J. Austral. Math. Soc. **20** (Series B) (1977), 13–30

THE STABILITY OF CONTINENTAL SHELF WAVES
I. SIDE BAND INSTABILITY AND LONG WAVE RESONANCE

R. GRIMSHAW

(Received 6 September 1976)
(Revised 20 December 1976)

Abstract

Continental shelf waves are examined for side band instability. It is shown that a modulated shelf wave is described by a nonlinear Schrödinger equation, from which the stability criterion is derived. Long shelf waves are stable to side band modulations, but as the wavenumber is increased there are regions of instability (in wavenumber space). A change of stability occurs at each long wave resonance, defined by the condition that the group velocity of the shelf wave equals a long wave speed. Equations describing the long wave resonance are derived.

1. Introduction

It has recently been established that the passage of large scale meteorological disturbances across a coastline will generate continental shelf waves (for example, Gill and Schumann [4]). The dispersion in these waves is due to topography. The simplest theories neglect this dispersion and use a long wavelength approximation; the results are generally consistent with observations (Gill and Schumann [4]; Kundu, Allen and Smith, [11]). The nonlinear effects in this long wavelength approximation have been described by Smith [14] and Grimshaw [6]. Nevertheless, there have been some observations of shelf waves at shorter wavelengths (for example, Cartwright [3]). The purpose of this paper is to examine the stability of shelf waves due to nonlinear effects.

We let L be a length scale (typical of the shelf width), f^{-1} be a time scale where f is the Coriolis parameter, and scale velocities by fL and the wave height by $\mu^2 h_0$. Here h_0 is the depth of the ocean beyond the shelf and $\mu^2 = f^2 L^2 (gh_0)^{-1}$ is the divergence parameter. Then the nonlinear shallow water equations are

$$\left.\begin{aligned} u_t - v + \zeta_x &= F = -uu_x - vu_y, \\ v_t + u + \zeta_y &= G = -uv_x - vv_y, \\ (hu)_x + (hv)_y + \mu^2 \zeta_t &= H = -\mu^2(\zeta u)_x - \mu^2(\zeta v)_y. \end{aligned}\right\} \tag{1.1}$$

Here x, y are the coordinates normal to and along the coast respectively, t is the time, u, v are the x- and y-components of velocity respectively and ζ is the wave height. h is the undisturbed depth, which we shall assume is a function of x only; we choose the origin so that $h(0) = 0$ and we shall assume that $h \to 1$ as $x \to \infty$ (see Fig. 1). For simplicity we shall assume that $|h_x|, |1-h|$ are $O(\exp(-Kx))$ as $x \to \infty$,

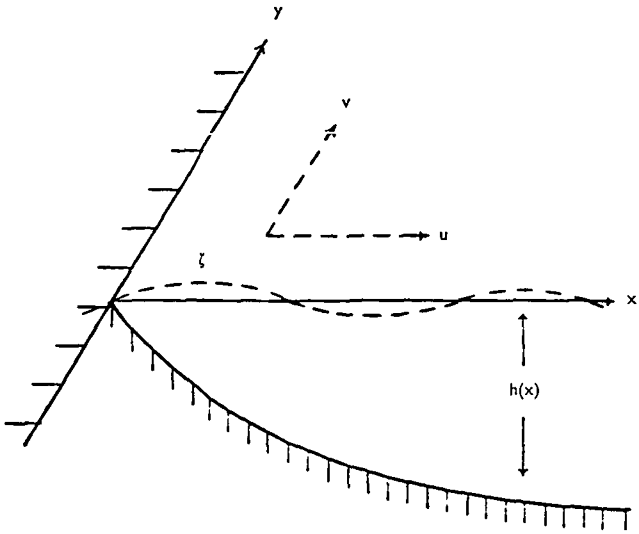

Fig. 1. A description of the coordinate system.

where K is a constant. We shall restrict attention to monotonic profiles so that $h_x \geqslant 0$ for all $x > 0$ and $h_x(0) \neq 0$. The boundary conditions associated with (1.1) are that $hu \to 0$ as $x \to 0$, and as $x \to \infty$. Equations (1.1) have been written with the linear terms on the left-hand side, and the nonlinear terms on the right-hand side. If we eliminate u, v from the left-hand side, we find that

$$L\zeta = M, \tag{1.2}$$

where L is the *linear* operator

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \equiv \mu^2\left(\frac{\partial^2}{\partial t^2}+1\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t\,\partial x}\left(h\frac{\partial}{\partial x}\right) - h\frac{\partial^3}{\partial t\,\partial y^2} - h_x\frac{\partial}{\partial y}, \tag{1.3}$$

and M is the nonlinear expression

$$M = \left(\frac{\partial^2}{\partial t^2}+1\right)H - \frac{\partial}{\partial x}\left(h\left(\frac{\partial F}{\partial t}+G\right)\right) - h\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial t} - F\right). \tag{1.4}$$

To obtain the linearized wave equation we replace M by zero in (1.2); then we seek a solution of the form

$$\zeta = \alpha A \phi(x)\exp(i\kappa y - i\omega t), \tag{1.5}$$

where A is a constant and α is a small parameter. It follows that

$$(h\phi_x)_x = \{\mu^2(1-\omega^2) + \kappa^2 h + c^{-1}h_x\}\phi,$$

where

$$c = \omega\kappa^{-1}. \tag{1.6}$$

https://doi.org/10.1017/S0334270000001417 Published online by Cambridge University Press

It may be shown that if $h_x(0))\neq 0$, then the boundary conditions for (1.6) are $\phi(0)$ is finite, and $\phi \to 0$ as $x \to \infty$. Huthnance [10] has shown that (1.6), with the associated boundary conditions, leads to the existence of dispersion relations (relating the frequency ω to the wavenumber κ) uniquely determining the frequencies of an infinite discrete set of shelf waves, a single Kelvin wave and an infinite discrete set of edge waves. A typical set of dispersion curves is shown in Fig. 2, [10]; the shaded

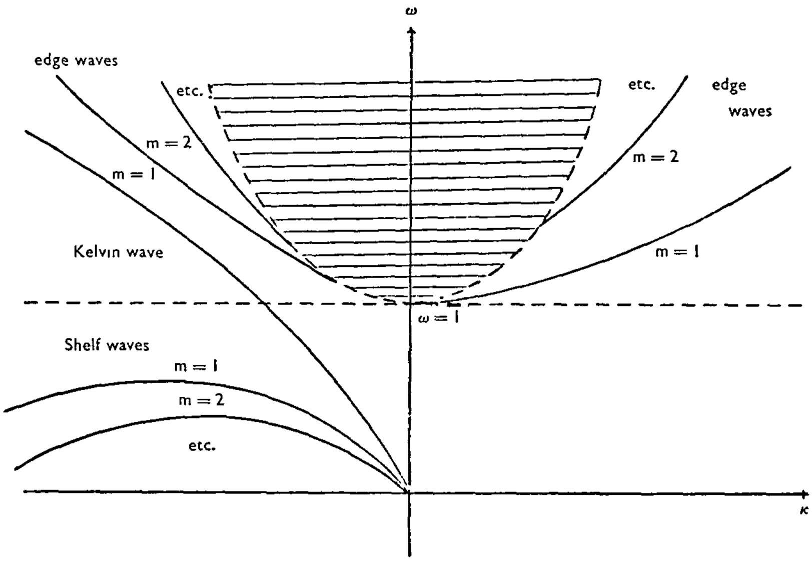

Fig. 2. A typical dispersion relation. The shaded region is $\mu^2(1-\omega^2)+\kappa^2 \leqslant 0$. The integers m refer to the mode numbers.

region (Poincaré's continuum) is the region in which $\mu^2(1-\omega^2)+\kappa^2 \leqslant 0$, and there are no trapped waves in this region. Shelf waves are distinguished by the criterion $\omega^2 < 1$ for all κ, and have negative phase velocities c. Typically we expect c to decrease as κ increases. A useful relation that follows from (1.6) is

$$-c^{-1}\int_0^\infty h_x \phi^2\,dx = \mu^2(1-\omega^2)\int_0^\infty \phi^2\,dx + \int_0^\infty h(\phi_x^2+\kappa^2\phi^2)\,dx. \tag{1.7}$$

The group velocity is $V = d\omega/d\kappa$; on differentiating (1.7) with respect to κ it may be shown that

$$V\left\{c^{-1}\int_0^\infty h_x\phi^2\,dx + 2\mu^2\omega^2\int_0^\infty \phi^2\,dx\right\} = 2\kappa^2 c\int_0^\infty h\phi^2\,dx + \int_0^\infty h_x\phi^2\,dx. \tag{1.8}$$

Typically, for shelf waves, V has the same sign as c for small wavenumbers, but the opposite sign for large wavenumbers.

In this paper we shall consider the stability of a single shelf wave mode ϕ_m, of wavenumber κ phase velocity $c_m(\kappa)$, and group velocity $V_m(\kappa)$. Here m is a positive integer designating the mode number; we shall assume that $\omega_m(\kappa) = \kappa c_m(\kappa)$ decreases as m increases (cf. Fig. 2). We shall consider a nearly monochromatic shelf wave, and subject it to the influence of weak nonlinearity and frequency modulation. Specifically we allow the amplitude αA to depend on $\alpha^2 t$ and $\alpha(y - V_m t)$.

It is now well known that in these circumstances the nonlinear Schrodinger equation governs the evolution of the wave amplitude (Benney and Newell [1]), and so it is no surprise that we obtain a nonlinear Schrödinger equation in the present case. Indeed, the theory developed in Sections 2, 3, 4 is very similar to the corresponding theory for internal gravity waves in a channel (Grimshaw [5], [8]). In Section 2 we develop the theory of a modulated shelf wave, and in Section 3 we discuss the mean flows generated by a modulated shelf wave. The equation governing the evolution of A is obtained in Section 4, and the stability of the wave to side band modulations deduced. We show that long shelf waves ($\kappa \approx 0$) are stable to these side band modulations, but that as $|\kappa|$ is increased regions of instability (in wave number space) will be encountered. In Section 5 we discuss long wave resonance which occurs when $V_m(\kappa) \approx c_s(0)$ for some integer s; there is then an interaction between the modulated shelf wave and a long wave of mode number s. This interaction takes place on a time scale $O(\alpha^{-\frac{4}{3}})$, and the equations describing the long wave resonance are derived. We show that the modulated shelf wave is unstable to this interaction with a long wave. Long wave resonance was first observed by McIntyre [12] for internal gravity waves in a channel, and the equations describing the resonance for that case were derived by Grimshaw [5], [8]; the equations are very similar to those obtained in the present case. The Appendix contains the derivation of a compatibility condition needed in Section 2.

2. Modulated waves

In order to describe a modulated wave, we introduce the long length and time scales

$$Y = \varepsilon(y - V_m t), \quad T = \varepsilon^2 t. \tag{2.1}$$

Here ε is a small parameter, chosen so that ε^{-1} is the appropriate length, or time, scale for the modulations. In the sequel ε will be determined by the nonlinear terms. We are anticipating that, to leading order, the wave travels at the group velocity V_m. We let α be an appropriate measure of wave amplitude, and we shall use α to measure nonlinear effects. To leading order in both ϵ and α,

$$\zeta = \alpha A(Y, T)\,\phi_m(x)\exp(i\kappa y - i\omega_m t) + \text{c.c.}, \tag{2.2}$$

where c.c. denotes complex conjugate. Here A is $O(1)$ with respect to ϵ, α. The primary aim of the subsequent analysis is to find an equation describing the evolution of A.

We seek a solution of (1.1) (or equivalently (1.2)) of the form

$$\left.\begin{aligned} \zeta &= \sum_{-\infty}^{\infty} \zeta_n(Y,T,x)\exp(ni\theta), \\ \text{where} \quad \theta &= \kappa y - \omega_m t \quad \text{and} \quad \zeta_n = \bar{\zeta}_{-n}. \end{aligned}\right\} \tag{2.3}$$

u, v are given by similar expansions. To leading order ζ_1 is given by (2.2), and we anticipate that ζ_0, ζ_2 are $O(\alpha^2)$, while ζ_n is $O(\alpha^n)$ for $n > 2$.

On substituting (2.3) into (1.2) and (1.3), it follows that

$$\left.\begin{aligned} L\zeta &\equiv \sum_{-\infty}^{\infty} \exp(ni\theta) L_n \zeta_n = M, \\ \text{where} \quad L_n &\equiv L\left(-in\omega_m - \varepsilon V_m \frac{\partial}{\partial Y} + \varepsilon^2 \frac{\partial}{\partial T}, in\kappa + \varepsilon \frac{\partial}{\partial Y}, \frac{\partial}{\partial x}\right). \end{aligned}\right\} \tag{2.4}$$

We may write the nonlinear term in the form

$$M = \sum_{-\infty}^{\infty} M_n \exp(in\theta), \tag{2.5}$$

and we anticipate that M_0, M_2 are $O(\alpha^2)$, M_1, M_3 are $O(\alpha^3)$ and the other M_n are of higher order in α. It follows that

$$L_n \zeta_n = M_n. \tag{2.6}$$

Consider first the case $n = 1$. It is shown in the Appendix that a necessary and sufficient condition for (2.7) to have a solution is the compatibility condition (A.9),

$$\int_0^\infty \phi_m L_1 \zeta_1 \, dx = \int_0^\infty M_1 \phi_m \, dx. \tag{2.7}$$

Since, as we shall verify *a posteriori*, M_1 is $O(\alpha^3)$ it follows that ζ_1 satisfies (2.6) within an error of $O(\alpha^3)$; it may then be shown that

$$\zeta_1 = \alpha A \phi_m + i\varepsilon\alpha \frac{\partial A}{\partial Y} \frac{\partial \phi_m}{\partial \kappa} + O(\varepsilon^2 \alpha, \alpha^3). \tag{2.8}$$

We let

$$D(\omega, \kappa) = \int_0^\infty \phi_m L(-i\omega, i\kappa) \phi_m \, dx. \tag{2.9}$$

From (1.6) it may be shown that

$$D(\omega, \kappa) = i\omega\left\{\mu^2(\omega^2 - \omega_m^2) \int_0^\infty \phi_m^2 \, dx + c_m^{-1}(1 - \omega_m/\omega) \int_0^\infty h_x \phi_m^2 \, dx\right\}. \tag{2.10}$$

Putting $D = 0$ recovers the dispersion relation $\omega = \omega_m(\kappa)$. It may be shown that (2.7) becomes

$$\alpha D\left(\omega_m - i\varepsilon V_m \frac{\partial}{\partial Y} + i\varepsilon^2 \frac{\partial}{\partial T}, \kappa - i\varepsilon \frac{\partial}{\partial Y}\right) A = \int_0^\infty M_1 \phi_m \, dx. \tag{2.11}$$

Expanding (2.11) it may be shown that

$$\left.\begin{aligned} \alpha\varepsilon^2 D_\omega \left\{ i \frac{\partial A}{\partial T} + \lambda \frac{\partial^2 A}{\partial X^2} \right\} &= \int_0^\infty M_1 \phi_m \, dx, \\ \text{where} \qquad \lambda = \frac{1}{2} \frac{\partial V_m}{\partial \kappa} &= \frac{1}{2} \frac{\partial^2 \omega_m}{\partial \kappa^2}. \end{aligned}\right\} \tag{2.12}$$

The second group of terms on the right-hand side describes the effects of frequency modulation. From (2.10) it follows that

$$D_{\omega|\omega=\omega_m} = i\left\{2\mu^2 \omega_m^2 \int_0^\infty \phi_m^2 \, dx + c_m^{-1} \int_0^\infty h_x \phi_m^2 \, dx\right\}. \tag{2.13}$$

Comparing (2.13) with (1.8) we see that D_ω can vanish only when the group velocity V_m is infinite; since typical dispersion curves (Fig. 2) have finite group velocities for all κ, we shall assume that D_ω is not zero. It remains to evaluate the nonlinear term M_1. Since we require M_1 only to $O(\alpha^3)$, it will be sufficient to consider only the contributions from the interactions of the harmonic $n = 1$ (ζ_1 etc.) with the harmonics $n = 2$ and $n = 0$.

In the remainder of this section we shall calculate ζ_2 and its contribution to M_1. First we note that, from (1.1).

$$\left.\begin{aligned} u_1 = i\alpha A g_m + O(\alpha\varepsilon), \quad v_1 &= \alpha A h_m + O(\alpha\varepsilon), \\ \text{where} \qquad g_m(1-\omega_m^2) &= (\omega_m \phi_{mx} - \kappa\phi_m) \\ \text{and} \qquad h_m(1-\omega_m^2) &= (\phi_{mx} - \omega_m \kappa\phi_m). \end{aligned}\right\} \tag{2.14}$$

Now to $O(\alpha^2)$ M_2 may be calculated using ζ_1, u_1, v_1 ((2.2) and (2.14)). We find that, using (1.1) and (1.4),

$$M_2 = i\alpha^2 A^2 \mathscr{M}_2, \tag{2.15}$$

where

$$\left.\begin{aligned} \mathscr{M}_2 &= (1-4\omega_m^2)\mathscr{H}_2 + 2\kappa(h\mathscr{F}_2 + 2\omega_m h\mathscr{G}_2) + (2\omega_m h\mathscr{F}_2 - h\mathscr{G}_2)_x, \\ \text{and} \qquad \mathscr{F}_2 &= g_m g_{mx} + \kappa g_m h_m, \\ \mathscr{G}_2 &= -g_m h_{mx} - \kappa h_m^2, \\ \mathscr{H}_2 &= -\mu^2(g_m\phi_m)_x - 2\mu^2\kappa\phi_m h_m. \end{aligned}\right\} \tag{2.16}$$

https://doi.org/10.1017/S0334270000001417 Published online by Cambridge University Press

Provided there is no second harmonic resonance ($c_m(\kappa) \neq c_s(2\kappa)$ for any integer s), it is shown in the Appendix that we may solve (2.6) uniquely for ζ_2 and we find that

$$\left.\begin{aligned} \zeta_2 &= \alpha^2 A^2 Z_2, \\ \text{where} \quad (hZ_{2x})_x - \mu^2(1-4\omega_m^2)Z_2 - 4\kappa^2 hZ_2 - c_m^{-1} h_x Z_2 &= (2\omega_m)^{-1}\mathscr{M}_2. \end{aligned}\right\} \tag{2.17}$$

Also, $\mu_2 = i\alpha^2 A^2 \mathscr{U}_2$ and $v_2 = \alpha^2 A^2 \mathscr{V}_2$ where $\mathscr{U}_2$ and $\mathscr{V}_2$ are real expressions defined in terms of Z_2, $\mathscr{F}_2$ and $\mathscr{G}_2$ (see Grimshaw [7], where further details of these calculations are given).

We shall use a superscript "2" to denote the contribution to M_1 due to the interaction of the harmonics $n = 2$ and $n = 1$. We find that

$$M_1^{(2)} = i\alpha^3 |A|^2 A \mathscr{M}_1^{(2)}, \tag{2.18}$$

where $\mathscr{M}_1^{(2)}$ is a complicated real expression involving ϕ_m and Z_2 (see [7]). Finally, the contribution of (2.18) to the right-hand side of (2.12) is

$$\left.\begin{aligned} D_\omega^{-1}\int_0^\infty M_1^{(2)}\phi_m\,dx &= \alpha^3\gamma_2|A|^2 A, \\ \text{where} \qquad \gamma_2 &= D_\omega^{-1}\int_0^\infty i\mathscr{M}_1^{(2)}\phi_m\,dx. \end{aligned}\right\} \tag{2.19}$$

Since $\mathscr{M}_1^{(2)}$ is real (all quantities in script variables are real) and D_ω (2.13) is pure imaginary, it follows that γ_2 is real.

3. Wave-induced mean flow

The equation which determines ζ_0 is (2.6) with $n = 0$. However, it is preferable to observe that this equation is just that obtained by averaging with respect to the phase θ, and an alternative procedure is to average (1.1).

To leading order, the nonlinear terms may be evaluated using only ζ_1, u_1 and v_1. We find that

$$\left.\begin{aligned} \varepsilon^2\frac{\partial u_0}{\partial T} - \varepsilon V_m\frac{\partial u_0}{\partial Y} - v_0 + \frac{\partial \zeta_0}{\partial x} &= \alpha^2|A|^2\mathscr{F}_0, \\ \varepsilon^2\frac{\partial v_0}{\partial T} - \varepsilon V_m\frac{\partial v_0}{\partial Y} + u_0 + \varepsilon\frac{\partial \zeta_0}{\partial Y} &= \varepsilon\alpha^2\frac{\partial}{\partial Y}|A|^2\mathscr{G}_0, \\ \frac{\partial}{\partial x}(hu_0) + \varepsilon h\frac{\partial v_0}{\partial Y} - \varepsilon\mu^2 V_m\frac{\partial \zeta_0}{\partial Y} + \varepsilon^2\mu^2\frac{\partial \zeta_0}{\partial T} &= \varepsilon\alpha^2\frac{\partial}{\partial Y}|A|^2\mathscr{H}_0. \end{aligned}\right\} \tag{3.1}$$

Here, to leading order, the nonlinear terms are given by, using (2.2) and (2.14) (with the terms of $O(\alpha\varepsilon)$ included),

$$\left.\begin{aligned}
\mathscr{F}_0 &= -2g_m g_{mx} + 2\kappa g_m h_m,\\
\mathscr{G}_0 &= -\frac{V_m \phi_{mx}(h_{mx}+\kappa g_m)}{1-\omega_m^2} + \frac{(\phi_m h_{mx} - \omega g_m \phi_{mx})}{1-\omega_m^2} - h_m^2,\\
&\quad + \frac{g_m \phi_m}{(1-\omega_m^2)}(2\mu^2 \omega_m V_m + 2\kappa h + h_x \omega_m^{-1}(1 - V_m c_m^{-1})) + \mathscr{J}_0,\\
\mathscr{H}_0 &= -\frac{\mu^2\{V_m(\phi_m \phi_{mx} + 2\omega_m \phi_m g_m) - \phi_m^2\}_x}{1-\omega_m^2} - 2\mu^2 \phi_m h_m + h\mathscr{J}_0,\\
\text{and}\qquad &\\
\mathscr{J}_0 &= \frac{\omega_m \mu^2\{\phi_m(\partial\phi_{mx}/\partial\kappa) - \phi_{mx}(\partial\phi_m/\partial\kappa)\}}{h(1-\omega_m^2)}.
\end{aligned}\right\}\tag{3.2}$$

Eliminating u_0, v_0 from (3.1) it follows that

$$\left.\begin{aligned}
&\left(\varepsilon\frac{\partial}{\partial T} - V_m\frac{\partial}{\partial Y}\right)\left\{\mu^2\zeta_0 - \frac{\partial}{\partial x}\left(h\frac{\partial\zeta_0}{\partial x}\right)\right\} - h_x\frac{\partial\zeta_0}{\partial Y} = \alpha^2\frac{\partial}{\partial Y}|A|^2\mathscr{M}_0,\\
\text{where}\qquad &\mathscr{M}_0 = \mathscr{H}_0 + h\mathscr{F}_0 - \frac{\partial}{\partial x}(h\mathscr{G}_0 - hV_m\mathscr{F}_0).
\end{aligned}\right\}\tag{3.3}$$

Note that $\mathscr{J}_0$ does not occur in $\mathscr{M}_0$. Hence we find that, to leading order,

$$\left.\begin{aligned}
&\zeta_0 = \alpha^2|A|^2\Phi_0(x),\\
\text{where}\qquad &\frac{\partial}{\partial x}\left(h\frac{\partial\Phi_0}{\partial x}\right) - \mu^2\Phi_0 - V_m^{-1}h_x\Phi_0 = V_m^{-1}\mathscr{M}_0.
\end{aligned}\right\}\tag{3.4}$$

Consider the homogeneous equation

$$\frac{\partial}{\partial x}\left(h\frac{\partial\phi}{\partial x}\right) - \mu^2\phi - c^{-1}h_x\phi = 0,\tag{3.5}$$

with the boundary conditions that ϕ is finite at $x=0$ and $\phi\to 0$ as $x\to\infty$. This is the equation for long shelf waves, and has a complete set of long wave modes $\phi_s^{(0)}$, with long wave speeds $c_s^{(0)} = c_s(0)$. Using a procedure similar to that used in the Appendix to solve (2.6), it may be shown that (3.4) has a unique solution Φ_0, provided that $V_m \neq c_s(0)$ for any $s = 1, 2, 3, \ldots$. Alternatively, let

$$\left.\begin{aligned}
&\Phi_0 = \sum_1^\infty a_s\phi_s^{(0)},\\
\text{where}\qquad &a_s\int_0^\infty h_x(\phi_s^{(0)})^2\,dx = \int_0^\infty h_x\phi_s^{(0)}\Phi_0\,dx.
\end{aligned}\right\}\tag{3.6}$$

Here we have used the orthogonality relation

$$\int_0^\infty h_x \phi_s^{(0)} \phi_r^{(0)}\, dx = 0 \quad \text{for } s \neq r. \tag{3.7}$$

Then, on multiplying (3.4) by $\phi_s^{(0)}$ and integrating with respect to x from 0 to ∞, it follows that

$$a_s\left(\frac{1}{c_s^{(0)}} - \frac{1}{V_m}\right) \int_0^\infty h_x (\phi_s^{(0)})^2\, dx = \frac{1}{V_m} \int_0^\infty \mathscr{M}_0 \phi_s^{(0)}\, dx. \tag{3.8}$$

Hence (3.6) gives the unique solution for Φ_0, provided that $V_m \neq c_s^{(0)}$, for any $s = 1, 2, 3, \ldots$. The long wave resonance which occurs when $V_m = c_s^{(0)}$ will be discussed in Section 6. Finally, it follows from (3.1) that

$$\left.\begin{aligned} u_0 &= \varepsilon\alpha^2 \frac{\partial}{\partial Y} |A|^2 \{V_m \Phi_{0x} - \Phi_0 + \mathscr{G}_0 - V_m \mathscr{F}_0\}, \\ v_0 &= \alpha^2 |A|^2 \{\Phi_{0x} - \mathscr{F}_0\}. \end{aligned}\right\} \tag{3.9}$$

Note that u_0 (the mean onshore velocity) is $O(\varepsilon)$ smaller than v_0 (the mean alongshore velocity).

We shall use a superscript "0" to denote the contribution to M_1 due to the interaction of the harmonic $n = 1$ with the mean flow (the harmonic $n = 0$). We find that

$$M_1^{(0)} = i\alpha^3 |A|^2 A \mathscr{M}_1^{(0)}, \tag{3.10}$$

where $\mathscr{M}_1^{(0)}$ is a complicated real expression involving ϕ_m and Φ_0 (see [7]). Finally, the contribution of (3.10) to the right-hand side of (2.12) is

$$\left.\begin{aligned} D_\omega^{-1} \int_0^\infty M_1^{(0)} \phi_m\, dx &= \alpha^3 \gamma_0 |A|^2 A, \\ \text{where} \qquad \gamma_0 &= D_\omega^{-1} \int_0^\infty i\mathscr{M}_1^{(0)} \phi_m\, dx. \end{aligned}\right\} \tag{3.11}$$

Since $\mathscr{M}_1^{(0)}$ is real, and D_ω (2.13) is pure imaginary, it follows that γ_0 is real.

4. The amplitude equation

We have now shown that the amplitude equation (2.12) is the nonlinear Schrödinger equation

$$\left.\begin{aligned} i\frac{\partial A}{\partial T} + \lambda \frac{\partial^2 A}{\partial Y^2} &= \gamma |A|^2 A, \\ \text{where} \qquad \gamma &= \gamma_0 + \gamma_2. \end{aligned}\right\} \tag{4.1}$$

https://doi.org/10.1017/S0334270000001417 Published online by Cambridge University Press

Here we have put $\epsilon = \alpha$ so that the nonlinear terms exactly balance the frequency modulation terms. The properties of this equation are well known (Zakharov and Shabat [15]). In particular, (4.1) has the plane wave solution

$$A = C\exp(-i\gamma|C|^2 T), \tag{4.2}$$

where C is a constant, which is unstable to side band modulations (Hasimoto and Ono [9]) if

$$\lambda\gamma < 0. \tag{4.3}$$

Indeed, if the plane wave (4.2) is perturbed by terms proportional to $\exp(iLY+pT)$, then the growth rate p is given by

$$p^2 = -\lambda L^2(2\gamma|C|^2 + \lambda L^2). \tag{4.4}$$

The instability, considered for fixed κ, has a maximum growth rate given by $p = |\gamma||C|^2$ when $\lambda L^2 = -\gamma|C|^2$. It is apparent from the typical dispersion curves (Figure 2), for positive frequencies ω_m, that $\lambda = \frac{1}{2}\partial V_m/\partial\kappa$ (2.16) will be negative for the range of κ of most interest (that is, values of κ ranging from zero to the vicinity of the turning point of the dispersion curve). Hence the waves will be unstable for positive values of γ.

We have been unable to obtain any general result concerning the sign of γ. However, for long waves (that is, $\kappa \to 0$), it will be shown below that γ is always negative (for positive frequencies), and hence long shelf waves are stable to side band modulations. Of course, the assumptions of the present theory prohibit the limit $\kappa \to 0$; nevertheless, it is useful to use the *approximation* $\kappa \approx 0$ in order to obtain some information about the sign of γ. The theory appropriate to the limit $\kappa \to 0$ was developed by Grimshaw [6]. When $\kappa \approx 0$,

$$\left.\begin{aligned} &\phi_m(z) = \phi_m^{(0)}(z) + O(\kappa^2) \\ \text{and}\qquad &c_m = c_m^{(0)} + \kappa^2 c_m^{(1)} + O(\kappa^4), \\ \text{where}\qquad &\frac{c_m^{(1)}}{c_m^{(0)2}}\int_0^\infty h_x \phi_m^{(0)2}\,dx = \int_0^\infty (h - \mu^2 c_m^{(0)2})\,\phi_m^{(0)2}\,dx. \end{aligned}\right\} \tag{4.5}$$

Here $\phi_m^{(0)}$ is the mth long wave mode and $c_m^{(0)}$ is the mth long wave speed. Also, from (2.14) it follows that

$$\left.\begin{aligned} g_m(z) &\approx \kappa(c_m^{(0)}\phi_{mx}^{(0)} - \phi_m^{(0)}) + O(\kappa^3), \\ h_m(z) &\approx \phi_{mx}^{(0)} + O(\kappa^2). \end{aligned}\right\} \tag{4.6}$$

We shall now proceed to use these approximations to evaluate γ.

First consider the calculation of Z_2 from (2.17). From the procedure outlined in the Appendix it may be shown that

$$\kappa^2 Z_2 \approx a\phi_m^{(0)} + O(\kappa^2). \tag{4.7}$$

Indeed, it is apparent that when $\kappa \to 0$ in (2.17) the homogeneous part of the equation for Z_2 becomes the long wave equation and hence Z_2 takes the form given by (4.7). Further, it may be shown that

$$-\frac{3c_m^{(1)}}{c_m^{(0)2}} a \int_0^\infty h_x \phi_m^{(0)2}\,dx \approx (2\omega_m)^{-1} \int_0^\infty \mathscr{M}_2 \phi_m^{(0)}\,dx. \tag{4.8}$$

Using (4.5) in (2.15) and (2.16), it follows that

$$\left.\begin{aligned} \mathscr{M}_2 &= \mathscr{H}_2 - (h\mathscr{G}_2)_x + O(\kappa^3), \\ \mathscr{G}_2 &= -g_m h_{mx} - \kappa h_m^2 + O(\kappa^3), \\ \mathscr{H}_2 &= -\mu^2 (g_m \phi_m^{(0)})_x - 2\kappa\mu^2 \phi_m^{(0)} h_m + O(\kappa^3). \end{aligned}\right\} \tag{4.9}$$

where

Substituting (4.9) into (4.8), we find that

$$a = \frac{\delta}{6c_m^{(1)}},$$

where

$$\delta \int_0^\infty h_x \phi_m^{(0)2}\,dx = c_m^{(0)} \int_0^\infty \{h(\phi_m^{(0)3} + c_m^{(0)} \phi_m^{(0)2} \phi_{mxx}^{(0)} - \phi_m^{(0)} \phi_{mx}^{(0)} \phi_{mxx}^{(0)})$$
$$- \mu^2 c_m^{(0)} \phi_{mx}^{(0)2} \phi_m^{(0)}\}\,dx - \mu^2 c_m^{(0)2} [\phi_m^{(0)2} \phi_{mx}^{(0)}]_{x=0}. \tag{4.10}$$

Indeed, δ is identical to the coefficient obtained by Grimshaw [6] (equation (3.12)) in another context. Next it may be shown that

$$\kappa^2 \mathscr{M}_1^{(2)} = a\mathscr{M}_2 + O(\kappa^3), \tag{4.11}$$

and hence it follows that

$$\gamma_2 = -\frac{6c_m^{(1)} a^2}{\kappa} + O(\kappa). \tag{4.12}$$

Here we have also used (2.13) to evaluate D_ω. Since κ is negative for shelf waves of positive frequency, we see that the second harmonic is destabilising as the contribution of γ_2 to γ is positive.

Next we shall use the approximations (4.5) in the calculation of the mean flow. It is apparent from (3.8) that

$$\Phi_0 = a_m \phi_m^{(0)} + O(1), \tag{4.13}$$

where a_m is $O(\kappa^{-2})$. Also it may be shown that

$$\mathscr{M}_0 = \kappa^{-1} \mathscr{M}_2 + O(\kappa^2). \tag{4.14}$$

Hence it follows from (3.8) that

$$\kappa^2 a_m = -2a + O(\kappa^2). \tag{4.15}$$

Also it may be shown that

$$\mathscr{M}_1^{(0)} = a_m \mathscr{M}_2 + O(\kappa), \tag{4.16}$$

https://doi.org/10.1017/S0334270000001417 Published online by Cambridge University Press

and hence it follows from (3.11) that

$$\gamma_0 = \frac{12 c_m^{(1)} a^2}{\kappa} + O(\kappa). \tag{4.17}$$

Finally, since $\gamma = \gamma_0 + \gamma_2$, we find that

$$\gamma = \frac{\delta^2}{6\kappa c_m^{(1)}}, \tag{4.18}$$

where we have used (4.10). Since κ is negative, γ is negative and long shelf waves are stable to side band modulations. Using (4.5) in (2.12) we see that $\lambda = 3\kappa c_m^{(1)} + O(\kappa^3)$, while (4.18) shows that γ is $O(\kappa^{-1})$ as $\kappa \to 0$. Thus nonlinear effects are enhanced as the wavenumber becomes small; the present theory remains valid provided that $\alpha \ll \kappa^2$, which is the criterion for ignoring the higher harmonics $n = 3$, etc. Grimshaw [6] computed values of δ for various profiles. In particular it was shown that for a profile used by Buchwald and Adams [2] to model the East Australian coast $\delta = 2.2$ for the first shelf wave mode, with larger values of δ for the higher modes; for the same profile $c_m^{(1)} = 1.3$, while $c_m^{(0)} = -0.4$.

The theory appropriate for the limit $\kappa \to 0$ was developed by Grimshaw [6]. It was shown there that in this limit

$$\zeta \sim Z(t, \eta)\, \phi_m^{(0)}(x), \quad \eta = y - c_m^{(0)} t, \tag{4.19}$$

where Z satisfies the Korteweg–de Vries equation

$$-Z_t + \delta Z Z_\eta + c_m^{(1)} Z_{\eta\eta\eta} = 0. \tag{4.20}$$

Here if the length scale is $O(\varepsilon^{-1})$ (that is, κ is $O(\varepsilon)$), then the time scale is $O(\epsilon^{-3})$, and the amplitude is $O(\varepsilon^2)$ (that is, the amplitude α is comparable with κ^2). If, in (4.20), we seek time harmonic solutions,

$$Z = \alpha A(T, Y) \exp(i\kappa y - i\omega t) + \text{c.c.}, \tag{4.21}$$

then it may readily be shown that A will satisfy the nonlinear Schrödinger equation (4.1) with γ given by (4.18) and $\lambda = 3\kappa c_m^{(1)}$. Thus the long wave theory agrees with the present theory in their common region of validity.

As $|\kappa|$ is increased from zero, it is clear from (3.8) that Φ_0 will be approximately given by (4.13) until the first long wave resonance occurs. Since V_m decreases as κ increases, this occurs at that value of κ for which $V_m(\kappa) = c_{m+1}(0)$. In the vicinity of this value of κ, Φ_0 will be approximately given by $a_{m+1}\phi_{m+1}^{(0)}$ where, from (3.8), a_{m+1} is proportional to $(V_m(\kappa) - c_{m+1}(0))^{-1}$; also γ_0 will then be approximately proportional to a_{m+1}, and is infinite at the long wave resonance. As $|\kappa|$ increases through the resonant value γ_0 will change sign and become positive. Since γ will be dominated by γ_0 for κ near the resonant value, γ will also change sign and become positive. Thus as $|\kappa|$ increases through the resonant value, the wave becomes unstable to side band modulations. As $|\kappa|$ increases further, other long wave resonances will be encountered, and at each such encounter there will be a change in stability behaviour. Similar remarks may be made about the second harmonic resonance ($c_m(\kappa) = c_s(2\kappa)$). At the first such resonance encountered as $|\kappa|$ is increased γ_2 will change sign and become negative. Thus if the first second harmonic resonance is encountered before the first long wave resonance γ will remain negative, and there will be no change in the stability behaviour. Finally, we note that for the Buchwald and Adams [2] model of the East Australian coast, the long wave resonance between the modes $m = 1$ and $s = 2$ occurs at $\kappa = -1.7$; this corresponds to a dimensional wavelength of 300 km, a group velocity of 110 km/day, and a period of approximately 2 days.

5. Long wave resonance

When $V_m(\kappa) = c_s(0)$ there is a resonant interaction between the wave, of wave-number κ and mode number m, and a long wave of mode number s. This resonance was observed by McIntyre [12] for internal gravity waves, and the equations governing this resonance were obtained by Grimshaw [5], [8]; recently Plumb [13] has discussed a similar resonance for Rossby waves. For typical dispersion curves for a shelf wave (Fig. 2), as κ varies from zero to the value at the turning point, $V_m(\kappa)$ varies from $c_m(0)$ to 0, and hence an infinity of long wave resonances are possible with $s = m+1, m+2, \ldots$. Once κ has passed the value at the turning point, no long wave resonances are possible. To leading order there are now two free waves present; one is the wave of wavenumber κ and is described by (2.2), while the other is the long wave

$$\zeta_0 = \alpha_0 A_0 \phi_s^{(0)}(x) + O(\alpha^2), \tag{5.1}$$

where α_0 is a parameter measuring the amplitude of the long wave. Both A, A_0 depend on Y, T, equation (2.1). The equation governing the evolution of A is again (2.12). Now, however, the contribution of the second harmonic to the right-hand side is $O(\alpha^3)$ as before, while the contribution of the long wave is $O(\alpha\alpha_0)$. Thus (2.12) becomes

$$\left.\begin{aligned} \alpha\varepsilon^2\left(i\frac{\partial A}{\partial T} + \lambda\frac{\partial^2 A}{\partial Y^2}\right) &= S + O(\alpha^3), \\ \text{where} \qquad S &= D_\omega^{-1}\int_0^1 M_1^{(0)}\phi_m\,dx. \end{aligned}\right\} \tag{5.2}$$

We note that

$$v_0 = \alpha_0 A_0 \phi_{sx}^{(0)} + O(\alpha^2), \tag{5.3}$$

while u_0 is $O(\varepsilon\alpha_0)$. Clearly S is $O(\alpha\alpha_0)$, and so we choose $\alpha_0 = \varepsilon^2$. The equation governing the long wave is (3.3). To leading order, ζ_0 is a free solution of (3.3). We

https://doi.org/10.1017/S0334270000001417 Published online by Cambridge University Press

determine α by requiring that, at the next order, (3.3) describes a balance between the time derivatives of ζ_0 and the right-hand side. Hence $\varepsilon\alpha_0 = \alpha^2$, and it follows that

$$\alpha = \varepsilon^{\frac{3}{2}}, \quad \alpha_0 = \varepsilon^2. \tag{5.4}$$

We seek a solution of (3.3) of the form

$$\left.\begin{aligned} \zeta_0 &= \sum_{r=1}^{\infty} a_r \phi_r^{(0)}, \\ \text{where} \qquad & \\ a_r &= \int_0^\infty h_x \phi_r^{(0)} \zeta_0 \, dx. \end{aligned}\right\} \tag{5.5}$$

Here a_r is $O(\alpha^2)$ for $r \neq s$, while $a_s = \alpha_0 A_0$. We put

$$V_m = c_s(0)(1+\sigma\varepsilon), \tag{5.6}$$

so that σ is a measure of the amount by which the resonance is tuned. On multiplying (3.3) by $\phi_r^{(0)}$, and integrating form 0 to ∞, it follows that

$$\left\{\left(\varepsilon\frac{\partial}{\partial T} - V_m\frac{\partial}{\partial Y}\right)\left(-\frac{a_r}{c_r(0)}\right) - \frac{\partial a_r}{\partial Y}\right\}\int_0^\infty h_x \phi_r^{(0)2}\, dx = \alpha^2 \frac{\partial}{\partial Y}|A|^2 \int_0^\infty \mathscr{M}_0 \phi_r^{(0)}\, dx. \tag{5.7}$$

For $r \neq s$, this equation confirms that a_r is $O(\alpha^2)$. However, for $r = s$, it becomes

$$\left.\begin{aligned} -\frac{1}{V_m}\frac{\partial A_0}{\partial T} + \sigma\frac{\partial A_0}{\partial Y} &= \mu\frac{\partial}{\partial Y}|A|^2, \\ \text{where} \qquad & \\ \mu\int_0^\infty h_x \phi_s^{(0)2}\, dx &= \int_0^\infty \mathscr{M}_0 \phi_s^{(0)}\, dx. \end{aligned}\right\} \tag{5.8}$$

Next, substituting (5.1) and (5.3) into $M_1^{(0)}$ we find that (see [7])

$$\left.\begin{aligned} & S = \nu\alpha_0 A_0, A \\ & \text{where} \\ & \nu = iD_\omega^{-1}\int_0^\infty [\omega_m \mu^2 \phi_s^{(0)}(\phi_{mx}^2 + \kappa^2\phi_m^2) + (1-\omega_m^2)\phi_{sx}^{(0)}\{(hh_m g_m)_x - \kappa h g_m^2 - \kappa h h_m^2\}]\, dx. \end{aligned}\right\} \tag{5.9}$$

Hence, equation (5.2) for A is

$$i\frac{\partial A}{\partial T} + \lambda\frac{\partial^2 A}{\partial Y^2} = \nu A_0 A. \tag{5.10}$$

The equations describing the long wave resonance are thus (5.8) and (5.10). They are identical in form to those obtained by Grimshaw [5], [8] for long wave resonance for internal gravity waves.

We have been unable to obtain the general solution of (5.8) and (5.10). However, there is an envelope solution for which

$$\left.\begin{aligned} A &= R\exp\left\{\frac{i\beta}{2\lambda}(Y-\gamma T)\right\}\operatorname{sech}\{k(Y-\beta T)\},\\ \nu A_0 &= m|A|^2 \end{aligned}\right\} \tag{5.11}$$

where

$$\left.\begin{aligned} 2\lambda k^2 &= -mR^2\\ m\left\{\frac{2\beta}{V_m}+\sigma\right\} &= \mu\nu,\\ \gamma-\tfrac{1}{2}\beta &= -2\lambda^2 k^2 \beta^{-1}. \end{aligned}\right\} \tag{5.12}$$

and

If we regard R, k as given, then these equations determine m, β and γ. Of course this solution requires that $\lambda(2\beta V_m^{-1}+\sigma)\mu\nu$ be negative. This condition should be compared with the instability condition (4.3), which, using (5.6), requires that $\lambda\sigma\mu\nu$ be negative (as when (5.6) holds, it may be shown that $\varepsilon\sigma\gamma\approx\mu\nu$). For example, at the first long wave resonance, $s = m+1$, the argument at the end of Section 4 indicates that we expect γ to be negative when σ is positive, and so $\mu\nu$ is negative. Thus, as λ is negative, the envelope solution requires that $(2\beta V_m^{-1}+\sigma)$ be positive. In particular, when σ is negative (so that the wave is unstable to side band modulations), β and V_m have the same sign, and the envelope solution propagates in the same sense as the wave.

Equation (5.8) and (5.10) have the plane wave solutions

$$A = C\exp(-i\nu C_0 T), \quad A_0 = C_0, \tag{5.13}$$

where C, C_0 are constants. If this plane wave is perturbed by terms proportional to $\exp(iLY+pT)$, then the growth rate p is given by

$$(p-i\sigma L V_m)(p^2+\lambda^2 L^4)+2i\mu\nu\lambda V_m|C|^2 L^3 = 0. \tag{5.14}$$

This has purely imaginary solutions for p, indicating stability, if and only if,

$$2\mu\nu\lambda V_m|C|^2 \gtrless \tfrac{2}{27}(\sigma V_m)^3-\tfrac{2}{3}\sigma V_m\lambda^2 L^2 \mp \tfrac{2}{27}(\sigma^2 V_m^2+3\lambda^2 L^2)^{\frac{3}{2}}. \tag{5.15}$$

For large values of $|\sigma|$, (5.15) requires that $\lambda\sigma\mu\nu$ be negative for instability, which agrees with (4.3) (as when (5.6) holds, it may be shown that $\varepsilon\sigma\gamma\approx\mu\nu$). However, for moderate values of $|\sigma|$, it may be shown that there is always a range of values of L for which (5.15) is not satisfied, and so the plane wave (5.13) is unstable. Indeed, as $\lambda^2 L^2 \to \infty$ for fixed σ (>0), (5.15) is satisfied. But if $\lambda^2 L^2$ is decreased the upper term on the right-hand side of (5.15) (i.e. the term with the negative sign) has a maximum of zero at $\lambda^2 L^2 = \sigma^2 V_m^2$, while the lower term has a minimum of zero at $\lambda^2 L^2 = 0$. If $\lambda\mu\nu\sigma$ is negative, it is the latter term which determines the stability behaviour, and the wave (5.13) is unstable for $L^2 < L_m^2$, where $\lambda^2 L_m^2$ is the value of $\lambda^2 L^2$ for which equality holds in (5.15) between the left-hand side and the lower term on the right-hand side. If $\lambda\mu\nu\sigma$ is positive, it is the upper term which determines the stability behaviour; the wave is unstable for $L_0^2 < L^2 < L_M^2$, where $L_0 = 0$ if $2\mu\nu\lambda\sigma|C|^2 > \frac{2}{27}|V_m|^2\sigma^4$; otherwise $\lambda^2 L_0^2$ and $\lambda^2 L_m^2$ are the values of $\lambda^2 L^2$ for which equality holds in (5.15) between the left-hand side and the upper term on the right-hand side. For fixed σ (<0), a similar analysis leads to the same conclusions regarding stability, but the role of the upper and lower terms in (5.15) is interchanged.

Appendix

Derivation of the compatibility condition

In this Appendix we shall sketch the procedure for solving equation (2.6) for $n \neq 0$. Expanding the operator on the left-hand side in powers of ϵ, (2.6) may be written in the form

$$L\left(-in\omega_m, in\kappa, \frac{\partial}{\partial x}\right)\zeta_n = \mathscr{F}_n. \tag{A.1}$$

Here $\mathscr{F}_n$ contains M_n and terms of $O(\varepsilon\zeta_n)$ arising from the expansion of the operator. If ζ_n is expanded in powers of ϵ (and α), we obtain a sequence of problems of the type (A.1) in each of which $\mathscr{F}_n$ may be regarded as known. Although we shall not carry out such an expansion explicitly, we may nevertheless proceed to solve (A.1), regarding $\mathscr{F}_n$ as known. Consider the homogeneous equation

$$\left.\begin{aligned} &L\left(-in\omega_m, in\kappa, \frac{\partial}{\partial x}\right)\psi = 0, \\ \text{or}\quad &in\omega_m\{(h\psi_x)_x - \mu^2(1-(n\omega_m)^2)\psi - \kappa^2 h\psi - c_m^{-1} h_x\psi\} = 0. \end{aligned}\right\} \tag{A.2}$$

We let ψ_1 be that solution of (A.2) which is finite at $x = 0$ (say $\psi_1(0) = 1$), and let ψ_2 be an independent solution of (A.2). Frobenius theory shows that we may put

$$\psi_2 = \psi_1 \log x + \tilde{\psi}_1, \tag{A.3}$$

where $\tilde{\psi}_1$ is finite at $x = 0$. Then the general solution of (A.1) is

$$in\omega_m W\zeta_n = C_1\psi_1 + C_2\psi_2 + \psi_1\int_0^x \mathscr{F}_n\psi_2\,dx - \psi_2\int_0^x \mathscr{F}_n\psi_1\,dx. \tag{A.4}$$

Here $W = h(\psi_2\psi_{1x} - \psi_1\psi_{2x})$ is the Wronskian and is a non-zero constant $(h_x(0))$, while C_1 and C_2 are arbitrary constants. Now $\mathscr{F}_n$ is finite at $x = 0$, and since we require ζ_n to be finite at $x = 0$, it follows that $C_2 = 0$.

We must also impose a condition as $x \to \infty$. Let

$$\gamma^2 = \mu^2(1-(n\omega_m)^2) + (n\kappa)^2. \tag{A.5}$$

Huthnance [10] has shown that for shelf waves $\mu^2 c_m^2 < 1$, and so γ is real and positive. Hence, as $x \to \infty$,

$$\left.\begin{aligned} &\psi_1 = \beta_{11}\chi_1 + \beta_{12}\chi_2, \quad \psi_2 = \beta_{21}\chi_1 + \beta_{22}\chi_2, \\ \text{where}\quad &\chi_1 \sim \exp(-\gamma x), \quad \chi_2 \sim \exp(\gamma x), \quad \text{as } x \to \infty. \end{aligned}\right\} \tag{A.6}$$

Here β_{11}, etc. are constants, and forming the Wronskian it may be shown that $(\beta_{11}\beta_{22}-\beta_{21}\beta_{12})$ is not zero. Substituting (A.6) into (A.4) it follows that

$$in\omega_m W\zeta_n = C_1(\beta_{11}\chi_1+\beta_{12}\chi_2)+(\beta_{11}\beta_{22}-\beta_{21}\beta_{12}) \times\left\{\chi_1\int_0^x \mathscr{F}_n\chi_2\,dx-\chi_2\int_0^x \mathscr{F}_n\chi_1\,dx\right\}. \tag{A.7}$$

Now $\mathscr{F}_n$ vanishes as $x\to\infty$, and we require that ζ_n should vanish as $x\to\infty$. It follows that

$$C_1\beta_{12} = (\beta_{11}\beta_{22}-\beta_{21}\beta_{12})\int_0^\infty \mathscr{F}_n\chi_1\,dx. \tag{A.8}$$

If β_{12} is not zero, then this equation determines C_1 and hence ζ_n uniquely; also, if M_n is $O(\alpha^n)$, it follows from (2.6) and (A.8) that ζ_n is $O(\alpha^n)$. However, if β_{12} vanishes, then (A.8) becomes a compatibility condition on $\mathscr{F}_n$. For $n=1$, ψ_1 is clearly $\phi_n(x)$ and β_{12} is zero. Then (A.8) is the compatibility condition

$$\int_0^\infty \mathscr{F}_1\phi_m\,dx = 0. \tag{A.9}$$

Without any loss of generality, we may put C_1 equal to zero and then (A.7) is the required solution. For $n\geqslant 2$, we are not free to impose a compatibility condition and so we must assume that β_{12} is not zero. Indeed, if β_{12} is zero, then ψ_1 is a shelf wave for the wavenumber $(n\kappa)$ and $c_m(\kappa)=c_s(n\kappa)$ for some integer s and $n\geqslant 2$. We shall assume that such nth harmonic resonances are absent and then (A 7) and (A 8) determine ζ_n uniquely for $n\geqslant 2$.

References

[1] D. J. Benney and A. C. Newell, "The propagation of nonlinear wave envelopes", *J. Math. and Physics* 46 (1967), 133–139.

[2] V. T. Buchwald and J. K. Adams, "The propagation of continental shelf waves", *Proc. Roy. Soc.* 305A (1968), 235–250.

[3] D. E. Cartwright, "Extraordinary tidal currents near St Kilda", *Nature* 223 (1969), 928–932.

[4] A. E. Gill and E. H. Schumann, "The generation of long shelf waves by the wind", *J. Phys. Ocean* 4 (1974), 83–90.

[5] R. Grimshaw, "The modulation and stability of an internal gravity wave", *Mém. Soc. Roy. Sc. Liège*, 6e serie, X (1976) 299–314.

[6] R. Grimshaw, "Nonlinear aspects of long shelf waves", *Geophys. Astrophys. Fluid Dynamics* 8 (1977), 3–16.

[7] R. Grimshaw, "The stability of continental shelf waves I. Sideband instability and long wave resonance", School of Mathematical Sciences Research Report No. 24, University of Melbourne (1976).

[8] R. Grimshaw, "The modulation of an internal gravity wave packet, and the resonance with mean motion", *Stud. Appl. Maths.* (1977) (to appear).

[9] H. Hasimoto and H. Ono, "Nonlinear modulation of gravity waves", *J. Phys. Soc. Japan* 33(1972), 805–811.

[10] J. M. Huthnance, "On trapped waves over a continental shelf", *J. Fluid Mech.* 69 (1975), 689–704.

[11] P. K. Kundu, J. S. Allen and R. L. Smith, "Modal decomposition of the velocity field near the Oregon coast", *J. Phys. Ocean.* 5 (1975), 683–704.
[12] M. E. McIntyre, "Mean motions and impulse of a guided internal gravity wave packet", *J. Fluid Mech.* 60 (1973), 801–811.
[13] R. A. Plumb, "The stability of small amplitude Rossby waves in a channel" *J. Fluid Mech.* (1977) (to appear).
[14] R. Smith, "Nonlinear Kelvin and continental-shelf waves", *J. Fluid Mech.* 52 (1972), 379–391.
[15] V. E. Zakharov and A. B. Shabat "Exact theory of two-dimensional self-forming and one-dimensional self-modulation of waves in nonlinear media", *Soviet Physics JETP* 34 (1972), 62–69.

Department of Mathematics
University of Melbourne
Parkville, Vic. 3052
Australia