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OSCILLATION CRITERIA FOR MATRIX DIFFERENTIAL INEQUALITIES(1)

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Several authors have recently considered the problem of obtaining sufficient conditions for the oscillation of the quasilinear matrix differential equation

$$(1) \qquad LV \equiv (A(x)V')' + B(x, V, V')V = 0$$

and the associated inequality $V^T LV \leq 0$ (as a form). Here A, B, and V are $m \times m$ matrix functions, $A(x)$ is symmetric, positive semidefinite and continuous in an interval $[a, \infty)$ and $B(x, V, V')$ is symmetric and continuous in $a \leq x < \infty$ for all V and V'.

Tomastik in 1968 [7], and more recently [8], obtained sufficient conditions for the oscillation of (1) under the additional assumption that both A and B are positive definite. Kreith [2], then obtained a different oscillation criterion for (1) without requiring B to be positive definite.

More recently, Noussair and Swanson [5], by using direct methods from the Calculus of Variations, earlier employed by Swanson [6] for systems of ordinary differential equations and by Allegretto and Swanson [1] for systems of partial differential equations, showed that (1) was oscillatory (without any additional hypotheses) if, in particular, there exist diagonal elements A_{kk}, B_{kk} of A and B respectively, such that

$$\int_a^\infty \frac{dx}{A_{kk}} = \int_a^\infty B_{kk}\, dx = +\infty.$$

This result which is established directly by Noussair and Swanson can also be obtained easily, unlike some of the other results in [5], by using methods employed by Kreith in [2], where a comparison theorem and Leighton's theorem [3] are employed.

It is the purpose of this note to obtain a class of more general oscillation criteria for equation (1) than those previously known. The results of Swanson and Noussair and Kreith, referred to above, will be obtained as special cases of our methods. Unlike many of the earlier results, our conditions, in general, will involve off-diagonal elements of A and B as well as their diagonal entries.

We remark that the techniques used here can also be extended to certain systems

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of partial differential inequalities and nonsymmetric operators by well-known methods.

The domain of L is assumed to be the set of $m\times m$ matrix functions $V\in C^1[a, \infty)$ such that LV exists and is continuous at every point.

It will also be convenient to apply L to m-vector functions ϕ, which is clearly possible under analogous assumptions on ϕ.

A matrix $V\in C^1[a, \infty)$ is said to be *prepared* relative to L iff

$$V^T(x)A(x)V' = (V')^T(x)A(x)V(x) \tag{2}$$

for all x in $[a, \infty)$.

A matrix differential inequality $V^TLV\leq 0$ is said to be oscillatory in $[a, \infty)$ iff the determinant of any prepared matrix solution V vanishes at some point in $[b, \infty)$ for all $b\geq a$.

It will also be useful to adopt the following terminology which was introduced by Marcus and Minc [4]. Let $Q_{k,n}$ denote the totality of strictly increasing sequences of k integers chosen from $1, \ldots, n$. If $T=(t_{j,l})$ denotes any $n\times n$ matrix and $\alpha=(i_1, \ldots, i_k)$ is any element of $Q_{k,n}$ then by $T(\alpha, \alpha)$ we shall mean the $k\times k$ submatrix of T whose (j, l) entry is given by (t_{i_j,i_l}) (i.e. $T(\alpha, \alpha)$ is obtained by deleting all rows and columns from T except for rows and columns $i_1, \ldots, i_k$).

By ΣT we shall mean the sum of all the entries of T.

We are now ready to state our main result:

THEOREM 1. *The inequality $V^TLV\leq 0$ is oscillatory in $[a, \infty)$ if there exists a vector function $\phi(x)$ such that, for every nonoscillatory differentiable matrix V in $[a, \infty)$ the scalar equation*

$$(pu')'+qu = 0 \tag{3}$$

is oscillatory in $[a, \infty)$, where

$$p(x) = \phi^T(x)A(x)\phi(x) > 0$$
$$q(x) = \phi^T(x)L(\phi)(x).$$

Proof. It is well known [5], [2], [6] that a sufficient condition for the oscillation of the inequality $V^TLV\leq 0$ is the existence, for any number b, of a number $c>b$ and a vector ψ with $\psi(b)=\psi(c)=0$ and for which

$$\int_b^c [(\psi')^TA\psi'-\psi^TB\psi]\,dx < 0.$$

Now if equation (3) is oscillatory, it follows from the Calculus of Variations that given any number b there exists a number $c>b$, and an admissible function u, such that $u(b)=u(c)=0$ and for which

$$\int_b^c [(u')^2p(x)-u^2q(x)]\,dx < 0.$$

But a simple calculation shows

$$\int_b^c [(u')^2\phi^T A\phi - u^2\phi^T L(\phi)]\,dx = \int_b^c [(u\phi)'^T A(u\phi) - (u\phi)^T B(u\phi)]\,dx$$

and oscillation of the inequality $V^T LV \leq 0$ follows.

REMARK. Inherent in the assumptions of Theorem 1 is the condition that $L(\phi)$ can be formed, i.e., that $A\phi'$ is differentiable. This condition can, in fact, be removed as follows: Let ϕ be an arbitrary C^1 vector function and let α, β, γ be C^1 scalar functions such that for every x and every nonoscillatory V the following inequality holds (in the sense of forms):

$$\begin{pmatrix} \phi^T A\phi & (\phi')^T A\phi \\ (\phi')^T A\phi & (\phi')^T A\phi' - \phi^T B\phi \end{pmatrix} \leq \begin{pmatrix} \alpha & \beta \\ \beta & -\gamma \end{pmatrix}$$

Then the theorem becomes: If the scalar equation $(\alpha u')' + (\gamma + \beta')u = 0$ is oscillatory, then so is the form $V^T LV \leq 0$.

Several oscillation criteria are now possible by specializing the choice of ϕ. Clearly the simplest choice for ϕ is ϕ=constant. By the well-known Leighton test [3], equation (3) is oscillatory if

$$\int^\infty \frac{1}{p}\,dt = \int^\infty q\,dt = +\infty.$$

Hence, we obtain:

COROLLARY 1. *If there exists an $\alpha \in Q_{k,m}$ such that for every nonoscillatory differentiable matrix V,*

$$\int_a^\infty [\sum A(\alpha, \alpha)]^{-1} = \int_a^\infty \sum B(\alpha, \alpha) = +\infty$$

then $V^T LV \leq 0$ is oscillatory.

Note that this criterion is better than the previous ones if, for example, the off-diagonal entries of $A(\alpha, \alpha)$ are negative, and those of $B(\alpha, \alpha)$ positive. If $\alpha \in Q_{1,m}$ then the Corollary is exactly the same as the result of Noussair and Swanson. The oscillation criterion of Kreith [2] can be obtained similarly.

The next simplest choice for ϕ is $\phi = f(x)\cdot\vec{c}$ where $f(x)$ is a scalar function and $\vec{c}$ is a constant vector. By such a choice we obtain the following corollary:

COROLLARY 2. *If there exists an $\alpha \in Q_{k,m}$ such that for every nonoscillatory differentiable matrix V,*

$$\int_a^\infty \frac{dx}{x\sum A(\alpha, \alpha)} = \int_a^\infty x\left(\sum B(\alpha, \alpha) - \frac{\sum A(\alpha, \alpha)}{4x^2}\right)dx = +\infty, \tag{4}$$

then $V^T LV \leq 0$ is oscillatory.

Proof. Let $\alpha=(i_1, \ldots, i_k)$ and set $\phi(x)=\sqrt{x}\,\vec{c}$ where $\vec{c}$ is a constant vector with ones in the $i_1, \ldots, i_k$ positions and zero elsewhere. If $A\phi' \in C^1$, then by Leighton's test [3], it follows that equation (3) will be oscillatory if

$$(5)\quad \int_a^\infty \frac{dx}{x\sum A(\alpha,\alpha)} = \int_a^\infty \sqrt{x}\left[\sum B(\alpha,\alpha)\sqrt{x}+\left(\sum A(\alpha,\alpha)\frac{1}{2\sqrt{x}}\right)'\right] dx = +\infty$$

But

$$\int_a^t \sqrt{x}\left[\sum B(\alpha,\alpha)\sqrt{x}+\left(\sum A(\alpha,\alpha)\frac{1}{2\sqrt{x}}\right)'\right] dx$$

$$= \int_a^t x\left(\sum B(\alpha,\alpha)-\frac{\sum A(\alpha,\alpha)}{4x^2}\right) dx+\tfrac{1}{2}\sum A(\alpha,\alpha)(t)-\tfrac{1}{2}\sum A(\alpha,\alpha)(a)$$

$$\geq \int_a^t x\left(\sum B(\alpha,\alpha)-\frac{\sum A(\alpha,\alpha)}{4x^2}\right) dx-\sum A(\alpha,\alpha)(a).$$

Therefore, (4) implies that (5) holds and hence $V^TLV\leq 0$ is oscillatory.

If $A\phi' \notin C^1$, then we employ the remark after Theorem 1 as follows: Let $\eta(x)$ denote a nonnegative C^1-function with

$$(6)\qquad \int_a^\infty (2\eta(x)-\sum A(\alpha,\alpha))^2\frac{1}{x\sum A(\alpha,\alpha)}\,dx < 1.$$

Clearly such a function exists. Note that

$$\begin{pmatrix} x\sum A(\alpha,\alpha) & \frac{1}{2}\sum A(\alpha,\alpha) \\ \frac{1}{2}\sum A(\alpha,\alpha) & \delta(x)\end{pmatrix} \leq \begin{pmatrix} \frac{3x}{2}\sum A(\alpha,\alpha) & \eta(x) \\ \eta(x) & \delta(x)+\gamma(x)\end{pmatrix}$$

where, for simplicity, we have set

$$\delta(x) = \frac{\sum A(\alpha,\alpha)}{4x}-x\sum B(\alpha,\alpha)$$

$$\gamma(x) = \left(\eta(x)-\frac{\sum A(\alpha,\alpha)}{2}\right)^2\cdot\frac{2}{x\sum A(\alpha,\alpha)}.$$

By the remark after Theorem 1, it follows that $V^TLV\leq 0$ will be oscillatory if

$$\int_a^\infty \frac{2\,dx}{3x\sum A(\alpha,\alpha)} = +\infty$$

and

$$(7)\qquad \int_a^\infty (-\delta-\gamma+\eta')\,dx = +\infty.$$

But $\int_a^\infty \gamma(x)\,dx$ is finite by (6) and $\int_a^t \eta'(x)\,dx=\eta(t)-\eta(a)\geq -\eta(a)$. Condition (7) then follows from (4). Note that if $\alpha \in Q_{1,m}$, $\alpha=(k)$ then (4) becomes

$$\int_a^\infty \frac{dx}{xa_{kk}} = \int_a^\infty x\left(b_{kk}-\frac{a_{kk}}{4x^2}\right) dx = +\infty.$$

If we further assume that a_{kk} is bounded above, then we can conclude that $V^TLV \leq 0$ is oscillatory if:

$$\int_a^\infty x\left(b_{kk} - \frac{a_{kk}}{4x^2}\right) dx = +\infty.$$

This Kneser-type result is better than the corresponding result of Noussair and Swanson, where it was also assumed that $b_{kk} - (a_{kk}/4x^2) > 0$.

Another interesting choice for ϕ can be made in terms of the eigenfunctions of A.

COROLLARY 3. *Let $\phi(x)$ denote a smooth normalized eigenvector of A with corresponding eigenvalue $\lambda(x)$ such that $\int_a^\infty dx/\lambda = +\infty$.*

If, for every nonoscillary differentiable matrix V,

$$\int_a^\infty [\phi^T B\phi - (\phi')^T A\phi']\, dx = +\infty,$$

then $V^TLV \leq 0$ is oscillatory.

Note that if $\|\phi\| = 1$, then $\phi^T A\phi' = \lambda\phi^T\phi' \equiv 0$. Hence,

$$\begin{aligned}\int_a^t \phi^T L(\phi)\, dx &= \int_a^t [\phi^T B + \phi^T(A\phi')']\, dx \\ &= \int_a^t [\phi^T B\phi - (\phi')^T A\phi' - (\phi^T A\phi')']\, dx \\ &= \int_a^t [\phi^T B\phi - (\phi')^T A\phi']\, dx.\end{aligned}$$

Therefore, the Leighton test [3] and Theorem 1 yield the result. If $A\phi'$ cannot be differentiated, we once again employ the method outlined after Theorem 1.

It is clear that the optimal choice of ϕ for any given problem will depend on the particular coefficients in question. To illustrate this remark, consider the following example:

Let $$A = \begin{pmatrix} 1 & \cos x \\ \cos x & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \delta(x) & \cos(x) \\ \cos x & \delta(x) \end{pmatrix}$$

where $\delta(x) \geq \delta_0 > -\frac{1}{4}$ and where $\delta_0 \leq 0$. Then if we choose $\phi = \begin{bmatrix} 1 \\ \cos x \end{bmatrix}$ we find that $\phi^T A\phi = 1 + 3\cos^2 x$, $\phi^T L\phi \geq 1 + 2\delta_0 - \cos^2 x$. Hence we can conclude that, with these coefficients the inequality $V^TLV \leq 0$ is oscillatory in $[1, \infty)$ by the Leighton test. This result does not seem to be easily obtainable by any of the previous criteria.

REFERENCES

1. W. Allegretto and C. A. Swanson, *Oscillation criteria for elliptic systems*, Proc. Amer. Math. Soc. **27** (1971), 325–330.

2. K. Kreith, *Oscillation criteria for nonlinear matrix differential equations*, Proc. Amer. Math. Soc. **26** (1970), 270–272.

3. W. Leighton, *On self-adjoint differential equations of second order*, J. London Math. Soc. **27** (1952), 37–47.

4. M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, Mass., 1964.

5. E. Noussair and C. A. Swanson, *Oscillation criteria for differential systems* (to appear).

6. C. A. Swanson, *Oscillation criteria for nonlinear matrix differential inequalities*, Proc. Amer. Math. Soc. **24** (1970), 824–827.

7. E. C. Tomastik, *Oscillation of nonlinear matrix differential equations of second order*, Proc. Amer. Math. Soc. **19** (1968), 1427–1431.

8. ———, *Oscillation of systems of second order differential equations*, J. Differential Equations, **9** (1971), 436–442.

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