ON PERIODIC SOLUTIONS OF $x''' + ax'' + bx' + g(x) = 0$

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In [1] J.O.C. Ezeilo asks whether the equation

$$x''' + ax'' + x' + a \sin x = 0 \tag{1}$$

has periodic solutions for $a \neq 0$. Since (1) has a two-dimensional space of solutions of period 2π if $\sin x$ is approximated by x, it is plausible to conclude, by analogy with $x'' + \sin x = 0$, that (1) does have periodic solutions. However, when one applies the standard theory of perturbation of periodic solutions (treating a as small, see [2]), one finds that the only real periodic solutions obtainable in this manner are the trivial ones $x(t, a) = n\pi$ for some integer n. The fact that these are the only real periodic solutions of (1) for $a \neq 0$ follows from the following elementary theorem on a somewhat generalized equation.

THEOREM. Let g be a real-valued continuously differentiable function defined for all x. Let a and b be real constants and suppose that $ab - g'(x) \geq 0$ for all x, with equality holding only on a discrete set. Then the only real periodic solutions of the equation

$$x''' + ax'' + bx' + g(x) = 0 \tag{2}$$

are the trivial ones $x(t) = c$ where $g(c) = 0$.

Proof. Suppose that $x(t)$ is a real periodic solution of (2) of period T and denote by G any function such that $G' = g$. Then since x', x'', and $G(x(t))$ all have period T, we have

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$$0 = \int_0^T x'\{x''' + ax'' + bx' + g(x)\}\,dt$$

$$= \{x'x'' + a(x')^2/2 + G(x(t))\}\Big|_{t=0}^{t=T} - \int_0^T (x'')^2 dt + b\int_0^T (x')^2 dt$$

$$= -\int_0^T (x'')^2 dt + b\int_0^T (x')^2 dt ,$$

or

$$(3) \qquad \int_0^T (x'')^2 dt = b\int_0^T (x')^2 dt .$$

Since $\int_0^T x''x'''dt = (x'')^2\Big|_{t=0}^{t=T} - \int_0^T x''x'''dt$ and x'' has period T, it follows that the integral on the left vanishes. Thus

$$0 = \int_0^T x''\{x''' + ax'' + bx' + g(x)\}\,dt$$

$$= \{x'g(x(t)) + b(x')^2/2\}\Big|_{t=0}^{t=T} + a\int_0^T (x'')^2 dt - \int_0^T (x')^2 g'(x(t))dt$$

$$= a\int_0^T (x'')^2 dt - \int_0^T (x')^2 g'(x(t))dt ,$$

and using (3) we obtain

$$(4) \qquad \int_0^T (x')^2\{ab - g'(x(t))\}\,dt = 0 .$$

Since the integrand of (4) is non-negative it follows from (4) that this integrand is identically zero. If s is a number such that $x'(s) \neq 0$ then x' is non-zero on some neighbourhood N of s. Thus $ab - g'(x(t)) = 0$ on N, and as x is continuous and the set of possible values is discrete, x must be constant on N. Thus x is constant everywhere, and as the only constant solutions of (2) are those in the statement of the theorem, the proof is complete.

In equation (1) we note that replacing t by $-t$ has the

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same effect as replacing a by $-a$. Thus we may assume that $a > 0$ if it is non-zero, and the bracketed term in the integrand of (4) is replaced by $a(1 - \cos x)$. As this is non-negative and vanishes only on a discrete set we have the immediate corollary:

COROLLARY. For real $a \neq 0$ the only real periodic solutions of (1) are the trivial ones $x(t) = n\pi$ for some integer n.

REFERENCES

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