On the "Flat" Regions of Integral Functions of Finite Order.

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1. The term "flat" is used to indicate that the minimum modulus of a function in a region is (in some sense) of the same order as the maximum modulus. Some properties concerned with this notion are described below. They came to light during an attempt to answer a question put to me by Professor Littlewood.

Can an integral function of order less than two be bounded at the lattice points,[1] *unless it is a constant?*

The problem is apparently one of interpolation and was at first attacked by expressing the function in terms of its values at the lattice points. The method proved inadequate but yielded a result of independent interest; that a function must be identically zero if, within a sector of angle $\frac{\pi}{2}$, it is regular, vanishes at the lattice points, and satisfies a condition

$$|f(z)| < Ae^{kr^2}, \ \left(k < \frac{\pi}{2}\right).$$

This theorem is of the same type as one of Carlson[2] concerning functions with zeros at the points 1, 2, 3, ...

The next result is related to Wiman's theorem[3] that an integral function of order $\rho < \frac{1}{2}$ cannot have a finite asymptotic value. It is shewn that there are annuli $r \leqslant |z| \leqslant r + r^{\sigma}$, $(\sigma < 1 - \rho)$, for large values of r, in which the function is "flat." Thus the function must be a constant if it is bounded at the points 1^2, 2^2, 3^2, ...

The final theorem is of the same general nature. It states that in the case of a function of genus 0 or 1 there are circles of fixed radius, arbitrarily distant from the origin, in which the function is "flat." This enables us to answer Professor Littlewood's question.

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[1] *i.e.* the points $\pm m \pm ni$ for integral values of m, n.

[2] Given in his dissertation "Sur une classe de séries de Taylor." See a paper of Hardy (Hardy, **7**) in which two proofs of Carlson's theorem are given. The proof of Theorem 2 was suggested by the second of these. See also Riesz, **12**, Hardy, **5**.

[3] Wiman, **17**. Lindelöf, **8**.

An integral function of genus 0 *or* 1 *is a constant if it is bounded at the lattice points.*

The principal results are enunciated at the end. It should be added that § 2 is independent of the subsequent work.

2. *Interpolation at the lattice points.*

The first formula corresponds to the cardinal series[1]

$$\sin \pi z \sum_{n=-\infty}^{\infty} \frac{(-)^n a_n}{z-n}$$

for interpolation over the set of points $z = n$. Define a function with zeros at the lattice points

(2 . 1) $$\phi(z) = e^{\pi z^2/2}\, \vartheta_1(\pi z \mid i).$$

By the properties of the ϑ-function,[2]

(2 . 2) $$\phi(z+m+ni) = (-)^{m+n+mn} \exp\{\tfrac{1}{2}\pi(m^2+n^2) + \pi(m-ni)z\}\, \phi(z)$$

and

$$c_{mn} = \lim_{z \to m+ni} \frac{\phi(z)}{z-m-ni}$$

(2 . 3) $$= (-)^{m+n+mn} \exp\{\pi(m^2+n^2)/2\}\, \phi'(0).$$

THEOREM 1. *Let* $f(z)$ *be an integral function for which*

(2 . 4) $$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^2} < \frac{\pi}{2}.$$

Then[3]

(2 . 5) $$f(z) = \phi(z) \sum_{m,\, n=-\infty}^{\infty} \frac{f(m+ni)}{c_{mn}(z-m-ni)}.$$

$M(r)$ denotes as usual the maximum of $|f(z)|$ for $|z| = r$.

Let K_p denote the square whose corners are $(\pm 1 \pm i)(p+\tfrac{1}{2})$. Then if ζ is a point inside K_p, other than one of the points $m+ni$,

(2 . 6) $$\frac{f(\zeta)}{\phi(\zeta)} + \sum_{K_p} \frac{f(m+ni)}{c_{mn}(m+ni-\zeta)} = \frac{1}{2\pi i}\int_{K_p} \frac{f(z)}{\phi(z)(z-\zeta)}\, dz.$$

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[1] For references to the literature connected with this formula see Ferrar, **3**, J. M. Whittaker, **16**.

[2] Whittaker and Watson, **15**, 464.

[3] The convergence of series of the form $\sum \frac{a_{mn}}{z + mw_1 + nw_2}$ has been discussed by Ferrar, **4**.

It follows from (2 . 2) and (2 . 4) that

$$M_p = \text{maximum of } \left| \frac{f(z)}{\phi(z)} \right| \text{ on } K_p,$$

$$\to 0, \qquad \text{as } p \to \infty$$

and thus the integral on the right of (2 . 6) $\to 0$, as $p \to \infty$.

Moreover, by (2 . 3) and (2 . 4),

$$\sum_{m,\, n=-\infty}^{\infty} \left| \frac{f(m+ni)}{c_{mn}} \right|$$

converges. These results contain a proof of the theorem.

It follows that an integral function must be identically zero if it vanishes at the lattice points and satisfies (2 . 4). In this it is enough to suppose that the conditions are satisfied in a quadrant of the plane. Thus,

THEOREM 2. *If*

(i) *$f(z)$ is regular at all points inside the angle* $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$;

(ii) $|f(z)| < Ae^{kr^2}$, $\left(k < \frac{\pi}{2}\right)$, *throughout the angle;*

(iii) *$f(m+ni) = 0$ at all points $m + ni$ inside the angle;*
then $f(z)$ is identically zero.

The theorem is not true for an angle $-\alpha \leqslant \theta \leqslant \alpha$, $\left(\alpha < \frac{\pi}{4}\right)$. For if

$$l > \frac{\pi \sin^2 \alpha}{\cos 2\alpha}$$

the function $e^{-lz^2} \vartheta_1(\pi z \mid i)$ tends to zero throughout the angle; moreover the example of $\phi(z)$ shows that the second condition cannot be replaced by

$$|f(z)| < Ae^{\pi r^2/2}.$$

The first step is to find an interpolation formula giving a function in terms of its values at the lattice points inside the angle.

Let $f(z)$ satisfy conditions (i), (ii), of Theorem 2. Then it will be shown that if $0 < z < 1$,

$$f(z) = -\phi(z) \int_0^\infty u^{-z-1} F(u)\, du, \tag{2.7}$$

where

(2 . 8) $$F(u) = \Sigma \frac{f(m+ni)}{c_{mn}} u^{m+ni},$$

the summation being taken over all values of m, n for which $m + ni$ lies inside the angle, *i.e.* for

$$n = -m+1, -m+2, \ldots, m-1; \; m = 1, 2, 3, \ldots.$$

Consider the triangle T_p whose vertices are the points A, B, C whose affixes are respectively κ, $\kappa + p - pi$, $\kappa + p + pi$ where p is a positive integer and $0 < \kappa < 1$.

It follows from (2 . 2) that on AB, CA

(2 . 9) $$|\phi(z)| \geqslant K e^{\pi r^2/2}, \qquad (K \text{ independent of } p),$$

while at a point $z = \kappa + p + yi$ on BC,

(2 . 10) $$|\phi(z)| \geqslant K e^{\pi(p^2+y^2)/2}.$$

Now, by Cauchy's theorem,

$$F_p(u) = \sum_{T_p} \frac{f(m+ni)}{c_{mn}} u^{m+ni}$$
$$= \frac{1}{2\pi i}\left\{ \int_{\kappa}^{\kappa+p-pi} + \int_{\kappa+p-pi}^{\kappa+p+pi} + \int_{\kappa+p+pi}^{\kappa} \right\} \frac{f(z)\, u^z}{\phi(z)}\, dz$$

and by (2 . 10) and the condition (ii)

$$\left| \int_{\kappa+p-pi}^{\kappa+p+pi} \frac{f(z)\, u^z}{\phi(z)}\, dz \right| \leqslant \int_{-p}^{p} \frac{A}{K} \exp\left\{ k(p+\kappa)^2 + ky^2 - \frac{\pi}{2}(p^2+y^2) \right\} |u|^r\, dy$$
$$\to 0, \qquad \text{as } p \to \infty.$$

In the same way it can be shown that the other two integrals tend to limits as $p \to \infty$, so that

$$F(u) = \lim_{p \to \infty} F_p(u)$$
$$= \frac{1}{2\pi i}\left\{ \int_{\kappa}^{\kappa+(1-i)\infty} + \int_{\kappa+(1+i)\infty}^{\kappa} \right\} \frac{f(z)\, u^z}{\phi(z)}\, dz.$$

If $0 < s < 1$, λ, μ can be found so that $0 < \lambda < s < \mu < 1$, and then

$$\int_0^1 u^{-s-1} F(u)\, du = \frac{1}{2\pi i} \int_0^1 u^{-s-1}\, du \left\{ \int_{\mu}^{\mu+(1-i)\infty} + \int_{\mu+(1+i)\infty}^{\mu} \right\} \frac{f(z)\, u^z}{\phi(z)}\, dz$$
$$= \frac{1}{2\pi i} \left\{ \int_{\mu}^{\mu+(1-i)\infty} + \int_{\mu+(1+i)\infty}^{\mu} \right\} \frac{f(z)}{(z-s)\,\phi(z)}\, dz,$$

provided that the integrals are absolutely convergent, and this is readily proved to be the case.

In the same way

$$\int_1^\infty u^{-s-1} F(u)\, du = -\frac{1}{2\pi i}\left\{\int_\lambda^{\lambda+(1-i)\infty} + \int_{\lambda+(1+i)\infty}^{\lambda}\right\} \frac{f(z)}{(z-s)\,\phi(z)}\, dz,$$

so that

$$\int_0^\infty u^{-s-1} F(u)\, du$$

$$= -\frac{1}{2\pi i}\left\{\int_\mu^{\mu+(1+i)\infty} + \int_{\lambda+(1+i)\infty}^{\lambda} + \int_\lambda^{\lambda+(1-i)\infty} + \int_{\mu+(1-i)\infty}^{\mu}\right\} \frac{f(z)}{(z-s)\,\phi(z)}\, dz$$

$$= -f(s)\,/\,\phi(s).$$

If the condition (iii) of Theorem 2 is satisfied, $F(u)$ is identically zero and the same is therefore true of $f(z)$.

3. *A theorem on integral functions of order less than* 1.

THEOREM 3. *Let $f(z)$ be an integral function of order $\rho < 1$. Let σ be a fixed number $< 1 - \rho$, and let $m_\sigma(r)$, $M_\sigma(r)$ be the bounds of $|f(z)|$ in the annulus $r \leqslant |z| \leqslant r + r^\sigma$. Then*

$$(3.1) \qquad \varlimsup_{r\to\infty} \frac{\log m_\sigma(r)}{\log M_\sigma(r)} \geqslant \cos \pi\rho.$$

This is an extension of a well known theorem, first stated by Littlewood,[1] to the effect that

$$(3.2) \qquad \varlimsup_{r\to\infty} \frac{\log m(r)}{\log M(r)} \geqslant \cos \pi\rho$$

where $m(r)$, $M(r)$ are the bounds of $|f(z)|$ on the circle $|z| = r$.

In its original form the proof given below applied only to functions of order zero. For other values of ρ the proof was of a different character and except in the case of functions of regular growth it was found necessary to suppose that $\sigma < 1 - 2\rho$. Dr Besicovitch, who read the work in manuscript, kindly pointed out to me that the complete result could be established with the aid of a theorem of his memoir,[2] which I had overlooked.

A function may be expected to be " flat " in the regions which lie farthest from its zeros, aud the first step is to pick out these regions.

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[1] Littlewood, **10**. Littlewood proved (3.2) with $\cos 2\pi\rho$ in place of $\cos \pi\rho$. (3.2) was afterwards proved by Wiman, **18**, and Valiron, **14**, at about the same time. Proofs have also been given by Pólya, **11**, and Besicovitch, **1**.

[2] Besicovitch, 1.

This can be effected by the shading process described in the following lemma.[1]

LEMMA 1. *Let* $r_1, r_2, r_3, \ldots$ *be an increasing sequence of positive numbers. Divide the real positive axis into segments of given length* λ *and mark the points* $r_1, r_2, r_3, \ldots$ *on it. Now shade every segment containing an* r_s *and its two neighbours. Of the remaining segments shade every one whose two neighbours on the right contain two or more r's. Then every segment whose three neighbours on the right contain three or more r's, and so on. Perform the same process for neighbouring segments on the left. Let* $n(r)$ *denote the number of points* $r_1, r_2, \ldots$ *in* $(0, r)$. *Then if*

$$\frac{n(r)}{r} \to 0, \text{ as } r \to \infty \tag{3.3}$$

almost every segment is unshaded.

By the last statement we mean that

$$\frac{N_s}{N} \to 0, \text{ as } N \to \infty \tag{3.4}$$

where N_s = number of shaded segments among the first N.

Suppose that this is false. Then there is a number h, $(0 < h < 1)$, such that

$$\frac{N_s}{N} > h \tag{3.5}$$

for arbitrarily large values of N.

Find r_0 so that

$$\frac{n(r)}{r} < \frac{h}{6\lambda}, \quad (r \geqslant r_0) \tag{3.6}$$

and let N_1 be a number $> 2\lambda r_0$ for which (3 . 5) is satisfied.

Let $r_1, r_2, \ldots, r_m$ be the r's in the first N_1 segments. Suppose now that the shading process is carried out in two stages. First mark in $r_1, r_2, \ldots, r_m$ only and perform the process for these points, and then mark in the other r's and complete the process. It is easy to see, by considering simple cases, that at most $3m$ segments will be shaded in the first stage. Let k additional segments among the first N_1 be shaded in the second stage. In the most unfavourable case

[1] Suggested by Boutroux's proof of his theorem on the minimum modulus of a polynomial. Boutroux, 2.

(*i.e.* the case implying the least number of r's) these will be the last k. Suppose that this is so. Then, for some p, the p segments immediately succeeding the N_1^{th} must contain at least $k+p$ of the points r_s. By (3.5)

$$3m + k > hN_1.$$

But

$$\begin{aligned} n\{(N_1+p+1)\lambda\} &\geqslant m+k+p \\ &> \frac{h}{3}N_1 + p \\ &> \frac{h}{3}(N_1+p), \end{aligned}$$

so that

$$n\{(N_1+p)\,2\lambda\} > \frac{h}{3}(N_1+p)$$

or

$$n(r) > \frac{h}{6\lambda}\, r,$$

for $r = (N_1+p)\,2\lambda$, which contradicts (3.6), since $r > r_0$.

LEMMA 2. *Let*

$$\frac{n(r)}{r^\tau} \to 0, \quad (\tau > 0). \tag{3.7}$$

Then if the axis is divided into segments by the points $(\lambda)^{1/\tau}$, $(2\lambda)^{1/\tau}$, $(3\lambda)^{1/\tau}$, ... *and the shading is carried out as before, almost every segment will be unshaded.*

For, let

$$\begin{aligned} n_\tau(r) &= \text{number of points } r_1^\tau,\ r_2^\tau,\ \ldots \text{ in } (0, r) \\ &= n(r^a), \quad a = \tau^{-1}, \end{aligned}$$

so that

$$\frac{n_\tau(r)}{r} = \frac{n(r^a)}{r} \to 0, \quad \text{as } r \to \infty. \tag{3.8}$$

Again, if r_p, r_{p+1}, ..., r_q are the r's in the segment $\{(m\lambda)^a, ((m+1)\lambda)^a\}$ then r_p^τ, ..., r_q^τ lie in $\{m\lambda, (m+1)\lambda\}$. Thus the shading with points r_1, r_2, ... and segments $\{(m\lambda)^a, ((m+1)\lambda)^a\}$ corresponds to the shading with points r_1^τ, r_2^τ, ... and segments $\{m\lambda, (m+1)\lambda\}$; and by (3.8) and Lemma 1, almost every segment of the latter set is unshaded.

Now consider an integral function[1] of order $\rho < 1$,

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[1] There is no loss of generality in taking the function to be of this form rather than $Az^q\,\Pi(1 + za_s^{-1})$.

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(3 . 9) $$f(z) = \prod_{s=1}^{\infty} \left(1 + \frac{z}{a_s}\right)$$

and write $|a_s| = r_s$.

Given $\sigma < 1 - \rho$, take τ so that $\rho < \tau < 1 - \sigma$. Then (3 . 7) is satisfied. Perform the shading process as in Lemma 2. Then,

LEMMA 3. *Let $I_p \{p^\alpha, (p+1)^\alpha\}$ be an unshaded segment. Then there is a positive constant A, depending only on σ, τ, such that*

(3 . 10) $$A^{-1} \leqslant \left| \frac{f(z_1)}{f(z_2)} \right| \leqslant A$$

provided that

$$p^\alpha \leqslant |z_1| \leqslant (p+1)^\alpha, \quad |z_1 - z_2| \leqslant (p+1)^{\sigma\alpha}.$$

Let I_p lie between r_n, r_{n+1}, and write $x_1 = |z_1|$, $x_2 = |z_2|$. Then under the conditions just stated,

$$\left| \left\{ \log \left| \frac{f(z_1)}{f(z_2)} \right| \right\} \right|$$

$$= \left| \left\{ \sum_{s=1}^{\infty} \log \left| 1 + \frac{z_1 - z_2}{z_2 - a_s} \right| \right\} \right|$$

$$\leqslant |z_1 - z_2| \sum_{s=1}^{n-1} \frac{1}{v_s} + \log \frac{(p+1)^\alpha - (p-1)^\alpha}{p^\alpha - (p-1)^\alpha} + |z_1 - z_2| \sum_{s=n+1}^{\infty} \frac{1}{v_s}$$

where v_s is the smaller of $|x_1 - r_s|$, $|x_2 - r_s|$. Since I_p is unshaded,

$$\left.\begin{aligned} x_1 - r_{n-t} &\geqslant p^\alpha - (p-t)^\alpha \\ r_{n+t} - x_1 &\geqslant (p+t+1)^\alpha - (p+1)^\alpha \end{aligned}\right\} \quad (t = 1, 2, 3, \ldots.)$$

and the same inequalities hold with x_2 in place of x_1. Thus

$$\left| \left\{ \log \left| \frac{f(z_1)}{f(z_2)} \right| \right\} \right| < (p+1)^{\sigma\alpha} \left\{ \sum_{t=1}^{p} \frac{1}{p^\alpha - (p-t)^\alpha} + \sum_{t=1}^{\infty} \frac{1}{(p+t+1)^\alpha - (p+1)^\alpha} \right\} + 1$$

$$< (p+1)^{\sigma\alpha} \left\{ \int_1^p \frac{du}{p^\alpha - (p-u)^\alpha} + \int_1^\infty \frac{du}{(p+u)^\alpha - p^\alpha} \right\} + 2$$

$$< 3p^{-\alpha(1-\tau-\sigma)} \log p + 2$$

$$< \frac{2(1-\sigma)}{1-\tau-\sigma}$$

and this is the result stated.

A theorem of Besicovitch[1] states that if $\rho' > \rho$, the inequality

$$(3.11) \qquad \log m(r) > \cos \pi\rho' \log M(r)$$

is satisfied for a set of values of r whose upper density is at least $1 - \rho/\rho'$, *i.e.* that given ϵ, arbitrarily large values of R can be found such that the measure of set of r in $(0, R)$ for which (3.11) is satisfied is greater than $(1 - \rho/\rho' - \epsilon) R$.

Since almost every segment is unshaded, it is easy to show that there are unshaded segments, arbitrarily distant from the origin, which contain values of r for which (3.11) is satisfied. Let I_p be such a segment and r' a value of r in I_p for which (3.11) is satisfied. Let z_1, z_2 be points at which $|f(z)|$ attains its bounds in the annulus

$$r' \leqslant |z| \leqslant r' + (r')^\sigma$$

so that $\qquad |f(z_1)| = m_\sigma(r'), \quad |f(z_2)| = M_\sigma(r').$

Since $p^\alpha \leqslant r' \leqslant (p+1)^\alpha$, so that $(r')^\sigma \leqslant (p+1)^{\sigma\alpha}$, there exist points z_3, z_4 on the circle $|z| = r'$ for which

$$|z_1 - z_3| \leqslant (p+1)^{\sigma\alpha}, \quad |z_2 - z_4| \leqslant (p+1)^{\sigma\alpha}.$$

Finally, by Lemma 3 and (3.11),

$$\begin{aligned} \log |f(z_1)| &\geqslant \log |f(z_3)| - |\log A| \\ &\geqslant \cos \pi\rho' \log |f(z_4)| - |\log A| \\ &\geqslant \cos \pi\rho' \log |f(z_2)| - 2|\log A| \end{aligned}$$

and this is equivalent to (3.1).

4. *The average value of a function in a circle.*

Let $f(z)$ be analytic in the circle $|z| < R$. If $r < R$, it is natural to define the average value of $|f|^\delta$ in the circle of radius r as

$$(4.1) \quad \sigma(|f|^\delta; r) = \sigma_\delta(r) = \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} |f(ue^{i\theta})|^\delta \, u \, du \, d\theta, \ (\delta > 0).$$

The properties of $\sigma_\delta(r)$ are as follows.

THEOREM 4. *$\sigma_\delta(r)$ is a continuous increasing differentiable function of r, and, unless $f(z)$ is a polynomial it increases more rapidly than any power of r. Moreover $\log \sigma_\delta(r)$ is a convex function of $\log r$.*

It is known[2] that

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta$$

[1] Besicovitch, 1. Theorem 1.

[2] Hardy, 6.

possesses all these properties, except that there may be isolated points at which it is not differentiable. Now

$$\sigma_\delta(r) = \frac{2}{r^2}\int_0^r \mu_\delta(u)\,u\,du$$
$$= \int_0^1 \mu_\delta(r\sqrt{t})\,dt.$$

The first four properties of $\sigma_\delta(r)$ are elementary deductions from this result. The convexity property is expressed by

$$\left(\log\frac{r_2}{r_1}\right)^{-1}\log\frac{\sigma_\delta(r_2)}{\sigma_\delta(r_1)} \leqslant \left(\log\frac{r_3}{r_1}\right)^{-1}\log\frac{\sigma_\delta(r_3)}{\sigma_\delta(r_1)}, \quad (r_1 < r_2 < r_3).$$

LEMMA 4. *Let* $a_s,\ b_s,\ c_s \geqslant 0,\quad 0 < \beta < \alpha$, *and*

$$\left(\frac{b_s}{a_s}\right)^\alpha \leqslant \left(\frac{c_s}{a_s}\right)^\beta, \quad (s = 1, 2, \ldots, n).$$

Then

$$\left(\frac{\sum_{s=1}^n b_s}{\sum_{s=1}^n a_s}\right)^\alpha \leqslant \left(\frac{\sum_{s=1}^n c_s}{\sum_{s=1}^n a_s}\right)^\beta.$$

For, by Hölder's inequality,

$$(\Sigma b_s)^\alpha \leqslant (\Sigma c_s^{\beta/\alpha}\, a_s^{1-\beta/\alpha})^\alpha \leqslant \{(\Sigma c_s)^{\beta/\alpha}(\Sigma a_s)^{1-\beta/\alpha}\}^\alpha$$
$$= (\Sigma c_s)^\beta (\Sigma a_s)^{\alpha-\beta}.$$

The corresponding result for integrals is as follows.

Let $f(x),\ g(x),\ h(x) \geqslant 0,\ (0 \leqslant x \leqslant a),\ 0 < \beta < \alpha$, *and*

$$\left\{\frac{g(x)}{f(x)}\right\}^\alpha \leqslant \left\{\frac{h(x)}{f(x)}\right\}^\beta, \quad (0 \leqslant x \leqslant a).$$

Then

$$\left\{\int_0^a g(x)\,dx\right\}^\alpha \left\{\int_0^a f(x)\,dx\right\}^{-\alpha} \leqslant \left\{\int_0^a h(x)\,dx\right\}^\beta \left\{\int_0^a f(x)\,dx\right\}^{-\beta}$$

The convexity property follows from this, on taking

$$f(x) = \mu_\delta(r_1\sqrt{x}),\ g(x) = \mu_\delta(r_2\sqrt{x}),\ h(x) = \mu_\delta(r_3\sqrt{x}),$$
$$\alpha = \left(\log\frac{r_2}{r_1}\right)^{-1},\ \beta = \left(\log\frac{r_3}{r_1}\right)^{-1},\ a = 1.$$

For the purpose which we have in view a more informative measure of the "surface density" is the average value of $\log|f(z)|$. Define

$$(4.2) \qquad \sigma(r) = \sigma(\log|f|;\ r) = \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} \log|f(ue^{i\theta})|\,u\,du\,d\theta.$$

Now if a is real and positive,

$$\left.\begin{aligned}\int_0^r\int_0^{2\pi}\log\left|1+\frac{ue^{i\theta}}{a}\right|u\,du\,d\theta &= \pi r^2\log\frac{r}{a}-\frac{\pi}{2}(r^2-a^2), \quad (0<a\leqslant r)\\ &= 0, \qquad\qquad\qquad (a\geqslant r)\end{aligned}\right\}$$

since

$$\left.\begin{aligned}\int_0^{2\pi}\log|a+ue^{i\theta}|\,d\theta &= 2\pi\log u, \quad (u\geqslant a)\\ &= 2\pi\log a, \quad (u\leqslant a)\end{aligned}\right\}.$$

Moreover if $h(z)$ is analytic,

$$\int_0^{2\pi}\log|e^{h(z)}|\,d\theta = 0.$$

Thus for the most general integral function[1]

$$f(z) = Ae^{g(z)}z^q\prod_{s=1}^{\infty}E\left(\frac{z}{a_s}, p_s-1\right)$$

the average value of $\log|f(z)|$ is

$$\begin{aligned}\sigma(r) &= \log|A|+\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}q\log u\,.\,u\,du\,d\theta\\ &\quad+\sum_{s=1}^{\infty}\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\log\left|1+\frac{ue^{i\theta}}{a_s}\right|u\,du\,d\theta\end{aligned}$$

$$(4\,.\,3) = \log|A|+q(\log r-\tfrac{1}{2})+\sum_{s=1}^{n}\left\{\log\frac{r}{r_s}-\tfrac{1}{2}\frac{r^2-r_s^2}{r^2}\right\}$$

where r_n is the largest r_s not greater than r. Now

$$\sum_{s=1}^{n}\log\frac{r}{r_s} = \int_{r_1}^{r}\frac{n(u)-q}{u}\,du, \quad n(r) = \text{number of } r_s\leqslant r,$$

$$\tfrac{1}{2}\sum_{s=1}^{n}(r^2-r_s^2) = \int_{r_1}^{r}\{n(u)-q\}\,u\,du$$

so that

$$(4\,.\,4)\qquad \sigma(r) = \log|A|+q\log\frac{r}{\sqrt{e}}+\int_{r_1}^{r}\frac{n(u)-q}{u}\left(1-\frac{u^2}{r^2}\right)du.$$

It follows that

$$(4\,.\,5)\qquad \varlimsup_{r\to\infty}\frac{\log\sigma(r)}{\log r} = \rho_1$$

$$= \text{exponent of convergence of zeros of } f(z).$$

There is a more precise result of the same nature.

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[1] Valiron, **13**, 13.

THEOREM 5. *If $f(z)$ is an integral function of finite non-integral order,*

$$\overline{\lim_{r \to \infty}} \frac{\sigma(r)}{\log M(r)} > 0.$$

If $f(z)$ is of finite integral order this property holds for the function $f(z) + d$, for every value of d except, possibly, one value.

If $f(z)$ is of order zero,

$$\overline{\lim_{r \to \infty}} \frac{\sigma(r)}{\log M(r)} = 1.$$

Take the first case, that of a function of order ρ, genus p, where ρ is not an integer. As usual we can suppose that $f(0) = 1$, so that (4 . 4) becomes

$$\sigma(r) = \int_{r_1}^{r} \frac{n(u)}{u} \left(1 - \frac{u^2}{r^2}\right) du. \tag{4.6}$$

Let $\rho(r)$ be a proximate order B of $f(z)$, that is to say, a function with the properties[1]

$$\overline{\lim_{r \to \infty}} \rho(r) = \rho, \qquad \underline{\lim} \rho(r) > p,$$

$$\lim_{r \to \infty} \{r\rho'(r) \log r\} = 0$$

$$\overline{\lim_{r \to \infty}} \frac{\log M(r)}{r^{\rho(r)}} = 1.$$

Then[2] there exist constants a, K and an indefinitely increasing sequence of values of r, say $R_1, R_2, R_3, \ldots.$, for which

$$n(aR_s) > K \log M(R_s) = KR_s^{\rho(R_s)}. \tag{4.7}$$

Thus if r has one of these values

$$\begin{aligned} \sigma(2ar) &> \int_{ar}^{2ar} \frac{n(u)}{u}\left(1 - \frac{u^2}{4a^2 r^2}\right) du \\ &> n(ar) \int_{ar}^{2ar} \frac{1}{u}\left(1 - \frac{u^2}{4a^2 r^2}\right) du \\ &= n(ar)(\log 2 - \tfrac{3}{8}) \\ &> K_1 r^{\rho(r)}. \end{aligned} \tag{4.8}$$

[1] Valiron, **13**, 64.

[2] Valiron, **13**, 69.

Moreover

(4.9) $\log M(2ar) < (2ar)^{\rho(2ar)} < K_2 r^{\rho(r)}$

by the properties of $\rho(r)$. K_1, K_2 are the same for all members of the sequence $R_1, R_2, \ldots$, and

(4.10) $$\frac{\sigma(2ar)}{\log M(2ar)} > \frac{K_1}{K_2}, \quad (r = R_1, R_2, \ldots).$$

The second case is proved in much the same way, on making use of a result concerning integral functions of integral order[1]; and the last part of the theorem can be established by means of Littlewood's method for treating functions of order zero.[2]

5. *Properties of functions of genus* 0 *and* 1.

The shading principle can be extended to two dimensions as follows.

LEMMA 5. *Divide the z-plane into squares of side λ by drawing lines parallel to the axes, and mark the points $a_1, a_2 \ldots$ on it. Shade every square containing an a and its eight neighbours. Of the remaining squares shade everyone whose twenty-four neighbours contain eight or more a's and generally every square whose $4q(q+1)$ neighbours contain $4q(q-1)$ or more a's. Then if*

$$\frac{n(r)}{r^2} \to 0$$

almost every square is unshaded.

$n(r)$ denotes the number of a's for which $r_s = |a_s| \leqslant r$. The proof is similar to that of Lemma 1 and the details may be omitted. From this there follows

LEMMA 6. *Let $f(z)$ be of genus 1 and let λ, ϵ be given positive numbers. Then for almost every square of side λ, drawn as above,*

(5.1) $$e^{-\epsilon r\sqrt{(\log r)}} \leqslant \left|\frac{f(z_1)}{f(z_2)}\right| \leqslant e^{\epsilon r\sqrt{(\log r)}}$$

where z_1, z_2 are any points of the square, and r is the distance of the centre of the square from the origin.

We may take

(5.2) $$f(z) = e^{kz} \prod_{s=1}^{\infty} \left\{\left(1+\frac{z}{a_s}\right) e^{-z/a_s}\right\}$$

[1] Valiron, **13**, 86.

[2] Littlewood, **9**.

since, as before, there is no loss of generality in taking $f(0) = 1$. Divide the plane into squares of side $\lambda_1 = N\lambda$, where N is an integer, and shade them. Then almost every square is unshaded, and in an unshaded square we have, if τ_s is the smaller of $|a_s + z_1|$ and $|a_s + z_2|$ and if $|z_1 - z_2| < \sqrt{2}\,\lambda$,

$$\left| \left\{ \log \left| \frac{f(z_1)}{f(z_2)} \right| \right\} \right|$$

$$\leqslant |k|\,|z_1 - z_2| + \sum_{s=1}^{\infty} \left| \left\{ \log \left| 1 + \frac{(z_2 - z_1)\, z_2}{a_s\,(a_s + z_2)} + O\left(\frac{|z_2 - z_1|^2}{|a_s|^2}\right) \right| \right\} \right|$$

$$< K|z_1 - z_2| + |z_1 - z_2|\,|z_2| \sum_{s=1}^{\infty} \frac{1}{r_s \tau_s}$$

$$(5\,.\,3) \quad < \sqrt{2}\,K\lambda + 2\lambda r \sum_{s=1}^{\infty} \frac{1}{r_s \tau_s}$$

$$< \sqrt{2}\,K\lambda + 2\lambda r \left(\sum_{s=1}^{\infty} \frac{1}{r_s{}^2} \right)^{\frac{1}{2}} \left(\sum_{s=1}^{\infty} \frac{1}{\tau_s{}^2} \right)^{\frac{1}{2}}$$

$$< K_1 \lambda r \left(\sum_{s=1}^{\infty} \frac{1}{\tau_s{}^2} \right)^{\frac{1}{2}}$$

where K is independent of λ, λ_1, z_1, z_2.

Consider a large square P of side $(2p+1)\,\lambda_1$, $(r < p < 2r)$, symmetrically disposed about the square of side λ_1 containing z_1, z_2. We have

$$\sum_{s=1}^{\infty} \frac{1}{\tau_s{}^2} = \sum_{P} \frac{1}{\tau_s{}^2} + \sum_{CP} \frac{1}{\tau_s{}^2}$$

$$< \frac{1}{\lambda_1{}^2} \left\{ 8 + \frac{25 - 9}{4} + \ldots + \frac{(2p+1)^2 - (2p-1)^2}{p^2} \right\} + \sum_{CP} \frac{1}{\tau_s{}^2}$$

since there are not more than seven zeros in the second ring of λ_1-squares surrounding that containing z_1, z_2, nor more than twenty-three in the third ring, and so on. Again, in CP, the region outside P,

$$|a_s + z_1|,\ |a_s + z_2| \geqslant p > |z_1|,\ |z_2|$$

so that

$$|a_s + z_1|,\ |a_s + z_2| > \tfrac{1}{2}\,|a_s| = \tfrac{1}{2}\, r_s\,.$$

Thus

$$\sum_{s=1}^{\infty} \frac{1}{\tau_s{}^2} < \frac{1}{\lambda_1{}^2} \sum_{q=1}^{p} \frac{(2q+1)^2 - (2q-1)^2}{q^2} + 4 \sum_{s=1}^{\infty} \frac{1}{r_s{}^2}$$

$$< \frac{8}{\lambda_1{}^2} \sum_{q=1}^{p} \frac{1}{q} + K$$

$$< \frac{8}{\lambda_1{}^2} \log p + K < \frac{K_0{}^2}{\lambda_1{}^2} \log r$$

and

$$\left| \left\{ \log \left| \frac{f(z_1)}{f(z_2)} \right| \right\} \right| < K_1 K_0 \frac{\lambda}{\lambda_1} r\sqrt{(\log r)}$$
$$= \frac{K_1 K_0}{N} r\sqrt{(\log r)}$$
$$< \epsilon r \sqrt{(\log r)}$$

by choice of N. This is true for points z_1, z_2 in every square of side λ contained in an unshaded square of side λ_1, and so for almost every square of side λ.

A sharper inequality holds in the case of functions of order 1.

LEMMA 7. *If $f(z)$ is of order 1 there is a constant l independent of λ, z_1, z_2 such that*

$$(5.4) \qquad e^{-l\lambda r^{3/4}} \leqslant \left| \frac{f(z_1)}{f(z_2)} \right| \leqslant e^{l\lambda r^{3/4}}$$

for almost every square of side λ.

In this case we surround the square containing z_1, z_2 with a large square Q of side $\sqrt{r}$.

Since $r_s > r - \sqrt{r} > \dfrac{r}{2}$ inside Q,

$$\sum_Q \frac{1}{r_s \tau_s} < \frac{2}{r} \sum_Q \frac{1}{\tau_s} < \frac{K}{r} \int_0^{\sqrt{r}} \int_0^{2\pi} \frac{u\, du\, d\theta}{u}$$
$$< K r^{-1/2}$$

while

$$\sum_{CQ} \frac{1}{r_s \tau_s} \leqslant \left(\sum_{CQ} r_s^{-4/3} \right)^{3/4} \left(\sum_{CQ} \tau_s^{-4} \right)^{1/4}$$
$$< K \left(\sum_1^\infty r_s^{-4/3} \right)^{3/4} \left(\int_{\sqrt{r}}^\infty u^{-4} . u\, du \right)^{1/4}$$
$$< K r^{-1/4}.$$

The result follows on combining these inequalities with (5.3).

The main theorem may now be stated.

THEOREM 6. *Let $f(z)$ be an integral function of genus 0 or 1, and let d be a given positive number. Then there is a positive constant h and a sequence $\zeta_1, \zeta_2, \ldots .$ such that*

$$|\zeta_s| \to \infty$$

and

$$(5.5) \qquad \log |f(z)| > h \log M(|\zeta_s|)$$

in the circle

$$|z - \zeta_s| \leqslant d.$$

There are four cases to consider. If ρ, p are respectively the order and genus of $f(z)$ these cases are

(i) $p = 0$, (ii) $p = 1$, $1 < \rho < 2$, (iii) $p = 1$, $\rho = 2$, (iv) $p = 1$, $\rho = 1$.

Case (i) is very simple. Reasoning similar to that in the proof of Lemma 3 shows that there are annuli $r - d \leqslant |z| \leqslant r + d$, for arbitrarily large values of r, in which

$$(5.6) \qquad \frac{1}{r} \leqslant \left| \frac{f(z_1)}{f(z_2)} \right| \leqslant r, \qquad |z_1 - z_2| \leqslant d.$$

Let ζ be a point on the circle $|z| = r$ such that

$$|f(\zeta)| = M(r).$$

Then by (5.6)

$$(5.7) \qquad |f(z)| \geqslant \frac{M(r)}{r}, \qquad |z - \zeta| \leqslant d.$$

Next take case (ii). Divide the z-plane into squares of side $\lambda > 2d$. Let a fraction $1 - \theta_1$ of a square consist of points distant at least d from the boundary of the square. It is possible to find a positive number θ and to choose λ so that (with the notation of (4.8), (4.9))

$$(1 - \theta)(1 - \theta_1)\frac{K_1}{2} + \{\theta_1(1 - \theta) + \theta\} K_2 < K_1.$$

Shade the squares for which (5.1) is satisfied with $\epsilon = 1$. Then almost every square will be shaded, and if s is chosen sufficiently large the fraction of the circle C_s, of radius $2a R_s$ (in the notation of (4.10)), covered by unshaded squares will be less than θ.

Then there is at least one point ζ_s in C_s in a shaded square and distant at least d from the boundary of the square for which

$$(5.8) \qquad \log |f(\zeta_s)| > \frac{K_1}{2} r^{\rho(r)}, \quad r = R_s.$$

For if this is not the case

$$\sigma(2ar) < (1 - \theta_1)(1 - \theta) \frac{K_1}{2} r^{\rho(r)} + \{\theta_1(1 - \theta) + \theta\} K_2 r^{\rho(r)}$$
$$< K_1 r^{\rho(r)}$$

and this contradicts (4.8). Thus if $|z - \zeta_s| \leqslant d$,

$$\log |f(z)| > \frac{K_1}{2} r^{\rho(r)} - r\sqrt{(\log r)}$$

using (5 . 1); and since

$$\underline{\lim}\, \rho(r) > 1$$

if s is sufficiently large

$$\log |f(z)| > \frac{K_1}{3} r^{\rho(r)} > \frac{K_1}{3K_2} \log M(r)$$

the result stated.

In case (iii) the argument just given applies either to $f(z)$ or to $f(z)+1$; and case (iv) is similar to case (iii), except that Lemma 7 is used instead of Lemma 6.

If $p = 0$, $g(z) = f(z^2)$ is of genus 0 or 1. On applying Theorem 6 to this function $g(z)$ another result follows.

THEOREM 7. *If $f(z)$ is of genus* 0, (5 . 5) *is satisfied in a sequence of circles* $|z - \zeta_s| \leqslant d\sqrt{|\zeta_s|}$.

6. *Summary.*

The principal results are as follows.

THEOREM 2. *If*

(i) *$f(z)$ is regular at all points inside the angle* $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$;

(ii) $|f(z)| < Ae^{kr^2}$, $\left(k < \frac{\pi}{2}\right)$, *throughout the angle*;

(iii) $f(m+ni) = 0$ *at all points* $m + ni$ *inside the angle*;

then $f(z)$ is identically zero.

THEOREM 3. *Let $f(z)$ be an integral function of order $\rho < 1$. Let σ be a fixed number $< 1 - \rho$, and let $m_\sigma(r)$, $M_\sigma(r)$ be the bounds of $|f(z)|$ in the annulus $r \leqslant |z| \leqslant r + r^\sigma$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log m_\sigma(r)}{\log M_\sigma(r)} \geqslant \cos \pi\rho.$$

THEOREM 6. *Let $f(z)$ be an integral function of genus* 0 *or* 1, *and let d be a given positive number. Then there is a positive constant h and a sequence $\zeta_1, \zeta_2, \ldots.$ such that*

$$|\zeta_s| \to \infty$$

and

$$\log |f(z)| > h \log M(|\zeta_s|)$$

in the circle $|z - \zeta_s| \leqslant d$.

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