ be a map from $P$ into itself. For every integer $1 \leq k \leq n$, let $J_k = [k, k+1]$. Assume that $f$ is the linearisation of $\sigma$ on $P$ such that the absolute value of the slope of some linear piece of $f^n$ is 1. Then there exist an integer $1 \leq m \leq n$ and $m$ distinct integers $i_1, i_2, \ldots, i_m$ in $[1, n]$ such that $f$ is linear on $J_{i_s}$ and $f(J_{i_s}) = J_{i_{s+1}}$ for every integer $1 \leq s \leq m$, where we define $i_{m+1} = i_1$. Furthermore, if $\sigma$ is a cyclic permutation on $P$, then the following also hold:

(a) $n + 1 = 2m$.

(b) $f^m(x) = -x + 4k - 1$ and $f^{2m}(x) = x$ for all $x \in J_{2k-1}$ and all integers $1 \leq k \leq m$. In particular, $f$ has periodic points of least period $(n + 1)/2$.

(c) There exists a cyclic permutation $\rho$ on the set of all integers in $[1, m]$ such that $\sigma^j(1) \in \{2\rho^j(1) - 1, 2\rho^j(1)\}$ and $\{\sigma^j(1), \sigma^{m+j}(1)\} = \{2\rho^j(1) - 1, 2\rho^j(1)\}$ for all integers $1 \leq j \leq m$.

(d) For every positive integer $k$, $N(f^k) = \text{tr} \left( A_f^k \right) = \sum_{j=1}^{n} \lambda_j^k$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the Markov matrix $A_f$ of $f$.

(e) The Artin-Mazur zeta function $\zeta_f(z)$ of $f$ is $\zeta_f(z) = 1/\left(1 + \sum_{k=1}^{n} \beta_{n-k}z^k\right)$, where $x^n + \sum_{k=0}^{n-1} \beta_k x^k$ is the characteristic polynomial of the Markov matrix of $f$.

**Remark.** We require $\sigma$ to be a cyclic permutation on $P$ in Theorem 3 while not in Theorem 2. This is because the requirement on the slope of $f$ in Theorem 3 is weaker than that in Theorem 2. If we do not make the stronger requirement on $\sigma$ in Theorem 3, there would be many trivial examples whose linearisations have well-defined Markov matrices while their Artin-Mazur zeta functions are not defined.

The following partial converse of Theorem 3 is easy to prove.

**Theorem 4.** Let $m$ and $n$ be positive integers such that $n+1 = 2m$. Let $\rho$ be a cyclic permutation on the set of all integers in $[1, m]$ and let $\sigma$ be a cyclic permutation on the set $P$ of all integers in $[1, n+1]$. For every integer $1 \leq i \leq n$, let $J_i = [i, i+1]$. 

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Assume that \( f \) is the linearisation of \( \sigma \) on \( P \) and \( \sigma^j(1) \in \{2\rho^j(1) - 1, 2\rho^j(1)\} \) and \( \{\sigma^1(1), \sigma^{m+1}(1)\} = \{2\rho^1(1) - 1, 2\rho^1(1)\} \) for all integers \( 1 \leq j \leq m \). Then, for every positive integer \( k \), the absolute value of the slope of \( f^k \) on \( J_{2j-1} \) is 1 for all integers \( 1 \leq j \leq m \). Consequently, the Artin-Mazur zeta function \( \zeta_f(z) \) of \( f \) is
\[
\zeta_f(z) = \frac{1}{1 + \sum_{k=1}^{n-1} \beta_{n-k} z^k},
\]
where \( x^n + \sum_{k=0}^{n-1} \beta_k x^k \) is the characteristic polynomial of the Markov matrix of \( f \).

REMARK. If \( \sigma \) is a cyclic permutation on the set \( P \) of all integers in \([1,n+1]\) and \( f \) is the linearisation of \( \sigma \) on \( P \), then Theorems 2, 3, & 4 give a complete solution of the Artin-Mazur zeta function of \( f \). In particular, the Artin-Mazur zeta function of \( f \) is rational with poles at the values \( 1/\lambda_j \) where \( \lambda_1, \lambda_2, \cdots, \lambda_n \) are the eigenvalues of the Markov matrix of \( f \).

Let \( f \) and \( g \) be two continuous maps from \([a,b]\) into itself. If they are (topologically) conjugate to each other, then it is clear that they have the same (if defined) Artin-Mazur zeta function. However, if they are not conjugate to each other, they may still have the same Artin-Mazur zeta function. For example, assume that both \( f \) and \( g \) satisfy the conditions in Theorem 2 above. If their respective Markov matrices are similar to each other, then, since similar matrices have the same characteristic polynomial \([4]\), they have, by Theorem 2, the same Artin-Mazur zeta function. In the following, we present some such examples. The following result is taken from \([2]\).

THEOREM 5. For every integer \( n \geq 3 \), let \( f_n(x) \) be the continuous map from \([1,n]\) onto itself defined by
\[
f_n(x) = \begin{cases} 
x + 1, & \text{for } 1 \leq x \leq n - 1, \\
-(n-1)x + n^2 - n + 1, & \text{for } n - 1 \leq x \leq n.
\end{cases}
\]
We also define sequences \( (b_{k,n}) \) as follows:
\[
b_{k,n} = \begin{cases} 
2k - 1, & \text{for } 1 \leq k \leq n - 1, \\
\sum_{t=1}^{n-1} b_{k-t,n}, & \text{for } n \leq k.
\end{cases}
\]
Then, for any integers \( k \geq 1 \) and \( n \geq 3 \), \( b_{k,n} \) is the number of distinct fixed points of the map \( f_n^k(x) \) in \([1,n]\). Moreover, the Artin-Mazur zeta function \( \zeta_{f_n}(z) \) of \( f_n \), for every integer \( n \geq 3 \), is
\[
\zeta_{f_n}(z) = \frac{1}{1 - \sum_{k=1}^{n-1} z^k}.
\]

THEOREM 6. For every odd integer \( m \geq 3 \), let \( g_m(x) \) and \( h_m(x) \) be the contin-
uous maps from $[1, m]$ onto itself defined by

\[
g_m(x) = \begin{cases} 
-x + m + 1, & \text{for } 1 \leq x \leq \frac{1}{2}(m - 1), \\
-\frac{1}{2}(m + 1)x + \frac{1}{4}(m + 1)^2 + 1, & \text{for } \frac{1}{2}(m - 1) \leq x \leq \frac{1}{2}(m + 1), \\
\frac{1}{2}(m - 1)x - \frac{1}{4}(m^2 - 1) + 1, & \text{for } \frac{1}{2}(m + 1) \leq x \leq \frac{1}{2}(m + 1) + 1, \\
-x + m + 2, & \text{for } \frac{1}{2}(m + 1) + 1 \leq x \leq m.
\end{cases}
\]

and

\[
h_m(x) = \begin{cases} 
x + \frac{1}{2}(m - 1), & \text{for } 1 \leq x \leq \frac{1}{2}(m + 1), \\
-(m - 1)x + \frac{1}{2}(m^2 + 2m - 1), & \text{for } \frac{1}{2}(m + 1) \leq x \leq \frac{1}{2}(m + 1) + 1, \\
x - \frac{1}{2}(m + 1), & \text{for } \frac{1}{2}(m + 1) + 1 \leq x \leq m.
\end{cases}
\]

Then, for any odd integer $m \geq 3$, both $g_m(x)$ and $h_m(x)$ have the same Artin-Mazur zeta function as $f_m(x)$, where $f_m(x)$ is defined as in Theorem 5 above.

**Proof:** Let $m \geq 3$ be an odd integer. It suffices to show that the Markov matrices of $f_m, g_m,$ and $h_m$ are similar to one another. Indeed, let $P$ be the set of all integers in $[1, m]$ and let $\sigma$ be a cyclic permutation on $P$. Let $\varphi$ be the linearisation of $\sigma$ on $P$ and let $V_{m-1}$ be the vector space over the field of real numbers with the set $Q_1 = \{J_1, J_2, \ldots, J_{m-1}\}$ as a basis, where, for every integer $1 \leq k \leq m - 1$, $J_k = [k, k + 1]$. Then, $\varphi$ determines a linear transformation (which we call $\overline{\varphi}$) on $V_{m-1}$ defined by $\overline{\varphi}\left(\sum_{k=1}^{m-1} r_k J_k\right) = \sum_{k=1}^{m-1} r_k \varphi(J_k)$, where $r_k$'s are real numbers and $\overline{\varphi}(J_k) = \sum_{s=i_k}^{j_k} J_s$ if $\varphi(J_k) = \bigcup_{s=i_k}^{j_k} J_s$ for some integers $1 \leq i_k \leq j_k \leq m - 1$. Furthermore, with respect to the basis $Q_1$, the linear transformation $\overline{\varphi}$ is also determined [4] by the $(m - 1) \times (m - 1)$ matrix $B_\varphi = (\beta_{ij})$ in such a way that, for every integer $1 \leq k \leq m - 1$, $\overline{\varphi}(J_k) = \sum_{j=1}^{m-1} \beta_{kj} J_j = \left(\sum_{s=i_k}^{j_k} J_s\right)$ which happens to be the same as the Markov matrix of the map $\varphi$ on $[1, m]$. Now, if we take $Q_2 = \{J_1, J_{m-1}, J_2, J_{m-2}, \ldots, J_t, J_{m-t}, \ldots, J_{(m-3)/2}, J_{(m+3)/2}, J_{(m-1)/2}, J_{(m+1)/2}, \sum_{k=1}^{(m+1)/2} J_k\}$ as a new basis for $V_{m-1}$, then it is easy to see that $\overline{g}_m$ acts on $Q_2$ like $\overline{f}_m$ on $Q_1$. Similarly, if we take $Q_3 = \{J_{(m-1)/2}, J_{m-1}, J_{(m-3)/2}, J_{m-2}, J_{(m-5)/2}, J_{m-3}, \ldots, J_3, J_{(m-1)/2+3}, J_2, J_{(m-1)/2+2}, J_1, J_{(m-1)/2+1}\}$ as a new basis for $V_{m-1}$, then $\overline{h}_m$ acts
on $Q_3$ like $f_m$ on $Q_1$. Therefore, the matrices of the linear transformations $\bar{f}_m, \bar{g}_m,$ and $\bar{h}_m$ on the respective bases $Q_1, Q_2,$ and $Q_3$ are the same. So, the matrices of $\bar{f}_m, \bar{g}_m,$ and $\bar{h}_m$ on the basis $Q_1$ are similar to one another [4]. Consequently, the Markov matrices of the maps $f_m, g_m,$ and $h_m$ on the interval $[1, m]$ are similar to one another and hence, by Theorem 2, $f_m, g_m,$ and $h_m$ have the same Artin-Mazur zeta function.

REFERENCES


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