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EXTENSIONS OF A THEOREM OF JORDAN ON PRIMITIVE PERMUTATION GROUPS

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Abstract

Let G be a primitive permutation group of finite degree n containing a subgroup H which fixes k points and has r orbits on Δ, the set of points it moves. An old and important theorem of Jordan says that if $r = 1$ and $k \geqslant 1$ then G is 2-transitive; moreover if H acts primitively on Δ then G is $(k + 1)$-transitive. Three extensions of this result are proved here: (i) if $r = 2$ and $k \geqslant 2$ then G is 2-transitive, (ii) if $r = 2$, $n > 9$ and H acts primitively on both of its two nontrivial orbits then G is k-primitive, (iii) if $r = 3$, $n > 13$ and H acts primitively on each of its three nontrivial orbits, all of which have size at least 3, then G is $(k - 1)$-primitive.

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Introduction

In 1871 Jordan proved the following result (see Theorem I of Jordan (1871) or Theorems 13.1, 13.2 of Wielandt (1964)): let G be a primitive permutation group on a finite set Ω and suppose that G has a subgroup H which fixes at least one point of Ω and is transitive on $\Delta = \operatorname{supp}(H)$, the set of points not fixed by H. Then G is 2-transitive; moreover if H is primitive on Δ then G is $(k + 1)$-fold transitive where $k = |\operatorname{fix} H|$.

This result has played a large part in the study of permutation groups since then (see for instance §13 of Wielandt (1964)). In fact Theorem I of Jordan (1871) gives more information than stated above: G is $(k + 1)$-fold transitive provided that H is transitive on Δ and for all $f \geqslant 1$, H admits at most one congruence on Δ of modulus f. Marggraff (1892) generalised this, showing that the same conclusion

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holds if H is transitive on Δ and for all $f \geqslant 1$, H admits at most f congruences on Δ of modulus f.

Two questions which arise from these results are: (a) what can be said about a primitive group G which has a subgroup H having r orbits on supp(H), where $r > 1$? (b) what more can be said if we make assumptions about the action of H on its orbits? The purpose of this paper is to give some answers to these questions in the cases $r = 2$ and $r = 3$. For $r = 2$ we prove Theorems 1 and 2:

THEOREM 1. *Let G be a primitive group of degree n on a finite set Ω and suppose that G has a subgroup H fixing at least two points of Ω and having two orbits Σ_1, Σ_2 on* supp(H). *Then G is 2-transitive.*

THEOREM 2. *If, with the hypotheses of Theorem 1, H is primitive on Σ_1 and on Σ_2 and $|\Sigma_1| \geqslant 3$, $|\Sigma_2| \geqslant 3$ then one of the following holds*:

(i) *G is k-fold primitive* (*where $k = |\text{fix } H|$*),

(ii) *$n = 9$, $k = 3$ and G is $ASL(2,3)$ or $AGL(2,3)$.*

Theorem 2 is best possible, for there are many simply primitive groups G in which the stabiliser G_α has two nontrivial orbits and acts primitively on both. The hypotheses of Theorem 2 are also considered in Antopolski (1971), where such a group G is shown to be 2-transitive.

For the case $r = 3$ we prove:

THEOREM 3. *Let G be primitive on a finite set Ω with a subgroup H such that $|\text{fix } H| = k$ and H has 3 orbits Σ_1, Σ_2, Σ_3 on* supp(H). *Suppose that H acts primitively on each Σ_i and $|\Sigma_i| \geqslant 4$ ($i = 1,2,3$). Then G is $(k-1)$-fold primitive.*

Again Theorem 3 is best possible, for there is a simply primitive group G in which $G_{\alpha\beta}$ has three nontrivial orbits and acts primitively on each of them (see Example 3.3). The general cases $r = 3$, $r = 4$, $r = 5$ are considered in Liebeck (1977), where it is proved that G has rank at most $r + 1$, thus verifying a conjecture of Wielandt (1971) in these cases. It is very likely that the methods of this paper will extend to prove further results for small values of r.

This paper is divided into four sections. In the first Theorem 1 is proved; the proof relies heavily on the graph theory associated with a permutation group as described in Neumann (1977). The second section consists of a proof of Theorem 2; here a different approach is taken—the orbits Σ_1, Σ_2 of H are 'built up' step by step until they become orbits of G_α for some $\alpha \in \Omega$. Then a theorem of O'Nan (1975) on 2-transitive but not 2-primitive groups is used to complete the proof. The proof of Theorem 3, given in the fourth section, runs along similar lines. In Section 3 some examples of primitive groups having subgroups with few orbits and several fixed points are presented.

The notation used is that of Wielandt (1964), except that for a subset Δ of Ω we write $G_{(\Delta)}$, $G_{\{\Delta\}}$ for the pointwise and setwise stabilisers of Δ in G, respectively.

1. Proof of Theorem 1

Before proving Theorem 1 we briefly outline the results in Neumann (1977) which we shall need. Let G be a transitive permutation group on a finite set Ω and let $\alpha \in \Omega$. There is a 1-1 correspondence between the orbits $\Delta_0, \Delta_1, \ldots, \Delta_s$ of G on $\Omega \times \Omega$ and the orbits $\Gamma_0, \Gamma_1, \ldots, \Gamma_s$ of G_α on Ω given by

$$\Gamma_i = \Gamma_i(\alpha) = \{\gamma \in \Omega \mid (\alpha, \gamma) \in \Delta_i\}.$$

For $i \geqslant 1$ we define the Γ_i-*graph* to be the directed graph on Ω in which there is an edge from β to γ if and only if $(\beta, \gamma) \in \Delta_i$. We denote by Γ_i^* the suborbit paired with Γ_i. If Γ_i is self-paired then the Γ_i-graph is undirected.

Suppose now that G is primitive on Ω. Then all the Γ_i-graphs are connected (Lemma 3 of Neumann (1977)). For any j, k define

$$\Delta_j \circ \Delta_k = \{(\beta, \gamma) \mid \beta \neq \gamma \text{ and there exists } \delta \in \Omega \text{ with } (\beta, \delta) \in \Delta_j, (\delta, \gamma) \in \Delta_k\}.$$

Let $\Gamma_j \circ \Gamma_k$ be the corresponding union of suborbits. The following result is taken from Theorems 3 and 4 of Neumann (1977).

RESULT A. (i) *Suppose that for $\beta \in \Gamma_i$, $G_{\alpha\beta}$ is transitive on Γ_j and that $|\Gamma_i| > 1$, $|\Gamma_j| > 1$. Then $\Gamma_i^* \circ \Gamma_j$ is a single suborbit of size greater than $|\Gamma_i|$ and $|\Gamma_j|$.*

(ii) *If $s \geqslant 2$ and G_α acts 2-transitively on Γ_i for some $i \geqslant 1$ then $\Gamma_i^* \circ \Gamma_i$ is a single suborbit of size greater than $|\Gamma_i|$.*

PROOF OF THEOREM 1. Let G be a primitive permutation group of degree n on Ω and assume that G has a subgroup H fixing at least 2 points and having 2 orbits Σ_1, Σ_2 on supp(H), the support of H (that is, $\Omega \setminus \text{fix } H$). Suppose for a contradiction that G is not 2-transitive. Let X be a subgroup of G with $|\text{supp}(X)|$ maximal subject to the following conditions:

(a) $|\text{fix } X| \geqslant 2$,

(b) X has at most 2 orbits on supp(X).

By Jordan's Theorem, X has precisely 2 orbits Δ_1, Δ_2 on supp(X). Write $\Delta = \Delta_1 \cup \Delta_2$ and $d_i = |\Delta_i|$ $(i = 1, 2)$.

LEMMA 1.0. *Suppose $g \in G$ is such that $\Delta_1 g \cap \Delta_1 \neq \varnothing$ and $\Delta_1 g \neq \Delta_1$. Then $\Delta_2 g \cap \Delta \neq \varnothing$ and $|\Delta g \cup \Delta| \geqslant n - 1$.*

PROOF. Let $Y = \langle X, X^g \rangle$. The nontrivial orbits of Y consist of unions among the sets Δ_1, Δ_2, $\Delta_1 g$ and $\Delta_2 g$. Suppose that $\Delta_2 g \cap \Delta = \varnothing$. Then the nontrivial orbits of Y are either

(i) $\Delta_2 g$ and $\Delta_1 g \cup \Delta_1 \cup \Delta_2$ (if $\Delta_1 g \cap \Delta_2 \neq \varnothing$), or

(ii) $\Delta_2 g$, Δ_2 and $\Delta_1 g \cup \Delta_1$ (if $\Delta_1 g \cap \Delta_2 = \varnothing$).

In case (i) we may choose $y \in X^g$ such that $\Delta_1 y \cap \Delta_1 \neq \varnothing$ and $\Delta_1 y \cap \Delta_2 \neq \varnothing$. Let $Z = \langle X, X^y \rangle$. If $|\mathrm{supp}(Z)| = |\mathrm{supp}(X)|$ then Z has just 1 nontrivial orbit, so G is 2-transitive by Jordan's Theorem, which is not so. Hence $|\mathrm{supp}(Z)| > |\mathrm{supp}(X)|$. However Z has at most 2 nontrivial orbits and certainly $\Delta_2 g \subseteq \mathrm{fix}\, Z$, so $|\mathrm{fix}\, Z| \geq 2$ and Z contradicts our choice of X. In case (ii) we may pick $y \in X^g$ with $\Delta_1 y \cap \Delta_1 \neq \varnothing$ and $\Delta_1 y \neq \Delta_1$; then $\langle X, X^y \rangle$ gives a similar contradicition.

Hence $\Delta_2 g \cap \Delta \neq \varnothing$. Finally, suppose that $|\Delta g \cup \Delta| \leq n - 2$. Then $|\mathrm{fix}\, Y| \geq 2$. Clearly Y has at most 2 nontrivial orbits, so by choice of X we must have $\mathrm{supp}(Y) = \mathrm{supp}(X)$. But then since $\Delta_1 g \cap \Delta_1 \neq \varnothing$ and $\Delta_1 g \neq \Delta_1$, Y has just 1 nontrivial orbit, so G is 2-transitive by Jordan's Theorem. This contradiction shows that $|\Delta g \cup \Delta| \geq n - 1$, proving the lemma.

We continue the proof of Theorem 1 in a series of steps.

STEP 1.1. *We have* $|\mathrm{fix}\, X| < \frac{1}{2}n < |\Delta|$. For suppose that $|\Delta| \leq \frac{1}{2}n$. Since G is primitive we may choose $g \in G$ such that $\Delta_1 g \cap \Delta_1 \neq \varnothing$ and $\Delta_1 g \neq \Delta_1$. By Lemma 1.0 then, $\Delta_2 g \cap \Delta \neq \varnothing$ and $|\Delta g \cup \Delta| \geq n - 1$. However

$$|\Delta g \cup \Delta| = |\Delta g| + |\Delta| - |\Delta g \cap \Delta| \leq \tfrac{1}{2}n + \tfrac{1}{2}n - 2 = n - 2,$$

which is a contradiction.

STEP 1.2. *Let* $\alpha \in \mathrm{fix}\, X$ *and let* $\Gamma_1, \ldots, \Gamma_s$ $(s \geq 2)$ *be the nontrivial orbits of* G_α. *Suppose that* Δ_1 *and* Δ_2 *lie in different orbits, say* $\Delta_1 \subseteq \Gamma_1$, $\Delta_2 \subseteq \Gamma_2$. *Then* Δ_1, Δ_2 *are blocks of imprimitivity for* $G_\alpha^{\Gamma_1}$, $G_\alpha^{\Gamma_2}$ *respectively. Further, either* $\Delta_1 = \Gamma_1$ *or* $\Delta_2 = \Gamma_2$.

Note that the last part of Step 1.2 follows from the first, for if $\Delta_1 \subset \Gamma_1$ and $\Delta_2 \subset \Gamma_2$ then by the first part, $|\Delta_i| \leq \frac{1}{2}|\Gamma_i|$ $(i = 1, 2)$, so $|\Delta| \leq \frac{1}{2}n$, contradicting Step 1.1.

Suppose that Δ_1 is not a block for $G_\alpha^{\Gamma_1}$ and pick $g \in G_\alpha$ such that $\Delta_1 g \neq \Delta_1$ and $\Delta_1 g \cap \Delta_1 \neq \varnothing$. By Lemma 1.0, $\Delta_2 g \cap \Delta \neq \varnothing$ and $\Delta g \cup \Delta = \Omega \setminus \{\alpha\}$, from which it follows that $s = 2$, $\Delta_1 g \cup \Delta_1 = \Gamma_1$, $\Delta_2 g \cup \Delta_2 = \Gamma_2$ and $\Delta_2 g \cap \Delta_2 \neq \varnothing$. Let $\Theta_i = \Gamma_i \setminus \Delta_i$ $(i = 1, 2)$. Then $\Theta_1 \neq \varnothing$ since Δ_1 is not a block for $G_\alpha^{\Gamma_1}$, so $1 \leq |\Theta_1| < d_1$.

We complete Step 1.2 in the following stages:

(a) Θ_1 *is a block for* $G_\alpha^{\Gamma_1}$. For otherwise there exists $h \in G_\alpha$ with $\Theta_1 h \neq \Theta_1$ and $\Theta_1 h \cap \Theta_1 \neq \varnothing$, forcing $|\Delta h \cup \Delta| < n - 1$, which contradicts Lemma 1.0.

(b) Γ_1, Γ_2 *are self-paired suborbits of* G. For G_α acts 2-transitively on the block system for $G_\alpha^{\Gamma_1}$ which contains Θ_1. Hence 2 divides $|G|$ and (b) follows.

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(c) *We have* $\Delta_2 \subset \Gamma_2$. Suppose on the contrary that $\Delta_2 = \Gamma_2$. Then if $\beta \in \Theta_1$ we have $X \leqslant G_{\alpha\beta}$, so $G_{\alpha\beta}$ is transitive on Γ_2. But then by Result A, $\Gamma_2 \circ \Gamma_1$ is a suborbit of G of size greater than $|\Gamma_1|$ and $|\Gamma_2|$, which is a contradiction.

Hence $1 \leqslant |\Theta_2| < d_2$ and as for (a), Θ_2 is a block for $G_\alpha^{\Gamma_2}$. For $i = 1, 2$ let $\mathfrak{B}_i$ be the block system for $G_\alpha^{\Gamma_i}$ containing Θ_i and let $k_i = |\mathfrak{B}_i|$.

(d) *We have* $G_{\alpha\{\Theta_1\}} = G_{\alpha\{\Theta_2\}}$. *Also* $k_1 = k_2 = k$ *and there exist* $g_2, \ldots, g_k \in G_\alpha$ *such that* $\mathfrak{B}_i = \{\Theta_i, \Theta_i g_2, \ldots, \Theta_i g_k\}$ $(i = 1, 2)$. For if there exists $h \in G_{\alpha\{\Theta_2\}} \setminus G_{\alpha\{\Theta_1\}}$ then $|\Delta h \cup \Delta| < n - 1$, contradicting Lemma 1.0. The last part follows if we take $\{1, g_2, \ldots, g_k\}$ to be a set of coset representatives for $G_{\alpha\{\Theta_1\}}$ in G.

As described at the beginning of this section, for $\beta \in \Omega$ we write $\Gamma_1(\beta)$, $\Gamma_2(\beta)$ for the nontrivial orbits of G_β (so $\Gamma_1 = \Gamma_1(\alpha)$, $\Gamma_2 = \Gamma_2(\alpha)$).

(e) *One of the following holds*:

(i) $\beta \in \Gamma_1(\gamma)$ *for all* $\beta \in \Theta_1, \gamma \in \Delta_2$;

(ii) $\beta \in \Gamma_2(\gamma)$ *for all* $\beta \in \Theta_2, \gamma \in \Delta_1$.

For suppose that (ii) is false. Then there exist $\beta \in \Theta_2, \gamma \in \Delta_1$ such that $\beta \in \Gamma_1(\gamma)$. Since $X \leqslant G_{\alpha\beta}$ and X is transitive on Δ_1 this means that $\beta \in \Gamma_1(\gamma)$ for all $\gamma \in \Delta_1$. It follows that $\beta g_2 \in \Gamma_1(\gamma g_2)$ for all $\gamma \in \Delta_1$; however $\beta g_2 \in \Delta_2$ and $\Theta_1 \subseteq \Delta_1 g_2$. Hence $\beta' \in \Gamma_1(\gamma')$ for some $\beta' \in \Theta_1$, $\gamma' \in \Delta_2$ and we see that (i) holds by considering the actions of $G_{\alpha\{\Theta_1\}}$ and X.

We assume without loss of generality that (i) holds.

(f) *Let* $t_i = |\Theta_i|$ $(i = 1, 2)$. *Then* $t_1 > d_2$. Suppose that $t_1 \leqslant d_2$. First we show that the Γ_1-graph has no triangles. To do this we choose $\beta \in \Theta_1$ and show that β is joined in the Γ_1-graph to no point of Γ_1. By (e), $\beta \in \Gamma_1(\gamma)$ for all $\gamma \in \Delta_2$. If $\beta \in \Gamma_1(\delta)$ for some $\delta \in \Delta_1$ then $\beta \in \Gamma_1(\delta)$ for all $\delta \in \Delta_1$ (by the action of X), so the valency $v_1(\beta)$ of β in the Γ_1-graph satisfies

$$v_1(\beta) \geqslant d_1 + d_2 + 1.$$

However $v_1(\beta) = |\Gamma_1| = d_1 + t_1$, from which it follows that $t_1 > d_2$, contrary to assumption. Hence β is joined in the Γ_1-graph to no point of Δ_1. If $\beta \in \Gamma_1(\delta)$ for some $\delta \in \Theta_1$ then by (e), β and δ have at least d_2 mutual adjacenices in the Γ_1-graph. However $\alpha \in \Gamma_1(\beta)$ and α, β have at most $t_1 - 1$ mutual adjacencies in the Γ_1-graph, so $t_1 - 1 \geqslant d_2$ which is not so. Hence β is joined in the Γ_1-graph to no point of Γ_1 and we have shown that the Γ_1-graph has no triangles.

Now let $\beta' \in \Theta_2$ and suppose $\beta' \in \Gamma_1(\gamma)$ for some $\gamma \in \Theta_2$. Then $\beta' g_2 \in \Gamma_1(\gamma g_2)$ and $\beta' g_2, \gamma g_2 \in \Delta_2$, so $\beta, \beta' g_2, \gamma g_2$ form a triangle in the Γ_1-graph, which is a contradiction. Since the Γ_2-graph is connected (Lemma 3 of Neumann (1977)) there exists $\delta \in \Gamma_2$ with $\beta' \in \Gamma_1(\delta)$; it must be the case that $\delta \in \Delta_2$. Hence $\beta' \in \Gamma_1(\delta)$ for all $\delta \in \Delta_2$.

Finally, if $\beta' \in \Gamma_1(\gamma)$ for some $\gamma \in \Theta_1$ then β', γ, δ form a triangle in the Γ_1-graph for any $\delta \in \Delta_2$. Hence $\beta' \in \Gamma_1(\gamma)$ for all $\gamma \in \Delta_1$. Consequently

$$v_1(\beta') = d_1 + d_2.$$

But then $v_1(\beta) = d_1 + d_2$, so β must be joined in the Γ_1-graph to some point of $\Gamma_1 \cup \Theta_2$. This we have seen above to be impossible. This contradiction establishes (f).

Now we can complete Step 1.2. Let $\beta \in \Theta_1$. Then $v_1(\beta) = |\Gamma_1| = kt_1$, and $k \geqslant 3$. By (f) we have $t_1 > d_2 > t_2$, so

$$kt_1 \geqslant t_1 + 2t_1 > t_1 + d_2 + t_2 = |\Theta_1| + |\Gamma_2|.$$

Thus $v_1(\beta) > |\Gamma_2 \cup \Theta_1|$ and so $\beta \in \Gamma_1(\gamma)$ for some $\gamma \in \Delta_1$, hence for all $\gamma \in \Delta_1$. By (e) then, $\beta \in \Gamma_1(\gamma)$ for all $\gamma \in \Delta$ $(= \Delta_1 \cup \Delta_2)$. It follows that for each j, $\delta \in \Gamma_1(\varepsilon)$ for all $\delta \in \Theta_1 g_j$, $\varepsilon \in \Delta g_j$. Also $\Delta g_j = (\Gamma_1 \cup \Gamma_2) \setminus (\Theta_1 \cup \Theta_2) g_j$. Consequently in the Γ_2-graph, points of $\Theta_1 g_j$ can be joined only to points of $(\Theta_1 \cup \Theta_2) g_j$. Let $\beta \in \Theta_1$, $\gamma \in \Theta_1 g_2$. Then $\beta \in \Gamma_1(\gamma)$ and β, γ have no mutual adjacencies in the Γ_2-graph. However $\gamma \in \Gamma_1(\alpha)$ and α, γ have at least one mutual adjacency in the Γ_2-graph since the Γ_2-graph is connected.

This final contradiction completes Step 1.2.

STEP 1.3. *Let* $\alpha \in \operatorname{fix} X$ *and let* $\Gamma_1, \ldots, \Gamma_s$ *be the nontrivial orbits of* G_α. *Then each* Γ_i $(i = 1, \ldots, s)$ *is a union of sets of the form* $\Delta_j g$ $(j = 1$ *or* $2)$ *where* $g \in G$ *and* $X^g \leqslant G_\alpha$.

To see this, let

$$W = \langle X^g \mid g \in G, X^g \leqslant G_\alpha \rangle.$$

Then $W \lhd G_\alpha$ and W is weakly closed in G_α with respect to G, so by Theorem 3.5 of Wielandt (1964), $G_\alpha = N_G(W)$ is transitive on fix W. Hence fix $W = \{\alpha\}$ and so for any $\gamma \in \Gamma_i$ there exists $g_\gamma \in G$ such that $X^{g_\gamma} \leqslant G_\alpha$ and $\gamma \in \operatorname{supp}(X^{g_\gamma}) = \Delta g_\gamma$. Step 1.3 follows.

We can now finish the proof of Theorem 1. First suppose that Δ_1, Δ_2 lie in different suborbits of G, say $\Delta_1 \subseteq \Gamma_1$, $\Delta_2 \subseteq \Gamma_2$. By Step 1.2 we may assume that $\Delta_1 = \Gamma_1$. Choose t such that $\Gamma_t \cap \operatorname{fix} X \neq \varnothing$ and

$$|\Gamma_t| = \max\{|\Gamma_i| : \Gamma_i \cap \operatorname{fix} X \neq \varnothing\},$$

so that $t \geqslant 2$. Let $\beta \in \Gamma_t \cap \operatorname{fix} X$; then $X \leqslant G_{\alpha\beta}$. Hence if $\Delta_2 = \Gamma_2$ then by Result A, one of $\Gamma_t^* \circ \Delta_1$ and $\Gamma_t^* \circ \Delta_2$ is a suborbit of G of size greater than $|\Gamma_t|$, $|\Delta_1|$ and $|\Delta_2|$, which is clearly impossible. And if $\Delta_2 \subset \Gamma_2$ then $\Gamma_t^* \circ \Delta_1$ gives a similar contradiction (for then $|\Gamma_t| \geqslant |\Gamma_2|$ by choice of Γ_t).

Thus we may suppose that Δ_1, Δ_2 lie in the same suborbit of G, say $\Delta_1 \cup \Delta_2 \subseteq \Gamma_1$. By Step 1.3 there exists $g \in G$ such that $X^g \leqslant G_\alpha$ and, say, $\Delta_1 g \subseteq \Gamma_2$. Certainly $|\Gamma_2| < |\Delta|$ by Step 1.1, so $\Delta_2 g \subseteq \Gamma_t$ for some $t \neq 2$. Hence $\Delta_1 g$, $\Delta_2 g$ lie in different suborbits of G and now the argument of the previous paragraph yields a contradiction with X^g replacing X and $\Delta_i g$ replacing Δ_i $(i = 1, 2)$.

This completes the proof of Theorem 1.

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2. Proof of Theorem 2

Before embarking on the proof of Theorem 2 we prove a lemma necessary to the proof.

LEMMA 2.0. *Let G be a 2-transitive permutation group of degree $n \geqslant 9$ on Ω, and suppose that for $\alpha \in \Omega$, G_α has a system $\mathfrak{B}$ of blocks of imprimitivity in $\Omega \setminus \{\alpha\}$ with blocks of size 2. Assume further that if $B = \{\beta, \gamma\} \in \mathfrak{B}$ then $G_{\alpha\beta\gamma}$ acts primitively on $\mathfrak{B} \setminus \{B\}$. Then $n = 9$ and G is $ASL(2,3)$ or $AGL(2,3)$.*

PROOF. If $n = 9$ the conclusion is easily seen to be true (see Sims (1970)), so assume that $n > 9$. Let $B_1 = \{\delta, \varepsilon\} \in \mathfrak{B} \setminus \{B\}$ and write $L = G_{\alpha\beta\gamma}$. Now if $L_{\{\delta\varepsilon\}}$ has an orbit on $\mathfrak{B} \setminus \{B, B_1\}$ of size 2 or 1 then since it is primitive, $L^{\mathfrak{B}\setminus\{B\}}$ has order p or $2p$ where $p = \frac{1}{2}(n-3)$ is prime. But then G contains an element which is a product of 1 or 2 p-cycles and fixes at least 3 points, contradicting Theorem 13.10 of Wielandt (1964). Hence every orbit of $L_{\{\delta\varepsilon\}}$ on $\mathfrak{B} \setminus \{B, B_1\}$ has size 3 or more, so $G_{\alpha\beta\gamma\delta\varepsilon}$ fixes no points in $\Omega \setminus \{\alpha, \beta, \gamma, \delta, \varepsilon\}$. This means that the unique third fixed point of $G_{\beta\delta}$ lies in $\{\alpha, \gamma, \varepsilon\}$, which is impossible as $G_{\beta\delta\alpha}$, $G_{\beta\delta\gamma}$, $G_{\beta\delta\varepsilon}$ fix γ, α, α respectively.

Now let G be primitive of degree n on Ω and suppose that G has a subgroup H which has 2 nontrivial orbits Σ_1, Σ_2 of size at least 3 and acts primitively on both of them. If $n = 9$ it is easy to see that (ii) of Theorem 2 holds, so we assume that $n > 9$. To prove Theorem 2 it is enough to show that G is 2-primitive if $k \geqslant 2$ (where $k = |\operatorname{fix} H|$), for then the obvious induction argument will establish the result. Suppose then that $k \geqslant 2$ but G is not 2-primitive. Let $\Sigma = \Sigma_1 \cup \Sigma_2$ and define

$$\mathfrak{S} = \{x \in G \mid \Sigma_i x \cap \Sigma \neq \varnothing \text{ and } \Sigma_i x \not\subseteq \Sigma \text{ for some } i \in \{1,2\}\}$$

so that $\mathfrak{S} \neq \varnothing$ by Theorem 8.1 of Wielandt (1964). Pick $g \in \mathfrak{S}$ with $|\Sigma g \cup \Sigma|$ as small as possible. We may assume that $\Sigma_1 g \cap \Sigma \neq \varnothing$ and $\Sigma_1 g \not\subseteq \Sigma$.

STEP 2.1. *We have $\Sigma_2 g \cap \Sigma \neq \varnothing$.* For suppose that $\Sigma_2 g \cap \Sigma = \varnothing$. Now $|\Sigma_1 g \cap \Sigma_j| > 0$ for some $j \in \{1,2\}$. If $|\Sigma_1 g \cap \Sigma_j| \geqslant 2$ choose $\alpha, \beta \in \Sigma_1 g \cap \Sigma_j$. Since H^g is primitive on $\Sigma_1 g$ there exists $x \in H^g$ such that

$$\alpha x \in \Sigma, \qquad \beta x \in \Sigma_1 g \setminus \Sigma.$$

Then $x \in \mathfrak{S}$. But $\Sigma_2 g \not\subseteq \Sigma x$, so $|\Sigma \cup \Sigma x| < |\Sigma \cup \Sigma g|$, contradicting the choice of g. And if $|\Sigma_1 g \cap \Sigma_j| = 1$, say $\Sigma_1 g \cap \Sigma_j = \{\alpha\}$, then any $x \in H^g$ with $\alpha x \in \Sigma_1 g \setminus \Sigma$ contradicts the choice of g.

Let $H_1 = \langle H, H^g \rangle$. By Step 2.1, H_1 has d nontrivial orbits $\Delta_1, \ldots, \Delta_d$ where d is 1 or 2. Write $\Theta_i = \Delta_i \setminus \Sigma$ $(i = 1, \ldots, d)$.

STEP 2.2. *If* $d = 2$ *then* $|\Theta_i| \leqslant 1$ *for* $i = 1, 2$. We prove this for $i = 1$; the case $i = 2$ is covered by the same argument. Suppose first that $|\Theta_1| \geqslant |\Sigma_1|$. Then by choice of g, Σ_1 is a block for $H_1^{\Delta_1}$. This forces $\Sigma_1 \subseteq \Sigma g$, for if not then there exists $x \in H^g$ fixing $\Sigma_1 \setminus \Sigma g$ and mapping an element of Σ_1 to an element of Θ_1, contradicting the fact that Σ_1 is a block for $H_1^{\Delta_1}$. However, since H^g is primitive on $\Sigma_i g$ we have $|\Sigma_1 \cap \Sigma_i g| \leqslant 1$ for $i = 1, 2$. This forces $|\Sigma_1| \leqslant 2$ as $\Sigma_1 \subseteq \Sigma g$. This contradiction shows that $|\Theta_1| < |\Sigma_1|$. Hence by choice of g, Θ_1 is a block for $H_1^{\Delta_1}$ and consequently $|\Theta_1| \leqslant 1$.

STEP 2.3. *If* $d = 1$ *then* $|\Theta_1| \leqslant 2$ *and* Θ_1 *is a block for* $H_1^{\Delta_1}$. To see this, suppose that $|\Theta_1| \geqslant 3$. Then $|\Sigma_i g \setminus \Sigma| \geqslant 2$ for some i, so since H^g is primitive on $\Sigma_i g$ there exists $h \in H^g$ such that $\Theta_1 h \neq \Theta_1$ and $\Theta_1 h \cap \Theta_1 \neq \varnothing$. By choice of g it must be the case that $h \notin \mathbb{S}$, so we may assume that

$$\Sigma_1 h \subseteq \Sigma \quad \text{and} \quad \Sigma_2 h \subseteq \Theta_1.$$

This implies that $\Sigma_2 \subseteq \Sigma g$, and so

$$|\Theta_1| = |\Sigma g| - |\Sigma g \cap \Sigma| = |\Sigma| - |\Sigma_2| - |\Sigma g \cap \Sigma_1| < |\Sigma_1|.$$

First we show that $H_1^{\Delta_1}$ is primitive. Let Γ be a block for $H_1^{\Delta_1}$ with $\Gamma \cap \Sigma_2 \neq \varnothing$ and $|\Gamma| > 1$. Now $\Gamma \cap \Sigma_2$ is a block for H^{Σ_2}, so either $\Sigma_2 \subseteq \Gamma$ or $|\Gamma \cap \Sigma_2| = 1$. If $|\Gamma \cap \Sigma_1| = 1$ then $\Gamma \cap \Theta_1 = \varnothing$, $|\Gamma \cap \Sigma_1| = 1$ and $|\Gamma| = 2$. This forces Γh (h as above) to contain a point of Σ and a point of Θ_1, which is impossible as Γh is a block for $H_1^{\Delta_1}$. Hence $\Sigma_2 \subseteq \Gamma$, from which it follows without difficulty that $\Gamma = \Delta_1$, using the fact that $\Sigma_j g$ intersects both Σ_1 and Σ_2 nontrivially for some j (since $d = 1$).

Hence $H_1^{\Delta_1}$ is primitive. It follows from Theorem 1 that $H_1^{\Delta_1}$ is also 2-transitive (this can also be deduced here using elementary arguments).

Finally, choose $\alpha \in \Theta_1$. Since $H_1^{\Delta_1}$ is 2-transitive there exists $k \in H_{1\alpha}$ such that $\Sigma_1 k \cap \Theta_1 \neq \varnothing$. Then certainly $\Sigma_1 k \not\subseteq \Sigma$. Since $|\Theta_1| < |\Sigma_1|$ we have $\Sigma_1 k \cap \Sigma \neq \varnothing$. Hence $k \in \mathbb{S}$. However, as $k \in H_{1\alpha}$ we have

$$\Sigma k \cup \Sigma \subseteq \Delta_1 \setminus \{\alpha\}$$

so that k contradicts the choice of g.

This contradiction shows that $|\Theta_1| \leqslant 2$. Clearly by choice of g, Θ_1 is a block for $H_1^{\Delta_1}$, so Step 2.3 is complete.

Note that in both cases $d = 1$ and $d = 2$ we have $|\Sigma g \setminus \Sigma| \leqslant 2$, so g is such that $|\Sigma g \cup \Sigma|$ is minimal subject to the condition $\Sigma g \neq \Sigma$. In particular $\Sigma g \cup \Sigma \subset \Omega$ since $k \geqslant 2$, and so $|\text{fix } H_1| \geqslant 1$.

Suppose now that $d = 1$, so that $|\Theta_1| \leqslant 2$ and Θ_1 is a block for $H_1^{\Delta_1}$. Let Γ be a proper block for $H_1^{\Delta_1}$ with $\Gamma \cap \Theta_1 \neq \varnothing$. If $\Sigma_1 \subseteq \Gamma$ or $\Sigma_2 \subseteq \Gamma$ then $\Gamma = \Delta_1$ since $\Sigma_i g$ intersects both Σ_1 and Σ_2 for some i. Hence $\Gamma \subseteq \Theta_1$. Thus the only proper block systems for $H_1^{\Delta_1}$ are the trivial one and the one containing Θ_1. If $|\operatorname{fix} H_1| \geqslant 2$ then Theorem I of Jordan (1871) (stated in the Introduction) shows that G is 3-transitive, which is not so. Hence $|\operatorname{fix} H_1| = 1$. But now if $|\Theta_1| = 1$ then $H_1^{\Delta_1}$ is primitive, so G is 2-primitive, which is not the case; and if $|\Theta_1| = 2$ then Lemma 2.0 gives a contradiction.

Consequently $d = 2$ and the orbits of H_1 are $\Sigma_i^1 = \Sigma_i \cup \Theta_i$ where $|\Theta_i| \leqslant 1$ $(i = 1, 2)$. Write $\Sigma^1 = \Sigma g \cup \Sigma$. Now define $g_1, g_2, \ldots$ inductively as follows: for $i = 1, 2, \ldots,$ $g_i \in G$ is chosen such that $|\Sigma^i g_i \cup \Sigma^i|$ is minimal subject to the condition $\Sigma^i g_i \neq \Sigma^i$; and $\Sigma^{i+1} = \Sigma^i g_i \cup \Sigma^i$. By the above considerations the group $H_{i+1} = \langle H_i, H_i^{g_i} \rangle$ has 2 nontrivial orbits $\Sigma_1^{i+1}, \Sigma_2^{i+1}$. Clearly there is a positive integer s such that $|\Omega \setminus \Sigma^s| = 1$. Then $\operatorname{fix} H_s = \{\alpha\}$, say, and H_s has the 2 nontrivial orbits

$$\Sigma_i^s = \Sigma_i \cup \Theta_i \cup \Theta_i^1 \cup \cdots \cup \Theta_i^{s-1} \qquad (i = 1, 2)$$

where all $|\Theta_i^j| \leqslant 1$. If $u_i = |\Theta_i \cup \Theta_i^1 \cup \cdots \cup \Theta_i^{s-1}|$ then H_s is $(u_i + 1)$-transitive on Σ_i^s. Since $k \geqslant 2$ we have $u_1 + u_2 \geqslant 1$.

We can now complete the proof of Theorem 2. If G is not 2-transitive then Σ_1^s, Σ_2^s are the nontrivial orbits of G_α, and G_α cannot be 2-transitive on both of them by Result A(ii). Hence we may assume that $\Sigma_1^s = \Sigma_1$. But then $\Sigma_1 \circ \Sigma_2^s$ is a suborbit of G of size greater than $|\Sigma_1|$ and $|\Sigma_2^s|$ by Result A(i), which is a contradiction.

Hence G is 2-transitive. Finally, since G is not 2-primitive, one of the following must hold:

(a) $\mathfrak{B} = \{\Sigma_1^s, \Sigma_2^s\}$ is a block system for G_α on $\Omega \setminus \{\alpha\}$,

(b) G_α has a block system on $\Omega \setminus \{\alpha\}$ with blocks of size 2, each containing 1 point from Σ_1^s and 1 from Σ_2^s.

In case (b) Lemma 2.0 gives a contradiction. In case (a) the kernel K of the action of G_α on $\mathfrak{B}$ contains H_s, so K acts 2-transitively on both of its orbits Σ_i^s $(i = 1, 2)$. Hence by Theorem D of O'Nan (1975), G is a normal extension of $PSL(m, q)$ in its natural 2-transitive representation on $PG(m-1, q)$ for some m, q. But the stabiliser of a point in such an extension of $PSL(m, q)$ cannot have a block system with 2 blocks.

This contradiction completes the proof of Theorem 2.

REMARK. An obvious modification of the proof (in Step 2.2) shows that the restrictions $|\Sigma_i| \geqslant 3$ in Theorem 2 are unnecessary providing we exclude the groups $PSL(2, 7)$ of degree 7 and $AGL(3, 2)$ of degree 8.

3. Some examples

In this section we present some examples of primitive groups having subgroups with several fixed points and few nontrivial orbits.

EXAMPLE 3.1. Let G be a group with $ASL(d, q) \leqslant G \leqslant AGL(d, q)$ acting on a d-dimensional vector space V over $GF(q)$ ($d \geqslant 2$, q a prime power). Then the pointwise stabiliser $G_{(W)}$ of a $(d-1)$-dimensional subspace W of V acts semiregularly with degree $q^d - q^{d-1}$ on $V \setminus W$ and has $|AGL(d, q): G|$ nontrivial orbits. This number of orbits can be any positive integer dividing $q - 1$; in particular it can be 2 (if q is odd) or 3 (if $q \equiv 1 (\bmod 3)$). Notice that the actions of $G_{(W)}$ on its orbits are primitive if and only if $d = 2$, q is prime and G is $ASL(d, q)$; if $d = 2$, $q = 3$ then $G_{(W)}$ has 2 orbits of size 3 and 3 fixed points (see conclusion (ii) of Theorem 2).

EXAMPLE 3.2. Let G be $PGL(d, q)$ ($d \geqslant 3$) acting on $PG(d-1, q)$, the set of 1-dimensional subspaces of V and let $\{v_1, \ldots, v_d\}$ be a basis for V. For $r \leqslant s$ put $\Delta_{rs} = \{\langle v \rangle | \langle v \rangle \subseteq \langle v_r, \ldots, v_s \rangle\}$. Then $G_{\alpha\beta}$ has two nontrivial orbits for any $\alpha, \beta \in PG(d-1, q)$. For $s \leqslant d-1$ let H_s be the subgroup $\{g \in G_{(\Delta_{1s})} | \Delta_{s+1, d}g \subseteq \Delta_{2d}\}$. Then H_s has two nontrivial orbits and fix $H_s = \Delta_{1s}$.

Since 'M_{21}' is $PSL(3, 4)$ acting on $PG(2, 4)$, considerations similar to the above give examples of subgroups of M_{22}, M_{23}, M_{24} (in their representations of degrees 22, 23, 24 respectively) having two nontrivial orbits of sizes either (a) 3 and 16 or (b) 8 and 8.

EXAMPLE 3.3. As promised in the Introduction here is an example of a simply primitive group G such that $G_{\alpha\beta}$ has 3 nontrivial orbits and acts primitively on each of them. Let G be the Higman-Sims simple group HS acting with degree 100, as described in Higman and Sims (1968). Then G has rank 3 and $G_\alpha \cong M_{22}$ has nontrivial orbits of sizes 22, 77. If β is a point in the orbit of size 22 then $G_{\alpha\beta} \cong PSL(3, 4)$ and $G_{\alpha\beta}$ has 3 nontrivial orbits of sizes 21, 21 and 56, acting 2-transitively on the orbits of size 21 and primitively on that of size 56.

In fact this is the only known example of a simply primitive group G in which $G_{\alpha\beta}$ has 3 primitive orbits; for it is easy to see that such a group must have rank 3 and act 2-primitively on one of its suborbits. The known examples of such rank 3 groups are listed in Atkinson (1977) and HS is the only one satisfying our requirements.

4. Proof of Theorem 3

As for the proof of Theorem 2 we require a preliminary lemma.

LEMMA 4.0. *Let G be a 2-transitive group of degree $n \geqslant 13$ on Ω and suppose that G_α has a system $\mathfrak{B}$ of blocks of imprimitivity with blocks of size 3 satisfying the following conditions*:

(i) *if $B = \{\beta, \gamma, \delta\} \in \mathfrak{B}$ the $G_{\alpha\beta\gamma\delta}$ has a subgroup H with 3 nontrivial orbits Σ_1, Σ_2, Σ_3 on $\Omega \setminus \{\alpha, \beta, \gamma, \delta\}$, each H^{Σ_i} being primitive*,

(ii) *for any $B' \in \mathfrak{B} \setminus \{B\}$, $|B' \cap \Sigma_i| = 1$ $(i = 1, 2, 3)$.*

Then $n = 13$ and G is $PSL(3,3)$.

PROOF. If $n = 13$ the conclusion is easily seen to be true (see Sims (1970)) so assume that $n \geqslant 14$. Let $B' = \{\beta', \gamma', \delta'\} \in \mathfrak{B} \setminus \{B\}$ with $\beta' \in \Sigma_1$, $\gamma' \in \Sigma_2$, $\delta' \in \Sigma_3$. Since $H \leqslant G_\alpha$ and B' is a block for G_α we have $H_{\beta'} = H_{\gamma'} = H_{\delta'}$. By the argument in the proof of Lemma 2.0 every nontrivial orbit of $G_{\alpha\beta\gamma\delta\beta'\gamma'\delta'}$ has size 3 or more. Now the G-translates of $\{\alpha\} \cup B$ form the blocks of a Steiner system $\mathfrak{S}(2, 4, n)$ (see Case 1 on page 274 of Atkinson (1973)); however by the previous sentence the unique block containing $\{\beta, \beta'\}$ must be contained in $\{\alpha, \beta, \gamma, \delta, \beta', \gamma', \delta'\}$, which is impossible.

Now let G be a primitive group of degree n on Ω and suppose that G has a subgroup H which has 3 nontrivial orbits Σ_1, Σ_2, Σ_3 of size at least 4 and acts primitively on each of them. To prove Theorem 3 it is sufficient to show that G is 2-primitive if $k \geqslant 3$. Suppose then that $k \geqslant 3$ and G is not 2-primitive. We mimic the proof of Theorem 2. Thus let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and define

$$\mathfrak{S} = \left\{x \in G | \Sigma_i x \cap \Sigma \neq \varnothing, \Sigma_i x \not\subseteq \Sigma \text{ for some } i \in \{1, 2, 3\}\right\}.$$

Choose $g \in \mathfrak{S}$ with $|\Sigma g \cup \Sigma|$ minimal. Let $H_1 = \langle H, H^g \rangle$ have nontrivial orbits $\Delta_1, \ldots, \Delta_d$ and write $\Theta_i = \Delta_i \setminus \Sigma$ $(i = 1, \ldots, d)$. Copying the proofs of Steps 2.1, 2.2 and 2.3 we have

STEP 4.1. *We have $\Sigma_i g \cap \Sigma \neq \varnothing$ for $i = 1, 2, 3$ (so $d \leqslant 3$). If $d = 3$ then $|\Theta_i| \leqslant 1$ $(i = 1, 2, 3)$ and if $d = 2$ then $|\Theta_1| \leqslant 2$, $|\Theta_2| \leqslant 1$ (where we take Δ_1 to contain two of the Σ_i).*

However the next step does not yield so easily.

STEP 4.2. *If $d = 1$ then $|\Theta_1| \leqslant 3$.* Suppose that $|\Theta_1| \geqslant 4$. As in the proof of Step 2.3 there exists $h \in H^g$ such that $\Sigma_m h \subseteq \Theta_1$ and $\Sigma_m \subseteq \Sigma g$ for some $m \in \{1, 2, 3\}$. We may take $m = 1$. Thus

$$\Sigma_1 h \subseteq \Theta_1, \qquad \Sigma_1 \subseteq \Sigma g.$$

Again it is easy to see that $H_1^{\Delta_1}$ is primitive, so by Jordan's Theorem $\Delta_1 = \Omega$.

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We first show that H_1 is 2-transitive. Let $\alpha \in \Theta_1$ and let $\Gamma_1, \ldots, \Gamma_s$ be the orbits of $H_{1\alpha}$ on $\Omega \setminus \{\alpha\}$. Suppose that H_1 is not 2-transitive, so that $s \geqslant 2$. For $i = 1, \ldots, s$ let $B_i = \{j \mid \Sigma_j \subseteq \Gamma_i\}$. We obtain a contradiction in the following three stages.

(A) $|B_i| \geqslant 2$ *for some* i. For suppose that $|B_i| \leqslant 1$ for all i. Then by choice of g, if $j \in B_i$ then Σ_j is a block for $H_{1\alpha}^{\Gamma_i}$. Since $|\Theta_1| < |\Sigma|$ it follows that $\Gamma_i = \Sigma_j$ for some i, j. Choose Γ_k of maximal size in

$$\{\Gamma_i | \Gamma_i = \Sigma_j \text{ for some } i,\ j\}$$

and choose Γ_l of maximal size in

$$\{\Gamma_i | \Gamma_i \cap \Theta_1 \neq \varnothing\}.$$

Then by Result A, $\Gamma_l^* \circ \Gamma_k$ is a suborbit of H_1 of size greater than $|\Gamma_l|$ and $|\Gamma_k|$, which is a contradiction.

(B) *If* j, $k \in B_i$ *for some* i *then* $|\Sigma_j| = |\Sigma_k|$. To see this, first suppose that $|B_i| = 2$ and $\Gamma_i \cap \Theta_1 \neq \varnothing$. Now if $\Sigma_j x \cap \Theta_1 \neq \varnothing$ for some $x \in H_{1\alpha}$ then $\Sigma_j x \subseteq \Theta_1$ by choice of g. Hence $H_{1\alpha}^{\Gamma_i}$ is imprimitive and it is easy to see that Σ_j, Σ_k are conjugate blocks for $H_{1\alpha}^{\Gamma_i}$, so that $|\Sigma_j| = |\Sigma_k|$.

Next, if $|B_i| = 3$ and $\Gamma_i \cap \Theta_1 \neq \varnothing$ then either Σ_j, Σ_k are conjugate blocks or $\mathfrak{B} = \{\Sigma_{i_1} \cup \Sigma_{i_2}, \Sigma_{i_3}, \Gamma_i \cap \Theta_1\}$ is a block system for $H_{1\alpha}^{\Gamma_i}$ (where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$). In the latter case the kernel K of the action of $H_{1\alpha}$ on $\mathfrak{B}$ contains H, so is primitive on Σ_{i_3} and hence on $\Sigma_{i_1} \cup \Sigma_{i_2}$. Thus we may pick $y \in K$ such that

$$\Sigma_{i_1} y \neq \Sigma_{i_1} \quad \text{and} \quad \Sigma_{i_1} y \cap \Sigma_{i_1} \neq \varnothing.$$

Then $\langle H, H^y \rangle$ has only 2 nontrivial orbits $\Sigma_{i_1} \cup \Sigma_{i_2}$ and Σ_{i_3}, so H_1 is 2-transitive by Theorem 1, contradicting our assumption. If $|B_i| = 3$ and $\Gamma_i \cap \Theta_1 = \varnothing$ we obtain a similar contradiction unless Σ_j, Σ_k are conjugate blocks for $H_{1\alpha}^{\Gamma_i}$.

Finally, if $|B_i| = 2$, $\Gamma_i \cap \Theta_1 = \varnothing$ and $|\Sigma_j| \neq |\Sigma_k|$ then $H_{1\alpha}^{\Gamma_i}$ is primitive. The third orbit Σ_l is contained in a block system $\mathfrak{B}$ for $H_{1\alpha}^{\Gamma_f}$ for some f, and the kernel K of the action of $H_{1\alpha}$ on $\mathfrak{B}$ contains an element y such that $\Sigma_j y \neq \Sigma_j$. Then $\langle H, H^y \rangle$ has only 2 nontrivial orbits and yields a contradiction by Theorem 1 again.

(C) *For some* i, H^{Σ_i} *is regular of prime degree*. Since $|\Theta_1| \geqslant 4$ there exists $i \in \{1, 2 3\}$ such that $|\Sigma_i g \setminus \Sigma| \geqslant 2$. Suppose that H^{Σ_i} is not regular of prime degree and choose $\alpha \in \Sigma_i g \setminus \Sigma$. We first show that $|\Sigma_1| = |\Sigma_2| = |\Sigma_3|$. If $|B_j| = 3$ for some j this follows from (B), so we suppose (using (A)) that

$$|B_1| = 1, \qquad |B_2| = 2.$$

Assume first that $1 \in B_1$, that is, $\Sigma_1 \subseteq \Gamma_1$. Recall that $\Sigma_1 \subseteq \Sigma g$ and $h \in H^g$ is such that $\Sigma_1 h \subseteq \Theta_1$. We have $|\Sigma_2| = |\Sigma_3|$ by (B).

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If $\Sigma_2 \subseteq \Sigma g$ then

$$|\Theta_1| = |\Sigma g| - |\Sigma g \cap \Sigma| < |\Sigma_3|$$

so $\Gamma_2 = \Sigma_2 \cup \Sigma_3$ (for otherwise there exists $x \in H_{1\alpha}$ such that $\Sigma_3 x \subseteq \Theta_1$, which is impossible). Hence for $\beta \in \Theta_1 \setminus \{\alpha\}$ we have

$$(\Sigma_2 \cup \Sigma_3) H_{1\beta} \neq \Sigma_2 \cup \Sigma_3$$

since otherwise $\Sigma_2 \cup \Sigma_3$ is a fixed set of $\langle H_{1\alpha}, H_{1\beta} \rangle = H_1$. Now (B) applied to the orbits of $H_{1\beta}$ gives $|\Sigma_1| = |\Sigma_2| = |\Sigma_3|$.

If $\Sigma_2 \not\subseteq \Sigma g$, $\Sigma_3 \not\subseteq \Sigma g$ then either $(\Sigma_2 \cup \Sigma_3)(H^g)_\alpha = \Sigma_2 \cup \Sigma_3$ or (B) applied to the orbits of $H_{1\alpha}$ gives $|\Sigma_1| = |\Sigma_2| = |\Sigma_3|$; so suppose $(\Sigma_2 \cup \Sigma_3)(H^g)_\alpha = \Sigma_2 \cup \Sigma_3$. Let $\beta \in \Sigma_i g \setminus \Sigma$ with $\beta \neq \alpha$. Since we have assumed that $(H^g)^{\Sigma_i g}$ is not regular we have

$$(H^g)^{\Sigma_i g} = \left\langle (H^g)_\alpha, (H^g)_\beta \right\rangle^{\Sigma_i g}.$$

It is easy to see that $\Sigma_i g$ must intersect $\Sigma_2 \cup \Sigma_3$ nontrivially, so

$$(\Sigma_2 \cup \Sigma_3)(H^g)_\beta \neq \Sigma_2 \cup \Sigma_3.$$

Now (B) applied to the orbits of $H_{1\beta}$ gives $|\Sigma_1| = |\Sigma_2| = |\Sigma_3|$.

We have now shown that $|\Sigma_1| = |\Sigma_2| = |\Sigma_3|$ if $1 \in B_1$. Similar arguments yield the same conclusion if $2 \in B_1$ or $3 \in B_1$. Hence in all cases,

$$|\Sigma_1| = |\Sigma_2| = |\Sigma_3|.$$

Now we may choose α ($\in \Sigma_i g \setminus \Sigma$) such that $\Sigma H_{1\alpha} \neq \Sigma$. We have

$$|\Theta_1| = |\Sigma g| - |\Sigma g \cap \Sigma| < |\Sigma g| - |\Sigma_1| = 2|\Sigma_1|.$$

By the argument of Step 1.3 any nontrivial orbit of $H_{1\alpha}$ has size at least $|\Sigma_1|$. Hence since $|\Theta_1| < 2|\Sigma_1|$ and $\Sigma H_{1\alpha} \neq \Sigma$, $H_{1\alpha}$ must have precisely 2 nontrivial orbits Γ_1, Γ_2 with either

(1) $\Sigma_{i_1} = \Gamma_1$, $\Sigma_{i_2} \cup \Sigma_{i_3} \subset \Gamma_2$, or

(2) $\Sigma_{i_1} \subset \Gamma_1$, $\Sigma_{i_2} \cup \Sigma_{i_3} = \Gamma_3$

where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. In case (1) the suborbit $\Gamma_2 \circ \Gamma_1$ gives the usual contradiction by Result A. In case (2), for any $\beta \in \Gamma_1 \setminus \Sigma_{i_1}$, one of the following possibilities must hold:

(a) the orbits of $H_{1\beta}$ are as in case (1) above,

(b) $H_{1\beta}$ has an orbit $\Sigma_i \cup \Sigma_j$ for some i, j,

(c) $\Sigma H_{1\beta} = \Sigma$.

Since $\langle H_{1\alpha}, H_{1\beta} \rangle = H_1$ for all $\beta \in \Gamma_1 \setminus \Sigma_{i_1}$ and $|\Gamma_1 \setminus \Sigma_{i_1}| \geq |\Sigma_{i_1}| \geq 4$, it is clear that (a) must hold for some β. This gives a contradiction as in case (1).

This finishes the proof of (C).

Now we can complete the proof of the 2-transitivity of H_1. By (C), some H^{Σ_i} is regular of prime degree p, say. Let K be the kernel of the action of H on Σ_i. If

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$K \neq 1$ then by Theorem 8.8 of Wielandt (1964), K has at most 2 nontrivial orbits, so H_1 is 2-transitive by Theorem 1. And if $K = 1$ then $|H| = p$ and so G contains an element which is a product of 1, 2 or 3 p-cycles, contradicting Theorem 13.10 of Wielandt (1964).

Hence H_1 is 2-transitive. We obtain a final contradiction, proving Step 4.2, by showing that H_1 is 2-primitive. It is easy to see that the only possible proper, nontrivial block systems $\mathfrak{B}$ for $H_{1\alpha}$ ($\alpha \in \Theta_1$) are

$$\{\Sigma_1, \Sigma_2, \Sigma_3, \Theta_1 \setminus \{\alpha\}\}, \qquad \{\Sigma_{i_1} \cup \Sigma_{i_2}, \Sigma_{i_3}, \Theta_1 \setminus \{\alpha\}\},$$
$$\{\Sigma_{i_1} \cup \Sigma_{i_2} \cup \Theta_1 \setminus \{\alpha\}, \Sigma_{i_3}\}$$

where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. In each case let K be the kernel of the action of $H_{1\alpha}$ on $\mathfrak{B}$. In the first two cases K does not restrict faithfully to its orbits on $\Omega \setminus \{\alpha\}$, and in the third case K is 2-transitive on each of its orbits (by Theorem 1). Now Theorem D and Proposition 4 of O'Nan (1975) show that none of these cases is possible.

This completes Step 4.2.

Note that in each of the cases $d = 1$, $d = 2$, $d = 3$ we have $|\Sigma g \setminus \Sigma| \leq 3$, so g is such that $|\Sigma g \cup \Sigma|$ is minimal subject to the condition $\Sigma g \neq \Sigma$. Hence $\Sigma g \cup \Sigma \subset \Omega$ since $k \geq 3$, and so $|\operatorname{fix} H_1| \geq 1$.

The argument presented after Step 2.3 (using Lemma 4.0 instead of Lemma 2.0) shows that d cannot be 1. Consequently either $d = 3$ and H_1 has orbits

$$\Sigma_i^1 = \Sigma_i \cup \Theta_i \qquad (|\Theta_i| \leq 1, i = 1, 2, 3)$$

or $d = 2$ and we may take H_1 to have orbits

$$\Sigma_1^1 = \Sigma_1 \cup \Sigma_2 \cup \Theta_1 \quad \text{and} \quad \Sigma_2^1 = \Sigma_3 \cup \Theta_2 \qquad (|\Theta_1| \leq 2, |\Theta_2| \leq 1).$$

Write $\Sigma^1 = \Sigma g \cup \Sigma$. Now for $i = 1, 2, \ldots$ choose $g_i \in G$ such that $|\Sigma^i g_i \cup \Sigma^i|$ is minimal subject to the condition $\Sigma^i g_i \neq \Sigma^i$. Write $\Sigma^{i+1} = \Sigma^i g_i \cup \Sigma^i$ and $H_{i+1} = \langle H_i, H_i^{g_i} \rangle$. For some r we have $\Sigma^r = \Omega \setminus \{\alpha\}$ for some $\alpha \in \Omega$. Then by Steps 4.1, 4.2 and the above reasoning (note however that we must use an obvious modification of Lemma 4.0 if some H_i has only 2 nontrivial orbits), the nontrivial orbits of H_r are of one of the following types:

(i) 3 orbits $\Sigma_i^r = \Sigma_i \cup \Theta_i \cup \Theta_i^1 \cup \cdots \cup \Theta_i^{r-1}$ ($i = 1, 2, 3$, all $|\Theta_i^j| \leq 1$),

(ii) 2 orbits $\Sigma_1^r = \Sigma_1 \cup \Sigma_2 \cup \Theta_1 \cup \Theta_1^1 \cup \cdots \cup \Theta_1^{r-1}$ and

$$\Sigma_2^r = \Sigma_3 \cup \Theta_2 \cup \Theta_2^1 \cup \cdots \cup \Theta_2^{r-1}, \quad (\text{all } |\Theta_1^j| \leq 2, |\Theta_2^j| \leq 1, |\Theta_1| \leq 2, |\Theta_2| \leq 1).$$

STEP 4.3. *G is 2-transitive*. Suppose false; then the rank of G is 3 or 4. If it is 4 then the nontrivial orbits of G_α are Σ_i^r ($i = 1, 2, 3$) and by Result A(ii), $\Sigma_i^r = \Sigma_i$ for some i. Hence if Σ_j is of maximal size in $\{\Sigma_i \mid \Sigma_i^r = \Sigma_i\}$ and Σ_k^r is of maximal

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size in $\{\Sigma_i^r \mid \Sigma_i \subset \Sigma_i^r\}$ then by Result A(i), $\Sigma_j^* \circ \Sigma_k^r$ is a suborbit of G of size greater than $|\Sigma_j|$ and $|\Sigma_k^r|$, which is impossible. Consequently G has rank 3 and G_α has as orbits either

$\Gamma_1 = \Sigma_1^r \cup \Sigma_2^r$, $\Gamma_2 = \Sigma_3^r$ (Σ_i^r as in (i) above), or

$\Gamma_1 = \Sigma_1^r$, $\Gamma_2 = \Sigma_2^r$ (Σ_i^r as in (ii) above).

We suppose that the first case holds; it will easily be seen that the ensuing argument also applies to the second case.

Thus we are assuming that $\Gamma_1 = \Sigma_1^r \cup \Sigma_2^r$, $\Gamma_2 = \Sigma_2^r$. Write $\Theta_i^0 = \Theta_i$ and let

$$\Delta_i = \Sigma_i^r \setminus \Theta_i^{r-1} \quad \text{and} \quad d_i = |\Delta_i| \qquad (i = 1, 2, 3).$$

Since H_r is 2-transitive on some Σ_i^r, $|G|$ is even and so Γ_1, Γ_2 are self-paired suborbits of G. We deal separately with the cases I. $|\Theta_1^{r-1} \cup \Theta_2^{r-1}| > 0$ and II. $|\Theta_1^{r-1} \cup \Theta_2^{r-1}| = 0$.

I. *The case* $|\Theta_1^{r-1} \cup \Theta_2^{r-1}| > 0$. Choose $\gamma \in \Theta_1^{r-1}$, say. If $|\Theta_3^{r-1}| = 0$ then $G_{\alpha\gamma}$ is transitive on Γ_2, giving the usual contradiction by Result A. Hence $\Theta_3^{r-1} = \{\beta\}$, say. If $\beta \in \Gamma_2(\delta)$ for some $\delta \in \Delta_3$ then, since $G_{\alpha\beta}$ is transitive on Δ_3 we have $\beta \in \Gamma_2(\delta)$ for all $\delta \in \Delta_3$ and $\Gamma_2 \cup \{\alpha\}$ is a component in the Γ_2-graph, which is impossible by Lemma 3 of Neumann (1977). Hence the Γ_2-graph has no triangles. If $\gamma \in \Gamma_2(\delta)$ for all $\delta \in \Delta_3$ then since the Γ_1-graph is connected, we have $\gamma \in \Gamma_1(\beta)$; since $|\Gamma_2(\gamma)| = |\Delta_3| + 1$ it follows that $\Theta_2^{r-1} = \{\varepsilon\}$, say, and $\gamma \in \Gamma_2(\varepsilon)$. But then γ, δ, ε form a triangle in the Γ_2-graph for any $\delta \in \Delta_3$, which is not so. Thus $\gamma \in \Gamma_1(\delta)$ for all $\delta \in \Delta_3$ and so $\gamma \in \Gamma_2(\beta)$. Now for some $i \in \{1, 2\}$ we have $\beta \in \Gamma_2(\delta)$ for all $\delta \in \Delta_i$; we may take $i = 1$. Hence, since there are no triangles in the Γ_2-graph, $\gamma \in \Gamma_2(\delta)$ for all $\delta \in \Delta_2$. Consequently $|\Theta_2^{r-1}| = 0$; for if $\Theta_2^{r-1} = \{\varepsilon\}$, say, then γ, ε have at least d_2 mutual adjacencies in the Γ_2-graph (the points of Δ_2), whereas α, γ have only 1 mutual adjacency (the point β). Since $\gamma \in \Gamma_1(\varepsilon)$ and $\gamma \in \Gamma_1(\alpha)$ this is a contradiction. Thus, writing $v_2(\beta)$ for the valency of β in the Γ_2-graph, we have

$$|\Gamma_2| = d_3 + 1 = v_2(\beta) = d_1 + 2 = v_2(\gamma) = d_2 + 1.$$

Counting edges in the Γ_2-graph between Γ_2 and Γ_1, we obtain

$$|\Gamma_2|(d_1 + 1) = |\Gamma_1|.$$

Since $|\Gamma_1| = d_1 + d_2 + 1$ this forces d_3 to be 1 or 0, which is not so.

II. *The case* $|\Theta_1^{r-1} \cup \Theta_2^{r-1}| = 0$. Certainly $|\Theta_3^{r-1}| = 1$ here, say $\Theta_3^{r-1} = \{\beta\}$. Since $|\text{fix } H| = k \geqslant 3$ we have

$$\bigcup_{i=1}^{3} \Theta_i^{r-2} \neq \varnothing.$$

Suppose first that $|\Theta_1^{r-2} \cup \Theta_2^{r-2}| > 0$, say $\Theta_1^{r-2} = \{\gamma\}$. If $\gamma \in \Gamma_2(\delta)$ for $\delta \in \Delta_3 \setminus \Theta_3^{r-2}$ then since $v_2(\gamma)$ is $d_3 + 1$ or $d_3 + 2$, we have $\Theta_2^{r-2} = \{\varepsilon\}$, say, and

$\gamma \in \Gamma_2(\varepsilon)$. Then γ, δ, ε form a triangle in the Γ_2-graph for any $\delta \in \Delta_3 \setminus \Theta_3^{r-2}$, which cannot be so. Hence $\gamma \in \Gamma_1(\delta)$ for all $\delta \in \Delta_3 \setminus \Theta_3^{r-2}$. If $\gamma \in \Gamma_2(\beta)$ then $\beta \in \Gamma_2(\delta)$ for all $\delta \in \Delta_1$, so $\gamma \in \Gamma_2(\delta)$ for all $\delta \in \Delta_2 \setminus \Theta_2^{r-2}$ and we have

$$|\Gamma_2| = d_3 + 1 = v_2(\beta) = d_1 + 1 = v_2(\gamma)$$

and $v_2(\gamma)$ is one of d_2, $d_2 + 1$ and $d_2 + 2$. However $|\Gamma_2| d_1$ is either $|\Gamma_1|$ or $2|\Gamma_1|$, which forces $d_1 \leqslant 3$, a contradiction. Thus $\gamma \in \Gamma_1(\beta)$; this yields a similar contradiction.

Finally, suppose that $|\Theta_1^{r-2} \cup \Theta_2^{r-2}| = 0$ and let $\Theta_3^{r-2} = \{\varepsilon\}$. We may assume that $\beta \in \Gamma_2(\delta)$ for all $\delta \in \Delta_1$ and $\beta \in \Gamma_1(\delta)$ for all $\delta \in \Delta_2$. Hence

$$\Delta_i G_{\alpha\beta} = \Delta_i \quad \text{for } i = 1, 2.$$

Now some points of Δ_2 are joined to points of Δ_3 in the Γ_2-graph, so $\varepsilon \in \Gamma_2(\delta)$ for all $\delta \in \Delta_2$. From the action of $G_{\alpha\beta}$ we see that $\gamma \in \Gamma_2(\delta)$ for all $\gamma \in \Delta_3$, $\delta \in \Delta_2$. Pick $\delta_1 \in \Delta_1$, $\delta_2 \in \Delta_2$. Then α, δ_1 have 1 mutual adjacency in the Γ_2-graph, but α, δ_2 have d_3 mutual adjacencies, which is a contradiction.

This completes Step 4.3.

Now we can finish the proof of Theorem 3. Suppose that the orbits of H_r are of type (i) described above (just before Step 4.3). Now G_α is imprimitive on $\Omega \setminus \{\alpha\}$ by assumption. Let Δ be a proper, nontrivial block for G_α. If $|\Delta| < 4$ then it is easy to see that $|\Delta| = 3$ and Lemma 4.0 gives a contradiction. And if $|\Delta| \geqslant 4$ we can take the block system $\mathfrak{B}$ containing Δ to be one of

$$\{\Sigma_1^r, \Sigma_2^r, \Sigma_3^r\} \quad \text{and} \quad \{\Sigma_1^r \cup \Sigma_2^r, \Sigma_3^r\}.$$

Let K be the kernel of the action of G_α on $\mathfrak{B}$, so that $H_r \leqslant K$. Then (using Theorem 1 in the second case) K is 2-transitive on each of its orbits and we have a contradiction by Theorem D of O'Nan (1975).

Similar arguments deal with the case where the orbits of H_r are of type (ii). Thus Theorem 3 is proved.

REMARK. Again (see the remark at the end of Section 2) we can relax the restrictions in Theorem 3 to $|\Sigma_i| \geqslant 3$, providing we exclude the group $PSL(3,3)$ of degree 13.

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