

Spreading primitive groups of diagonal type do not exist

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The synchronization hierarchy of finite permutation groups consists of classes of groups lying between 2-transitive groups and primitive groups. This includes the class of spreading groups, which are defined in terms of sets and multisets of permuted points, and which are known to be primitive of almost simple, affine or diagonal type. In this paper, we prove that in fact no spreading group of diagonal type exists. As part of our proof, we show that all non-abelian finite simple groups, other than six sporadic groups, have a transitive action in which a proper normal subgroup of a point stabilizer is supplemented by all corresponding two-point stabilizers.

Keywords: spreading groups; primitive permutation groups; diagonal type

1. Introduction

An important fact in the characterization of a finite permutation group G on a finite set Ω is that all 2-transitive groups are primitive. However, there is a large gap between primitivity and 2-transitivity, in the sense that relatively few primitive groups are 2-transitive. In [1], Araújo, Cameron and Steinberg introduced a hierarchy of classes of permutation groups that provides a finer characterization. These classes include *spreading* groups, defined in terms of sets and multisets of elements of Ω (see § 2 for a precise definition); *separating* groups, defined in terms of subsets of Ω , or graphs on Ω preserved by G ; and *synchronizing* groups, originally defined in [2] in terms of transformation semigroups on Ω containing G . (See also [6] for equivalent definitions of these properties in terms of G -modules.) In particular, these members of the hierarchy are related as follows (see [1, p. 151]):

2-transitive \implies spreading \implies separating \implies synchronising \implies primitive.

Moreover, the finite primitive groups can be divided into five types via the O’Nan–Scott Theorem, and by [1, Theorem 2.11 & Proposition 3.7], each synchronizing group is one of three types: almost simple, affine or diagonal

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(the relevant diagonal type groups will be defined in § 2). It is also well-known that each finite 2-transitive group is almost simple or affine.

Now, [1, § 6–7] classifies the affine spreading groups, and provides examples of almost simple groups that are spreading but not 2-transitive, separating but not spreading, or synchronizing but not separating, while [5] describes affine synchronizing groups that are not spreading. In addition, [8, Theorem 1.4] states that a primitive group that is not almost simple is synchronizing if and only if it is separating. The first (and so far only) known examples of synchronizing groups of diagonal type were presented in [4]: the groups $\text{PSL}_2(q) \times \text{PSL}_2(q)$ acting diagonally on $\text{PSL}_2(q)$, with $q = 13$ and $q = 17$.

Many open questions involving this hierarchy of permutation groups remain, and in this paper, we work towards solving Problem 12.4 of [1], the classification of spreading groups (of almost simple or diagonal type). Specifically, we completely resolve the diagonal case, with the aid of the following theorem.

THEOREM 1.1 [8, Theorem 1.5]. *Let G be a synchronizing primitive group of diagonal type. Then G has socle $T \times T$, where T is a non-abelian finite simple group.*

In addition, it is shown in [4, Theorem 2] that the diagonal type group $\text{PSL}_2(q) \times \text{PSL}_2(q)$ is non-spreading for every prime power q . We generalize this result to all groups of diagonal type, as follows.

THEOREM 1.2. *Each primitive group of diagonal type is non-spreading.*

Therefore, the types of spreading groups are precisely the types of finite 2-transitive groups: almost simple and affine.

For all but six possible socles $T \times T$, we will prove Theorem 1.2 as a consequence of the following result, which may be of general interest, and which distinguishes six sporadic simple groups from all other non-abelian finite simple groups. For the six remaining socles, we employ elementary character theory.

THEOREM 1.3. *Let T be a non-abelian finite simple group. Then the following are equivalent.*

- (i) *There exists a proper subgroup A of T and a proper normal subgroup B of A such that $A = B(A \cap A^\tau)$ for all $\tau \in \text{Aut}(T)$.*
- (ii) *There exists a proper subgroup A of T and a proper normal subgroup B of A such that $A = B(A \cap A^t)$ for all $t \in T$.*
- (iii) *The group T does not lie in the set $\{J_1, M_{22}, J_3, \text{McL}, \text{Th}, \text{M}\}$ of sporadic groups.*

Property (ii) of this theorem implies that, in the transitive action of T on the set of right cosets of A , the proper normal subgroup B of the point stabilizer A is supplemented by all corresponding two-point stabilizers. This is a stronger property than all two-point stabilizers being nontrivial, which is equivalent to the action having base size at least three.

2. Background

In this section, we provide the background on spreading permutation groups, and on primitive groups of diagonal type, that will be necessary to prove Theorems 1.2 and 1.3.

Given a set Ω , a multiset (or set) J of elements of Ω and a point $\omega \in \Omega$, we write $\mu_J(\omega)$ to denote the multiplicity of ω in J . We say that J is *trivial* if either μ_J is constant on Ω , or there exists a unique point $\omega \in \Omega$ such that $\mu_J(\omega) \neq 0$. Additionally, a sum $J + K$ of multisets is defined so that $\mu_{J+K}(\omega) = \mu_J(\omega) + \mu_K(\omega)$ for all ω , and similarly for scalar products of multisets. As usual, the cardinality of J is $|J| := \sum_{\omega \in \Omega} \mu_J(\omega)$.

We are now able to define spreading permutation groups.

DEFINITION 2.1 [1, § 5.5]. *A transitive permutation group G on Ω is non-spreading if there exist a nontrivial subset X of Ω and a nontrivial multiset J of elements of Ω , such that $|J|$ divides $|\Omega|$ and $\sum_{x \in Xg} \mu_J(x)$ is constant for all $g \in G$. Otherwise, G is spreading.*

We shall call a pair (X, J) satisfying the above properties a *witness* to G being non-spreading. Recall from § 1 that each imprimitive group is non-spreading. It is also immediate from Definition 2.1 that if G is non-spreading, then so is each transitive subgroup of G . The following theorem provides a useful method for finding a witness to a group being non-spreading.

THEOREM 2.2 [3]. *Let G be a finite group acting transitively on a finite set Ω , let A be a subgroup of G , and let B a proper normal subgroup of A . Additionally, let $\omega_1, \dots, \omega_k \in \Omega$ with $k \geq 2$, such that $\omega_i^B \neq \omega_j^B$ for $i \neq j$, and*

$$\omega_1^A = \omega_1^B \cup \omega_2^B \cup \dots \cup \omega_k^B.$$

Finally, let $\Delta := \{Y \in X^G \mid Y \cap \omega_1^A \neq \emptyset\}$ for some non-empty $X \subsetneq \Omega$. If B is transitive on each A -orbit of Δ , then $(X, \Omega + k\omega_1^B - \omega_1^A)$ is a witness to G being non-spreading.

Now, let T be a non-abelian finite simple group. In the remainder of this section, we shall consider the primitive groups of diagonal type with socle $T \times T$. To define these groups, let $T \times T$ act on the set $\Omega := T$ by $x^{(t_1, t_2)} := t_1^{-1}xt_2$. In addition, let $\rho_1 : T \times T \rightarrow \text{Sym}(\Omega)$ be the corresponding permutation representation, and let $\rho_2 : \text{Aut}(T) \rightarrow \text{Sym}(\Omega)$ be the natural permutation representation of $\text{Aut}(T)$ on Ω . Then $\rho_2(\text{Inn}(T)) = \rho_1(\{(t, t) \mid t \in T\})$. Finally, let $\sigma \in \text{Sym}(\Omega)$ such that $x^\sigma = x^{-1}$ for all $x \in T$. Define $W(T) := \langle \rho_1(T \times T), \rho_2(\text{Aut}(T)), \sigma \rangle \leq \text{Sym}(\Omega)$, and note that $W(T) \cong (T \times T) \cdot (\text{Out}(T) \times C_2)$. The *groups of diagonal type* with socle $T \times T$ are precisely the subgroups of $W(T)$ containing $\rho_1(T \times T)$ (see, for example, [17, § 1]). We will often drop the use of ρ_1 and ρ_2 , and allow elements (and subgroups) of $T \times T$ and $\text{Aut}(T)$ to denote permutations (and subgroups) of $\text{Sym}(\Omega)$.

In order to derive Theorem 1.2 from Theorem 1.3 (for most non-abelian finite simple groups), we require the following corollary of Theorem 2.2. Throughout the remainder of this section, we will use the elementary fact that if G is a transitive

permutation group on a set Σ and $H \leq G$, then H acts transitively on Σ if and only if $G = HG_\alpha$ for each $\alpha \in \Sigma$.

COROLLARY 2.3. *Let T be a non-abelian finite simple group, and suppose that there exists a proper subgroup A of T and a proper normal subgroup B of A such that $A = B(A \cap A^\tau)$ for all $\tau \in \text{Aut}(T)$. Then $(A, \Omega + |A : B|B - A)$ is a witness to the diagonal type group $W(T)$ being non-spreading in its action on $\Omega = T$.*

Proof. Let $G := W(T)$, $M := \rho_1(1 \times T)$, $\omega_1 := 1_T$, and $D := G_{\omega_1} = \langle \rho_2(\text{Aut}(T)), \sigma \rangle \cong \text{Aut}(T) \times C_2$. Note that $G = DM$, as M acts transitively on Ω . Let $X := A \subsetneq \Omega$. Then $\omega_1^{1 \times B} = B < A = \omega_1^{1 \times A}$, and $X^G = X^{DM} = \{A^\tau t \mid \tau \in \text{Aut}(T), t \in T\}$. Next, define $\Delta := \{Y \in X^G \mid Y \cap \omega_1^{1 \times A} \neq \emptyset\}$, and observe that $\Delta = \{A^\tau a \mid \tau \in \text{Aut}(T), a \in A\}$. Thus the orbits of $1 \times A$ on Δ are $\Delta_\tau = \{A^\tau a \mid a \in A\}$ for each $\tau \in \text{Aut}(T)$. Additionally, the stabilizer in $1 \times A$ of $A^\tau \in \Delta_\tau$ is $1 \times (A \cap A^\tau)$. As $A = B(A \cap A^\tau)$, we observe that $1 \times B$ is transitive on Δ_τ . The result now follows from Theorem 2.2, applied to the groups G , $1 \times A$ and $1 \times B$. □

Our next lemma translates Property (i) of Theorem 1.3 into the language of coset actions.

LEMMA 2.4. *Let T be a non-abelian finite simple group, $R \in \{T, \text{Aut}(T)\}$, and A a proper subgroup of T . Suppose also that A has a proper normal subgroup B that acts transitively on each orbit of A in its action on the right cosets of A in R . If $R = \text{Aut}(T)$, or if all $\text{Aut}(T)$ -conjugates of A are conjugate in T , then $A = B(A \cap A^\tau)$ for all $\tau \in \text{Aut}(T)$.*

Proof. Let Σ be the set of right cosets of A in R . Then the point stabilizers in the action of A on Σ are precisely the subgroups $A \cap A^t$ with $t \in R$. As B is transitive on each orbit in this action, it follows that $A = B(A \cap A^t)$ for all $t \in R$. Hence we are done if $R = \text{Aut}(T)$. Assume therefore that all $\text{Aut}(T)$ -conjugates of A are conjugate in T . Then for each $\tau \in \text{Aut}(T)$, there exists $t \in T$ such that $A^\tau = A^t$, and so $B(A \cap A^\tau) = B(A \cap A^t) = A$. □

In the case where T is one of the six sporadic groups listed in Property (iii) of Theorem 1.3, we will show that $W(T)$ is non-spreading using the following lemma, which involves elementary character theory. The multiset in the witness here is similar to that of Theorem 2.2.

LEMMA 2.5. *Let T be a non-abelian finite simple group, and let $\text{Irr}(T)$ be the set of irreducible complex characters of T . Suppose also that there exist pairwise non-conjugate elements r, s_1 and s_2 of T , such that:*

- (i) $|s_1^T| = |s_2^T|$; and
- (ii) $\chi(r^\tau) = 0$ for all $\tau \in \text{Aut}(T)$ and all $\chi \in \text{Irr}(T)$ with $\chi(s_1) \neq \chi(s_2)$.

Then $(r^T, \Omega + s_1^T - s_2^T)$ is a witness to the diagonal type group $W(T)$ being non-spreading in its action on $\Omega = T$.

Proof. The set $X := r^T$ and the multiset $J := \Omega + s_1^T - s_2^T$ are clearly nontrivial, and (i) implies that $|J| = |\Omega|$. Thus by Definition 2.1, it remains to prove that $\sum_{x \in X^g} \mu_J(x)$ is constant for all $g \in G := W(T)$. In fact, we will show that $|X^g \cap s_1^T| = |X^g \cap s_2^T|$ for all g , and it will follow immediately that $\sum_{x \in X^g} \mu_J(x) = |X|$ for all g .

As above, $G_{1_T} = \langle \rho_2(\text{Aut}(T)), \sigma \rangle$, and the transitivity of $M := \rho_1(1 \times T)$ on Ω implies that $G = G_{1_T}M$. Hence for each $g \in G$, there exist $\tau \in \text{Aut}(T)$, $\varepsilon = \pm 1$ and $t \in T$ such that $X^g = ((r^\tau)^\varepsilon)^T t$. Letting $m := (r^\tau)^\varepsilon$, $i \in \{1, 2\}$ and $h \in s_i^T$, we observe from [16, Problem 3.9] that the set $\{(x, y) \mid x \in m^T, y \in t^T, xy = h\}$ has size

$$a_{g,i} := \frac{|m^T||t^T|}{|T|} \sum_{\chi \in \text{Irr}(T)} \frac{\chi(m)\chi(t)\overline{\chi(s_i)}}{\chi(1)},$$

where $\overline{\chi(s_i)}$ is the complex conjugate of $\chi(s_i)$. Since $a_{g,i}$ is constant for all $h \in s_i^T$, and $|m^T t^u \cap s_i^T|$ is constant for all $u \in T$, it follows that $|X^g \cap s_i^T| = a_{g,i}|s_i^T|/|t^T|$. As $\chi((r^\tau)^{-1}) = \overline{\chi(r^\tau)}$ for each character χ of T , (i) and (ii) yield $|X^g \cap s_1^T| = |X^g \cap s_2^T|$ for all $g \in G$, as required. \square

REMARK 2.6. For many small non-abelian finite simple groups T , it is possible to choose an element $r \in T$ satisfying Property (ii) of Lemma 2.5 when s_1 and s_2 are non-conjugate elements of T such that $s_1^{-1} = s_2$, or more generally, such that $\langle s_1 \rangle = \langle s_2 \rangle$. The non-abelian finite simple groups with all elements conjugate to their inverses are classified in [18, Theorem 1.2], and [14, Corollary B.1] shows that $\text{PSP}_6(2)$ and $\text{P}\Omega_8^+(2)$ are the only such groups where $u, v \in T$ are conjugate whenever $\langle u \rangle = \langle v \rangle$. Denoting conjugacy classes as in the ATLAS [13], we observe by inspecting the ATLAS character tables of these two groups that we can choose $r \in 7A$, $s_1 \in 6A$ and $s_2 \in 6B$ when $T = \text{PSP}_6(2)$, and $r \in 7A$, $s_1 \in 15A$ and $s_2 \in 15B$ when $T = \text{P}\Omega_8^+(2)$. We leave open the problem of determining which non-abelian finite simple groups, if any, do not contain pairwise non-conjugate elements r, s_1 and s_2 satisfying Properties (i) and (ii) of Lemma 2.5.

3. Non-abelian finite simple groups and groups of diagonal type

In this section, we show that Property (i) of Theorem 1.3 holds for all non-abelian finite simple groups other than the sporadic groups $J_1, M_{22}, J_3, \text{McL}, \text{Th}$ and \mathbb{M} , and then prove Theorems 1.3 and 1.2. Note that we address the Tits group ${}^2F_4(2)'$ together with the sporadic groups. Throughout, we use ATLAS [13] notation for the structures of groups.

We first consider the finite simple groups of Lie type, beginning with the Chevalley groups (i.e. the untwisted groups of Lie type) defined over fields of size $q \geq 3$, then the twisted groups of Lie type, and finally the Chevalley groups with $q = 2$.

PROPOSITION 3.1. *Let T be a finite simple Chevalley group $X_\ell(q)$, with $q \geq 3$. Then Property (i) of Theorem 1.3 holds for T .*

Proof. Let p be the prime dividing q , and let U be a Sylow p -subgroup of T . Then T contains a subgroup N such that $A := N_T(U)$ and N form a (B, N) -pair, and

the normal subgroup $H := A \cap N$ of N complements U in A (see [12, Ch. 7.2–9.4]). Since $q \geq 3$ (and since T is not the soluble group $A_1(3) = \text{PSL}_2(3)$), we observe from [12, pp. 121–122] that $|H| > 1$, and so U is a proper subgroup of A . Furthermore, as A is the normalizer in T of the Sylow subgroup U , all $\text{Aut}(T)$ -conjugates of A are conjugate in T .

By Lemma 2.4, it suffices to show that A and U have the same orbits in the action of T on the set of right cosets of A . For each element w of the Weyl group $W := N/H$, fix a preimage $n_w \in N$ of w under the natural homomorphism from N to W . In addition, let x and y be elements of T so that Ax and Ay lie in a common A -orbit. Then the double cosets AxA and AyA are equal. By [12, Theorem 8.4.3], there exist $a_x, a_y \in A$, $w_x, w_y \in W$ and $u_x, u_y \in U < A$ such that $x = a_x n_{w_x} u_x$ and $y = a_y n_{w_y} u_y$. It follows that $An_{w_x}A = An_{w_y}A$. As A and N form a (B, N) -pair, [12, Proposition 8.2.3] yields $n_{w_x} = n_{w_y}$. Thus $AxU = An_{w_x}U = An_{w_y}U = AyU$, i.e. Ax and Ay lie in the same U -orbit, as required. \square

PROPOSITION 3.2. *Let T be a finite simple twisted group of Lie type ${}^tX_\ell(q)$. Then Property (i) of Theorem 1.3 holds for T .*

Proof. Note that $T \not\cong {}^2F_4(2)'$. As above, let p be the prime dividing q , and let U be a Sylow p -subgroup of T . By [12, Ch. 13–14], T contains subgroups N , H and W analogous to those in the proof of Proposition 3.1, so that $A := U : H$ and N form a (B, N) -pair, and $W = N/H$. Furthermore, H is nontrivial for all q , as T is not isomorphic to ${}^2A_2(2) = \text{PSU}_3(2)$, ${}^2B_2(2) = \text{Sz}(2)$ or ${}^2F_4(2)$. Although [12] directly states only that $A \leq N_T(U)$, we can show that $A = N_T(U)$. Indeed, if this were not the case, then some preimage $n_w \in N$ of a *fundamental reflection* $w \in W$ would normalise U (see [12, Proposition 8.2.2 & Theorem 8.3.2]). However, we deduce from Proposition 13.6.1 and the proof of Theorem 13.5.4 in [12] that no such w exists, and so $A = N_T(U)$. We now proceed exactly as in the proof of Proposition 3.1, this time using Proposition 13.5.3 of [12] instead of Theorem 8.4.3. \square

PROPOSITION 3.3. *Let T be a finite simple Chevalley group $X_\ell(2)$. Then Property (i) of Theorem 1.3 holds for T .*

Proof. Since $A_2(2) = \text{PSL}_3(2) \cong \text{PSL}_2(7)$ is addressed by Proposition 3.1, and since the remaining groups $X_\ell(2)$ with $\ell \leq 2$ are not simple, we shall assume that $\ell \geq 3$. Let $\Phi \subseteq \mathbb{R}^\ell$ be a root system for T , with $\Phi^+ \subseteq \Phi$ a system of positive roots and $\Pi \subseteq \Phi^+$ a system of simple roots (see, for example, [12, Ch. 2]). Then Φ is the disjoint union of Φ^+ and $-\Phi^+$, and each node in the Dynkin diagram D corresponding to T is associated with a unique root in Π (and vice versa). Hence each diagram automorphism of D induces a permutation of Π . Let r and s be the simple roots associated with a leaf in D and the adjacent node, respectively. Additionally, let S be the closure of the set $\{s\}$ under the group of diagram automorphisms of D , so that $|S| \in \{1, 2\}$.

Next, let U , N and W be as in the proof of Proposition 3.1. In this case, the group H from that proof is trivial, and so $N_T(U) = U$ and $W \cong N$. By [12, Proposition 2.1.8, p. 68 & p. 93], T is generated by a set $\{x_u \mid u \in \Phi\}$ of involutions (x_u is written as $x_u(1)$ in [12]), with $U = \langle x_u \mid u \in \Phi^+ \rangle$, and N is generated by a

set $\{n_v \mid v \in \Pi\}$ of involutions. Let $A := \langle U, n_v \mid v \in \Pi \setminus S \rangle$ be the parabolic subgroup of T corresponding to the subset $\Pi \setminus S$ of Π . Then by [12, Ch. 8.5] and the well-known structure of parabolic subgroups of Chevalley groups, A has shape $[2^m] : (L_{\{r\}} \times L_{\Pi \setminus (\{r\} \cup S)})$ for some positive integer m , where $L_{\{r\}} \cong A_1(2) \cong S_3$ and $L_{\Pi \setminus (\{r\} \cup S)}$ are Levi subgroups. Since $q = 2$, all automorphisms of T are products of inner and graph automorphisms. Moreover, $\Pi \setminus S$ is fixed setwise by all diagram automorphisms of D , which correspond to the graph automorphisms of T , and so all $\text{Aut}(T)$ -conjugates of A are conjugate in T . In addition, A contains an index two subgroup B of shape $[2^m] : (F \times L_{\Pi \setminus (\{r\} \cup S)})$, where $F := L'_{\{r\}} \cong C_3$. We will show that $AtA = AtB$ for all $t \in T$, and the result will follow from Lemma 2.4.

Since $N_T(U) = U \leq A$, we deduce as in the proof of Proposition 3.1 that, for each $t \in T$, there exists $n \in N$ such that $AtA = AnA$. Thus it suffices to prove that $AnA = AnB$ for all $n \in N$ (it will immediately follow that $AnB = AtB$ for the corresponding $t \in T$). We also observe using [12, Ch. 8.5] (and the fact that F has index two in $L_{\{r\}}$) that $x_r \in A \setminus B$ and $x_{-r}x_r \in B$. The former containment implies that $AnA = AnB \cup Anx_rB$. To complete the proof, we will show that $Anx_rB = AnB$.

The Weyl group W acts linearly on \mathbb{R}^ℓ and fixes Φ . Let w be the element of W corresponding to n . By [12, Lemma 7.2.1(i)], $nx_r = x_{w(r)}n$ and $x_{-w(r)}n = x_{w(-r)}n = nx_{-r}$. If $w(r) \in \Phi^+$, then $x_{w(r)} \in \langle x_u \mid u \in \Phi^+ \rangle = U \leq A$, and so $Anx_r = An$. Otherwise, $-w(r) \in \Phi^+$ and $x_{-w(r)} \in A$, yielding $An = Anx_{-r}$ and hence $Anx_r = Anx_{-r}x_r$, which lies in AnB by the previous paragraph. In either case, we see that $Anx_rB = AnB$, as required. \square

Next, we consider the alternating groups, followed by the sporadic groups and the Tits group.

PROPOSITION 3.4. *Let T be a finite simple alternating group A_n . Then Property (i) of Theorem 1.3 holds for T .*

Proof. Since $A_5 \cong \text{PSL}_2(4) = A_1(4)$ and $A_6 \cong \text{PSL}_2(9) = A_1(9)$ are addressed by Proposition 3.1, we shall assume that $n \geq 7$. Let Σ be the set of 3-subsets of a set of size n , and let $\alpha \in \Sigma$. Then $A := T_\alpha = (S_3 \times S_{n-3}) \cap T$, and all $\text{Aut}(T)$ -conjugates of A are conjugate in T . Additionally, A has four orbits on Σ , namely $\{\alpha\}$ and $\{\beta \in \Sigma \mid |\beta \cap \alpha| = i\}$ for $i \in \{0, 1, 2\}$. Since the index two subgroup $B := A_3 \times A_{n-3}$ of A acts transitively on each of these orbits, and since the action of T on Σ is equivalent to its action on the right cosets of A , the result follows from Lemma 2.4. \square

PROPOSITION 3.5. *Let T be the Tits group ${}^2F_4(2)'$, or a sporadic simple group that does not lie in $\{J_1, M_{22}, J_3, \text{McL}, \text{Th}, \text{M}\}$. Then Property (i) of Theorem 1.3 holds for T .*

Proof. We observe from the ATLAS [13] that T has a maximal subgroup A , as specified in Table 1, such that either $T = O'N$ or all $\text{Aut}(T)$ -conjugates of T_α are conjugate in T . It is also clear that the group B in the table is a proper normal subgroup of A . Note that if $T = M_{12}$, then A has two normal subgroups isomorphic to S_5 ; in what follows, B may be chosen as either.

Table 1. A maximal subgroup A of a group T from Proposition 3.5, with B a proper normal subgroup of A

T	A	B	$ T : A $
M_{11}	$A_6.2$	A_6	11
M_{12}	$S_5 \times 2$	S_5	396
J_2	$A_5 \times D_{10}$	$A_5 \times 5$	1008
M_{23}	$(2^4 : (3 \times A_5)) : 2$	$2^4 : (3 \times A_5)$	1771
${}^2F_4(2)'$	$(2^2.[2^8]) : S_3$	$(2^2.[2^8]) : 3$	2925
HS	S_8	A_8	1100
M_{24}	$M_{12} : 2$	M_{12}	1288
He	$PSp_4(4) : 2$	$PSp_4(4)$	2058
Ru	$(2^6 : PSU_3(3)) : 2$	$2^6 : PSU_3(3)$	188500
Suz	$(3.PSU_4(3)) : 2$	$3.PSU_4(3)$	22880
O'N	$PSL_3(7) : 2$	$PSL_3(7)$	122760
Co_3	McL : 2	McL	276
Co_2	$PSU_6(2) : 2$	$PSU_6(2)$	2300
Fi_{22}	$P\Omega_8^+(2) : S_3$	$P\Omega_8^+(2) : 3$	61776
HN	$PSU_3(8) : 3$	$PSU_3(8)$	16500000
Ly	$(3.McL) : 2$	$3.McL$	9606125
Fi_{23}	$P\Omega_8^+(3) : S_3$	$P\Omega_8^+(3) : 3$	137632
Co_1	$(3.Suz) : 2$	$3.Suz$	1545600
J_4	$(2_+^{1+12}.3M_{22}) : 2$	$2_+^{1+12}.3M_{22}$	3980549947
Fi'_{24}	$(3 \times P\Omega_8^+(3) : 3) : 2$	$3 \times P\Omega_8^+(3) : 3$	14081405184
\mathbb{B}	$(2.^2E_6(2)) : 2$	$2.^2E_6(2)$	13571955000

Now, let $R := \text{Aut}(T)$ if $T = O'N$, and $R := T$ otherwise. Additionally, let χ be the permutation character corresponding to the action of R on the set Σ of right cosets of A in R . Then for each $t \in R$, the number of points in Σ fixed by t is equal to $\chi(t)$. By the Cauchy–Frobenius Lemma, A and B have $c_A := 1/|A| \sum_{a \in A} \chi(a)$ and $c_B := 1/|B| \sum_{b \in B} \chi(b)$ orbits on Σ , respectively. It is straightforward to compute c_A and c_B in GAP [15] using the Character Table Library [9] (c_A is most readily calculated as the inner product $[\chi, \chi]$), and in each case we obtain $c_A = c_B$. Thus Lemma 2.4 yields the result. \square

We are now able to prove Theorem 1.3. Given a group G and a non-trivial (core-free) subgroup H of G , we will write $b(G, H)$ to denote the base size of G in its action on the set Σ of right cosets of H , i.e., the minimum size of a subset Δ of Σ such that the pointwise stabilizer $G_{(\Delta)}$ is trivial. Notice that $b(G, H) \geq 2$, as H is a point stabilizer in this action.

Proof of Theorem 1.3. Propositions 3.1–3.5 show that (iii) implies (i), which clearly implies (ii). To complete the proof, we will show that if (iii) does not hold, then neither does (ii). If $T \in \{J_1, M_{22}, J_3, \text{McL}\}$, then we construct T in Magma [7] via the AutomorphismGroupSimpleGroup and Socle functions, and show that (ii) does not hold via fast, direct computations.

Table 2. Elements r , s_1 and s_2 that satisfy Properties (i) and (ii) of Lemma 2.5, for each of six sporadic groups T

T	r	s_1	s_2
J_1	7A	5A	5B
M_{22}	5A	7A	7B
J_3	5A	19A	19B
McL	4A	9A	9B
Th	7A	39A	39B
M	110A	119A	119B

Each element is specified by its corresponding conjugacy class, labelled as in the ATLAS [13].

Suppose next that $T = \text{Th}$, let C be a proper subgroup of T , and let A be a subgroup of C . Observe that if $b(T, C) = 2$, then (as discussed below the statement of Theorem 1.3) the two-point stabilizer $C \cap C^t$ is trivial for some t . Hence $A \cap A^t = 1$, and so A does not satisfy (ii). In general, $b(T, A) \leq b(T, C)$. By [11, Theorem 1], T has only two maximal subgroups (up to conjugacy) with corresponding base size greater than two, namely $M_1 := {}^3D_4(2) : 3$ and $M_2 := 2^5 \cdot \text{PSL}_5(2)$, and $b(T, M_1) = b(T, M_2) = 3$. Hence any proper subgroup A of T satisfying (ii) is a non-simple subgroup of M_1 or M_2 with $b(T, A) = 3$.

Now, let Σ be the set of right cosets in T of a proper subgroup A , and let c be the number of orbits of A on Σ . If $b(T, A) \geq 3$, then $|A \cap A^t| \geq 2$ for all $t \in T$, and it follows from the Orbit-Stabiliser Theorem that $c|A|/2 \geq |\Sigma|$. If A is a maximal subgroup of either M_1 or $M'_1 \cong {}^3D_4(2)$, then the character table of A is included in The GAP Character Table Library. Thus for every such $A \neq M'_1$, we can compute c as in the proof of Proposition 3.5, and we see that in fact $c|A|/2 < |\Sigma|$, and hence $b(T, A) = 2$. To show that $b(T, A) = 2$ for each maximal subgroup A of M_2 , we construct T and M_2 in Magma as subgroups of $\text{GL}_{248}(2)$ using the respective generating pairs $\{x, y\}$ and $\{w_1, w_2\}$ given in [19]. Magma calculations (with a runtime of less than 10 minutes and a memory usage of 1.5 GB) then show that $A \cap A^y = 1$ for all representatives A of the conjugacy classes of maximal subgroups of M_2 returned by the `MaximalSubgroups` function. Therefore, M_1 and M_2 are the only non-simple subgroups of T with corresponding base size at least three. Further character table computations in GAP show that if $A \in \{M_1, M_2\}$, then for each proper normal subgroup B of A , the number of B -orbits on Σ is greater than the number of A -orbits. An additional application of the Orbit-Stabiliser Theorem yields $B(A \cap A^t) < A$ for some $t \in T$. Therefore, (ii) does not hold.

Finally, suppose that $T = M$. As above, any proper subgroup A of T satisfying (ii) also satisfies $b(T, A) \geq 3$. By [10, Theorem 3.1], the unique (up to conjugacy) proper subgroup A of T with $b(T, A) \geq 3$ is the maximal subgroup $K := 2.B$, with $b(T, K) = 3$. Since K is quasisimple, its centre Z of order two is its unique non-trivial proper normal subgroup, and Z lies in each maximal subgroup of K . Hence $Z(K \cap K^t) < K$ for all $t \in T \setminus K$, and so (ii) does not hold. \square

We now prove our main theorem.

Proof of Theorem 1.2. As each spreading primitive group is synchronizing, Theorem 1.1 implies that a spreading primitive group of diagonal type has socle $T \times T$, for some non-abelian finite simple group T . For each T , let $\Omega := T$, and let $W(T)$ be the subgroup of $\text{Sym}(\Omega)$ defined in § 2. Recall also that each subgroup of a non-spreading group is non-spreading. Thus it suffices to show that $W(T)$ is non-spreading for each T . If $T \notin \{J_1, M_{22}, J_3, \text{McL}, \text{Th}, \mathbb{M}\}$, then this is an immediate consequence of Theorem 1.3 and Corollary 2.3.

For each of the six remaining groups T , let r, s_1 and s_2 be members of the conjugacy classes of T given in Table 2. By inspecting the character table for T in [13], we observe that these elements satisfy Properties (i) and (ii) of Lemma 2.5. Therefore, that lemma shows that $W(T)$ is non-spreading. \square

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