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Abstract

Despite the presence of many famous examples, the precise interplay between basic hypergeometric series and modular forms remains a mystery. We consider this problem for canonical spaces of weight 3/2 harmonic Maass forms. Using recent work of Zwegers, we exhibit forms that have the property that their holomorphic parts arise from Lerch-type series, which in turn may be formulated in terms of the Rogers–Fine basic hypergeometric series.

1. Introduction and statement of results

There are a number of famous examples of q-series which essentially coincide with modular forms when $q := e^{2\pi i\tau}$. For example, the celebrated Rogers–Ramanujan identities

$$R_1(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}, \tag{1.1}$$

$$R_2(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{1.2}$$

provide series expansions of infinite products which are essentially weight 0 modular forms. As another example, the partition generating function satisfies

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}. \tag{1.3}$$

Since $q^{-1}P(q^{24}) = 1/\eta(24\tau)$, the reciprocal of Dedekind's weight 1/2 modular form, (1.3) is another instance of a modular form which is a q-series.

Although there are many such identities (see, for example, [And86a, And86b]), these scattered results fall far short of a comprehensive theory which describes the interplay between such combinatorial series and modular forms. At the Millennial Conference on Number Theory at the University of Illinois in 2000 [And03], Andrews discussed this conundrum, and he urged research in this direction.

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The situation is further complicated by Ramanujan's mock theta functions, a collection of 22 series such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{1.4}$$

Despite its resemblance to $P(q)$, the mock theta function $f(q)$, like all of the mock theta functions, does not arise as the minor modification of the Fourier expansion of a modular form. Nevertheless, a wealth of evidence, such as identities involving mock theta functions and modular forms, suggested a strong connection between these objects (see, for example, [And89]).

This quandary was resolved by Zwegers in his 2002 PhD thesis [Zag, Zwe01, Zwe02]. Zwegers related the mock theta functions to certain real analytic modular forms, which, following Bruinier and Funke [BF04], we refer to as harmonic Maass forms (for more on harmonic Maass forms in number theory see [Ono]). To make this precise, suppose that $k \in \frac{1}{2} + \mathbb{Z}$. If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{1.5}$$

As usual, we let

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1.6}$$

be the weight k hyperbolic Laplacian, where $\tau = x + iy$ with $x, y \in \mathbb{R}$. If N is a positive integer and Γ a congruence subgroup of level $4N$, then a *harmonic Maass form* of weight k on Γ is any smooth function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H}$ is the upper half-complex plane, satisfying:

(i) for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have[1]

$$f(A\tau) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \ (c\tau + d)^k \ f(\tau);$$

(ii) we have that $\Delta_k f = 0$;

(iii) the function $f(\tau)$ has at most linear exponential growth at all cusps.

Loosely speaking, Zwegers completed the mock theta functions to obtain weight $1/2$ harmonic Maass forms. Each mock theta function is the holomorphic part of a weight $1/2$ harmonic Maass form. Following the work of Zwegers, the first and third authors investigated Andrews' problem for families of weight $1/2$ harmonic Maass forms. They proved [BO] that the basic hypergeometric series

$$R(w, q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 - wq)(1 - w^{-1}q) \cdot (1 - wq^2)(1 - w^{-1}q^2) \cdots (1 - wq^n) \cdot (1 - w^{-1}q^n)}, \tag{1.7}$$

when $w \neq 1$ is a root of unity, is essentially the holomorphic part of a weight $1/2$ harmonic Maass form. These specializations include $f(q) = R(-1, q)$, and they play an important role in the study of Dyson's rank partition functions [BO, BOR08]. All of these recent results concern weight $1/2$ harmonic Maass forms.

Here we describe a similar theory for weight $3/2$ harmonic Maass forms. The holomorphic parts of these forms, which we also call *mock theta functions*, arise naturally from Lerch-type

[1] This transformation law agrees with Shimura's notion of half-integral weight modular forms [Shi73].

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series, which in turn may be described in terms of the basic hypergeometric-type series

$$F(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) := \sum_{n=0}^{\infty} \frac{(\alpha,\zeta)_n \delta^{n^2}\epsilon^n}{(\beta,\zeta)_n(\gamma,\zeta)_n}. \tag{1.8}$$

As usual, we have that

$$(u,w)_n := \begin{cases} (1-u)(1-uw)(1-uw^2)\cdots(1-uw^{n-1}) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Remark. The definition of F is motivated by the examples above. Indeed, we have that

$$R_1(q) = F(0,q,0,q,1,q), \quad R_2(q) = F(0,q,0,q,q,q),$$
$$P(q) = F(0,q,q,q,1,q), \quad R(w,q) = F(0,wq,w^{-1}q,q,1,q).$$

A famous identity of Rogers and Fine (see [Fin88, p. 15]) implies that

$$\frac{1}{1+\alpha}F(\alpha,-\alpha\beta,0,1,\alpha,\beta) = \frac{1}{1+\alpha}\sum_{n=0}^{\infty}\frac{(\alpha,\beta)_n\alpha^n}{(-\alpha\beta,\beta)_n} = \sum_{n=0}^{\infty}(-1)^n\alpha^{2n}\beta^{n^2}.$$

We differentiate it to obtain

$$F_{RF}(\alpha,\beta) := \frac{\alpha}{2}\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{1+\alpha}F(\alpha,-\alpha\beta,0,1,\alpha,\beta)\right) = \sum_{n=1}^{\infty}(-1)^n n\alpha^{2n}\beta^{n^2}. \tag{1.9}$$

For formal parameters a and b, we then define

$$F_{RF}(a,b;\alpha,\beta) := \sum_{n=0}^{\infty}(-1)^{(bn+a)}(bn+a)\alpha^{2(bn+a)}\beta^{(bn+a)^2}. \tag{1.10}$$

For each pair of integers $0 < a < b$, where $b \geq 4$ is even, we shall obtain a weight $3/2$ harmonic Maass form with the property that its holomorphic part is an expression involving the Rogers–Fine hypergeometric function. To make this precise, we let $\theta(a,b;\tau)$ be the classical theta function

$$\theta(a,b;\tau) := \sum_{n\equiv a \pmod{b}} q^{n^2/2b}. \tag{1.11}$$

We define $\mathcal{M}^{+}_{a,b}(\tau)$ by

$$\mathcal{M}^{+}_{a,b}(\tau) := \frac{-1}{b\Theta_0(b\tau)}\sum_{n\equiv a \pmod{b}}\frac{nq^{n^2/2b+n(1-a/b)}}{1-q^n}, \tag{1.12}$$

where $\Theta_0(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2/2}$, and we define $\mathcal{M}^{-}_{a,b}(\tau)$ by

$$\mathcal{M}^{-}_{a,b}(\tau) := \frac{1}{4\pi i\sqrt{b}}\int_{-\overline{\tau}}^{i\infty}\frac{\theta(a,b;w)}{(-i(\tau+w))^{\frac{3}{2}}}dw. \tag{1.13}$$

Using these two functions, we define the real analytic function $\mathcal{M}_{a,b}(\tau)$ by

$$\mathcal{M}_{a,b}(\tau) := \mathcal{M}^{-}_{a,b}(\tau) + \mathcal{M}^{+}_{a,b}(\tau). \tag{1.14}$$

Theorem 1.1. *If $0 < a < b$ are integers, where $b \geq 4$ is even, then the following are true.*

(i) *For $\tau \in \mathbb{H}$ we have that $\mathcal{M}_{a,b}(\tau)$ is a weight $3/2$ harmonic Maass form on $\Gamma(2b)$.*

(ii) *The holomorphic part of $\mathcal{M}_{a,b}(\tau)$ satisfies*

$$\mathcal{M}^{+}_{a,b}(\tau) = \frac{-1}{b\Theta_0(b\tau)} \times \sum_{k=0}^{\infty} \left(F_{RF}\left(a, b; q^{k/2+1/2-a/2b}, -q^{1/2b}\right) + F_{RF}\left(b-a, b; q^{k/2+a/2b}, -q^{1/2b}\right)\right).$$

Remark. One may determine how the forms $\mathcal{M}_{a,b}(\tau)$ transform under $\tau \mapsto -1/\tau$ by proceeding in a similar way as in the proof of Theorem 1.1(i), in particular making use of Theorem 2.2(iii). The resulting forms may be written as a linear combination of derivatives of specializations of the functions $\widehat{\mu}$, and they are analogous to the Lambert series in (3.1).

Remark. Although Theorem 1.1 does not include the theta function $\theta(0, b; \tau)$, a slightly modified version of its conclusion holds, and it involves the generating function of the Hurwitz class numbers. Namely, an appropriate Maass form to study is the so-called 'Zagier–Eisenstein series', which can be thought of as a prototype for the new Maass forms constructed here. We discuss this in further detail in § 4.

We obtain Theorem 1.1 using recent work of Zwegers, which we recall in § 2. In § 3 we prove Theorem 1.1. In § 4 we discuss two roles that such Maass forms play in number theory. We discuss the case where $a = 0$ which corresponds to Zagier's weight 3/2 non-holomorphic Eisenstein series, and we give an application to Andrews' *spt*-function [And].

2. The work of Zwegers

In his PhD thesis on mock theta functions [Zwe02], Zwegers constructed weight 1/2 harmonic Maass forms by making use of the transformation properties of Lerch sums. Here we briefly recall some of his results.

For $\tau \in \mathbb{H}$, and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, Zwegers defines the function

$$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v; \tau)} \cdot \sum_{n\in\mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}, \tag{2.1}$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$ and

$$\vartheta(v; \tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i \nu} w^{\nu} q^{\nu^2/2}. \tag{2.2}$$

Zwegers (see [Zwe02, § 1.3]) proves that $\mu(u, v; \tau)$ satisfies the following important properties.

Lemma 2.1. *Assuming the notation above, we have that*

$$\mu(u, v; \tau) = \mu(v, u; \tau),$$
$$\mu(u+1, v; \tau) = -\mu(u, v; \tau),$$
$$z^{-1} w q^{-\frac{1}{2}} \mu(u+\tau, v; \tau) = -\mu(u, v; \tau) - i z^{-\frac{1}{2}} w^{\frac{1}{2}} q^{-\frac{1}{8}},$$

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$$\mu(u, v; \tau + 1) = \zeta_8^{-1}\mu(u, v; \tau),$$
$$(\tau/i)^{-\frac{1}{2}} e^{\pi i(u-v)^2/\tau}\mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = -\mu(u, v; \tau) + \frac{1}{2i}h(u - v; \tau),$$

where

$$h(z; \tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i x^2\tau - 2\pi x z}dx}{\cosh \pi x},$$

and $\zeta_N := e^{2\pi i/N}$.

Remark. The integral $h(z; \tau)$ is known as a *Mordell integral*.

Lemma 2.1 shows that $\mu(u, v; \tau)$ is nearly a weight 1/2 Jacobi form, where τ is the modular variable. Zwegers then uses μ to construct weight 1/2 harmonic Maass forms. He achieves this by modifying μ to obtain a function $\widehat{\mu}$ which he then uses as building blocks for such Maass forms. To make this precise, for $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$, let $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$, and let

$$R(u; \tau) := \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\nu-\frac{1}{2}}\left\{\mathrm{sgn}(\nu) - E\left((\nu + c)\sqrt{2\mathrm{Im}(\tau)}\right)\right\}e^{-2\pi i\nu u}q^{-\nu^2/2}, \tag{2.3}$$

where $E(x)$ is the odd function

$$E(x) := 2\int_0^x e^{-\pi u^2}du = \mathrm{sgn}(x)(1 - \beta(x^2)), \tag{2.4}$$

where for positive real x we let $\beta(x) := \int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du$.

Using μ and R, Zwegers defines the real analytic function

$$\widehat{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2}R(u - v; \tau). \tag{2.5}$$

Zwegers' construction of weight 1/2 harmonic Maass forms depends on the following theorem (see [Zwe02, § 1.4]).

THEOREM 2.2. *Assuming the notation and hypotheses above, we have that:*

(i) $\widehat{\mu}(u, v; \tau) = \widehat{\mu}(v, u; \tau)$;

(ii) $\widehat{\mu}(u + 1, v; \tau) = z^{-1}wq^{-\frac{1}{2}}\widehat{\mu}(u + \tau, v; \tau) = -\widehat{\mu}(u, v; \tau)$;

(iii) $\zeta_8\widehat{\mu}(u, v; \tau + 1) = (-\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\widehat{\mu}\big(u/\tau, v/\tau; -1/\tau\big) = \widehat{\mu}(u, v; \tau)$;

(iv)
$$\widehat{\mu}\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \chi(A)^{-3}(\gamma\tau + \delta)^{\frac{1}{2}}e^{-\pi i\gamma(u-v)^2/(\gamma\tau+\delta)} \cdot \widehat{\mu}(u, v; \tau),$$

where $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *and* $\chi(A) := \eta(A\tau)/\big((\gamma\tau + \delta)^{\frac{1}{2}}\eta(\tau)\big)$.

Theorem 2.2 shows that $\widehat{\mu}(u, v; \tau)$ is essentially a weight 1/2 non-holomorphic Jacobi form. In analogy with the classical theory of Jacobi forms, one may then obtain harmonic Maass forms by making suitable specializations for u and v by elements in $\mathbb{Q}\tau + \mathbb{Q}$, and by multiplying by appropriate powers of q. We shall consider specializations of a certain derivative to obtain weight 3/2 harmonic Maass forms.

Harmonic Maass forms of weight k are mapped to classical modular forms, their so-called *shadows*, by the differential operator

$$\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

The following lemma will be important for establishing that the non-holomorphic part of a certain weight 3/2 harmonic Maass form, which we shall prove equals $\mathcal{M}_{a,b}(\tau)$, is indeed the period integral of the weight 1/2 theta function $\theta(a, b; \tau)$.

Lemma 2.3 [Zwe02, Lemma 1.8]. *The function R is real analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{2}y^{-\frac{1}{2}}e^{-2\pi c^2 y}\vartheta(\overline{u}; -\overline{\tau}),$$

where $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$. Moreover, we have that

$$\frac{\partial}{\partial \overline{\tau}}R(a\tau - b;\tau) = -\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y}\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}(-1)^{\nu-\frac{1}{2}}(\nu+a)e^{-\pi i\nu^2\overline{\tau}-2\pi i\nu(a\overline{\tau}-b)}.$$

3. Proof of Theorem 1.1

Here we prove Theorem 1.1. We first prove an elementary q-series identity.

3.1 A q-series identity

For convenience, we consider the Lambert-type series

$$L(a, b; z, d, q) := \sum_{n\equiv a \ (\mathrm{mod}\ b)} \frac{nz^n q^{n^2}}{1-q^{dn}}, \tag{3.1}$$

where $d \neq 0$, $a \not\equiv 0 \pmod{b}$. The proof of Theorem 1.1(ii) follows from the following simple proposition which relates the Lambert series $L(a, b; z, d, q)$ to the Rogers–Fine hypergeometric series F_{RF}.

Proposition 3.1. *Suppose that $0 < a < b$ are integers. The following q-series identity is true:*

$$L(a, b; z, d, q) = \sum_{k\geq 0}(F_{RF}(a, b; z^{1/2}q^{kd/2}, -q) + F_{RF}(b-a, b; z^{-1/2}q^{(k+1)d/2}, -q)).$$

Proof. To prove the identity observe the following:

$$\begin{aligned} L(a, b; z, d, q) &= \sum_{n=0}^{\infty}\frac{(bn+a)z^{bn+a}q^{(bn+a)^2}}{1-q^{d(bn+a)}} - \sum_{n=1}^{\infty}\frac{(bn-a)z^{-(bn-a)}q^{(bn-a)^2}}{1-q^{-d(bn-a)}} \\ &= \sum_{n=0}^{\infty}\frac{(bn+a)z^{bn+a}q^{(bn+a)^2}}{1-q^{d(bn+a)}} + \sum_{n=1}^{\infty}\frac{(bn-a)z^{-(bn-a)}q^{(bn-a)^2+d(bn-a)}}{1-q^{d(bn-a)}} \\ &= \sum_{n=0}^{\infty}\bigg((bn+a)z^{bn+a}q^{(bn+a)^2}\sum_{k=0}^{\infty}q^{d(bn+a)k}\bigg) \\ &\quad + \sum_{n=1}^{\infty}\bigg((bn-a)z^{-(bn-a)}q^{(bn-a)^2}\sum_{k=1}^{\infty}q^{d(bn-a)k}\bigg). \end{aligned} \tag{3.2}$$

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One next makes changes of variables in the indices of summation k and n in (3.2) so that the series begin at $k=0$ and $n=0$. Proposition 3.1 follows by re-ordering summation and applying (1.10). $\square$

3.2 Proof of Theorem 1.1

Using Zwegers' function $\widehat{\mu}$, we define

$$\Phi_{a,b}(u;\tau) := q^{-a^2/2b^2+a/2b-1/8} z^{a/b-1/2} \cdot \widehat{\mu}\bigg(\bigg(\frac{a}{b}-1\bigg)\tau, u-\frac{\tau}{2}+\frac{1}{2};\tau\bigg), \tag{3.3}$$

where $z := e^{2\pi i u}$. If $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ satisfies $\alpha, \delta \equiv 1 \pmod{2b}$, $\beta \equiv 0 \pmod{2b}$, and $\gamma \in 2\mathbb{Z}$, then one may derive the following:

$$\Phi_{a,b}\bigg(\frac{u}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\bigg) = \Psi_{a,b}(A) \cdot (\gamma\tau+\delta)^{\frac{1}{2}} e^{-\pi i \gamma u^2/(\gamma\tau+\delta)} \cdot \Phi_{a,b}(u;\tau), \tag{3.4}$$

with

$$\Psi_{a,b}(A) := \chi(A)^{-3} \cdot \exp(\pi i \cdot \Omega(A)).$$

Here the quantity $\Omega(A)$ is defined by

$$\begin{aligned}\Omega(A) := {}& -\alpha\beta\bigg(\frac{a}{b}-\frac{1}{2}\bigg)^2 + \beta\gamma\bigg(\frac{a}{b}-\frac{1}{2}\bigg) - \frac{\gamma\delta}{4} + (1-\alpha)\bigg(\frac{a}{b}-\frac{1}{2}\bigg)\\ &+ \frac{\gamma}{2} + \bigg(\frac{a}{b}-1\bigg)(\alpha-1) + \frac{1}{2}(\gamma-\alpha+1) + \bigg(\frac{a}{b}-1\bigg)\beta + \frac{1}{2}(\delta-\beta-1).\end{aligned}$$

One obtains the transformation given in (3.4) by first applying Theorem 2.2(iv), followed by repeatedly applying Theorem 2.2(ii). With this, the form $\Phi_{a,b}(u;\tau)$ becomes visible, and a tedious but straightforward calculation simplifies the factor of automorphy, resulting in (3.4).

Although Zwegers' $\widehat{\mu}$-function is essentially a weight $1/2$ non-holomorphic Jacobi form, one may obtain higher weight harmonic Maass forms by specializing some of its images under various differential operators (e.g. the heat operator). *A priori*, such images are in fact higher dimensional Jacobi forms. To obtain our results, the relevant function is

$$f_{a,b}(\tau) := \frac{1}{2\pi i} \cdot \frac{\partial}{\partial u} \Phi_{a,b}(u; b\tau)|_{u=0}. \tag{3.5}$$

By (3.4), we find that

$$f_{a,b}\bigg(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\bigg) = \Psi_{a,b}\bigg(\begin{pmatrix}\alpha & \beta b\\ \gamma/b & \delta\end{pmatrix}\bigg)(\gamma\tau+\delta)^{\frac{3}{2}} f_{a,b}(\tau),$$

where we now additionally require $\gamma \equiv 0 \pmod{b}$. By the classical theory of Dedekind's eta-function [Rad73], it is known that if γ is even, then

$$\chi\bigg(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\bigg) = \bigg(\frac{\gamma}{\delta}\bigg) i^{-\frac{1}{2}} \cdot \exp\bigg(\frac{\pi i}{12}(\alpha\gamma(1-\delta^2) + \delta(\beta-\gamma+3))\bigg).$$

If $A \in \Gamma(2b)$, then a lengthy straightforward calculation shows that

$$\Psi_{a,b}(A) = \bigg(\frac{\gamma}{\delta}\bigg).$$

This then shows that $f_{a,b}(\tau)$ is a weight $3/2$ harmonic Maass form on $\Gamma(2b)$. That it is harmonic follows from Zwegers' work, or by a simple inspection of the Fourier expansion (see, for example, [BF04, § 3]).

To complete the proof of Theorem 1.1(i), it suffices to show that $f_{a,b}(\tau)=\mathcal{M}_{a,b}(\tau)$. By the definition of $\widehat{\mu}$, we find that the non-holomorphic part of $\Phi_{a,b}$ is given by

$$\frac{i}{2}q^{-a^2/2b^2+a/2b-1/8}z^{a/b-1/2}R\bigg(\bigg(\frac{a}{b}-\frac{1}{2}\bigg)\tau-u-\frac{1}{2};\tau\bigg).$$

By the definition of R, one finds that this equals

$$-\frac{1}{2}\sum_{n\in a/b+\mathbb{Z}}\bigg(\operatorname{sgn}\bigg(n+\frac{1}{2}-\frac{a}{b}\bigg)-E\bigg(\bigg(n-\frac{u_2}{y}\bigg)\sqrt{2y}\bigg)\bigg)q^{-n^2/2}e^{2\pi inu},$$

where $u=u_1+iu_2$, and $\tau=x+iy$ as usual. Letting $\tau\mapsto b\tau$, differentiating with respect to u, setting $u=0$, and dividing by $2\pi i$ gives the non-holomorphic part of $f_{a,b}(\tau)$ which equals

$$-\frac{1}{2}\sum_{n\in a/b+\mathbb{Z}}\bigg(\operatorname{sgn}\bigg(n+\frac{1}{2}-\frac{a}{b}\bigg)-E(n\sqrt{2by})\bigg)nq^{-bn^2/2}+\frac{1}{4\pi\sqrt{2by}}\sum_{n\in a/b+\mathbb{Z}}E'(n\sqrt{2by})q^{-bn^2/2}.$$

Using the facts that

$$E(x)=\operatorname{sgn}(x)(1-\beta(x^2)),$$
$$\beta(x)=\frac{1}{\pi}x^{-\frac{1}{2}}e^{-\pi x}-\frac{1}{2\sqrt{\pi}}\cdot\Gamma(-1/2;\pi x),$$
$$E'(x)=2e^{-\pi x^2},$$

where $\Gamma(\kappa;X)$ is the usual incomplete Gamma-function, we then find that the non-holomorphic part of the weight $3/2$ non-holomorphic modular form is

$$\frac{1}{4\sqrt{\pi}}\sum_{n\in a/b+\mathbb{Z}}\operatorname{sgn}(n)n\Gamma(-1/2;2b\pi yn^2)q^{-bn^2/2}$$

(compare with Lemma 2.3). This is easily seen to equal $\mathcal{M}^-_{a,b}(\tau)$, thanks to the fact that for every positive integer n we have

$$\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi inw}}{(-i(\tau+w))^{\frac{3}{2}}}dw=i(2\pi n)^{\frac{1}{2}}\cdot\Gamma(-1/2,4\pi ny)q^{-n}.$$

To complete the proof that $f_{a,b}(\tau)=\mathcal{M}_{a,b}(\tau)$, we must now equate the holomorphic parts. Again by the definition of $\widehat{\mu}$, we find that the holomorphic part of $\Phi_{a,b}(u;\tau)$ is

$$q^{-a^2/2b^2+a/2b-1/8}z^{a/b-1/2}\cdot\mu\bigg(\bigg(\frac{a}{b}-1\bigg)\tau,u-\frac{\tau}{2}+\frac{1}{2};\tau\bigg)=\frac{q^{-a^2/2b^2+a/b-5/8}z^{a/b-1/2}}{\vartheta(u-\tau/2+1/2;\tau)}\cdot\sum_{n\in\mathbb{Z}}\frac{e^{2\pi iun}q^{n^2/2}}{1-q^{n+a/b-1}}.$$

Using the fact that

$$\frac{\partial}{\partial u}z^{-\frac{1}{2}}\vartheta(u;\tau)|_{u=(1-\tau)/2}=0,$$

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we find that the holomorphic part of $f_{a,b}(\tau) = \mathcal{M}^+_{a,b}(\tau)$. To make this last claim we need the fact that

$$\vartheta\left(\frac{1-\tau}{2};\tau\right) = -q^{-\frac{1}{8}}\Theta_0(\tau).$$

Therefore, it follows that $\mathcal{M}_{a,b}(\tau) = f_{a,b}(\tau)$ is a weight 3/2 harmonic Maass form on $\Gamma(2b)$, and this completes the proof of Theorem 1.1(i).

The proof of Theorem 1.1(ii) follows from Proposition 3.1 thanks to the fact that

$$L(a,b;q^{1-a/b},2b,q^{1/2b}) = \sum_{n\equiv a \pmod b} \frac{n(q^{1-a/b})^n \cdot q^{n^2/2b}}{1-q^n}.$$

4. Number theoretic applications of weight 3/2 harmonic Maass forms

Here we give two important examples of these weight 3/2 harmonic Maass forms. First we recall Zagier's weight 3/2 non-holomorphic Eisenstein series, which we reformulate in terms of the Rogers–Fine functions. We also consider the *spt*-function of Andrews [And]. We give a closed formula for its generating function in terms of weight 3/2 harmonic Maass forms, and we describe a recent result of the second two authors on the p-adic properties of this function.

4.1 Zagier's form

Zagier's weight 3/2 Eisenstein series is perhaps the first prominent example of a weight 3/2 harmonic Maass form. In view of the combinatorial nature of the other Maass forms in this paper, it is important to consider the combinatorial nature of this key form. Here we address this, and relate our results to earlier work of Zagier [Zag75], Hirzebruch and Zagier [HZ76] and Eichler [Eic55] pertaining to Hurwitz class numbers, which imply that the Zagier Eisenstein series $\mathcal{F}(\tau)$ (as defined in § 1) is a weight 3/2 Maass form on $\Gamma_0(4)$.

In view of Theorem 1.1, it is natural to expect that the generating function $H(\tau)$ for the Hurwitz class numbers, itself a mock theta function, may be represented in terms of basic hypergeometric series. This is indeed the case, and it turns out that [Eic55, formula (5)] implies that

$$\begin{aligned}\Theta_0(2\tau)\cdot H(\tau) = -\frac{1}{12} - \frac{1}{6}\sum_{n=1}^{\infty}\sum_{d|n}(-1)^n\, d\cdot q^n + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ d\leq\sqrt{n}}}(-1)^n dq^n \\ + \sum_{m=0}^{\infty}\biggl(2F_{RF}\Bigl(iq^{2(m+1)},1\Bigr) - 2F_{RF}\Bigl(iq^{2m},q^4\Bigr) + \frac{1}{6}F_{RF}\Bigl(iq^{(m+1)/2},1\Bigr) \\ - \frac{1}{2}F_{RF}\Bigl(iq^{m/2},q\Bigr)\biggr) + F_{RF}(i,q^4) + \frac{1}{4}F_{RF}(i,q) - \frac{1}{4}F_{RF}(1,q),\end{aligned}$$

where $F_{RF}(\alpha,\beta)$ is as defined in (1.9).

4.2 Andrews' *spt*-function

Recently, Andrews [And] introduced the function $s(n) = spt(n)$ which counts the number of smallest parts among the integer partitions of n. As an illustration, one sees that $s(4) = 10$ by

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examining the partitions of 4 below:

$$\underline{4},\quad 3+\underline{1},\quad \underline{2}+\underline{2},\quad 2+\underline{1}+\underline{1},\quad \underline{1}+\underline{1}+\underline{1}+\underline{1}.$$

The generating function for $s(n)$ is

$$S(z) := \sum_{n=0}^{\infty} s(n)q^n = \frac{1}{(q)_\infty} \cdot \sum_{n=1}^{\infty} \frac{q^n \cdot \prod_{m=1}^{n-1}(1-q^m)}{1-q^n} = q + 3q^2 + 5q^3 + 10q^4 + \cdots, \tag{4.1}$$

where $(q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$.

Recently, the first author [Bri08] related $q^{-1}S(24z)$ to weight 3/2 harmonic Maass forms, and she obtained congruences modulo primes $\ell > 3$. Further results for such primes are contained in [BGM09, Gar].

These works do not reveal any information about $s(n)$ modulo 2 and 3. To place this in proper context, we note that very little is known about $p(n)$ modulo 2 and 3. This difficulty stems from the fact that there are no known methods of mapping the generating function for $p(n)$ to spaces of holomorphic modular forms mod 2 or 3, where the powerful theory of Galois representations gives results. These technical difficulties persist for $s(n)$.

Using the results of this paper, the second two authors solved conjectures of Garvan and Sellers concerning $s(n)$ modulo 2 and 3 [FO08]. We require two further mock theta functions. The first arises from the nearly modular Eisenstein series $E_2(\tau)$. Define $D(\tau)$ by

$$D(\tau) := \frac{q^{-\frac{1}{24}}}{(q)_\infty}\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = \frac{q^{-\frac{1}{24}}}{(q)_\infty}E_2(z) = q^{-\frac{1}{24}} - 23q^{\frac{23}{24}} - \cdots. \tag{4.2}$$

The second mock theta function $L(\tau)$ is defined by

$$L(\tau) := \frac{(q^6)_\infty^2 (q^{24})_\infty^2}{(q^{12})_\infty^5} \cdot \left(\sum_{n\in\mathbb{Z}} \frac{(12n-1)q^{6n^2-\frac{1}{24}}}{1-q^{12n-1}} - \sum_{n\in\mathbb{Z}} \frac{(12n-5)q^{6n^2-\frac{25}{24}}}{1-q^{12n-5}}\right) \tag{4.3}$$

$$= q^{\frac{23}{24}} + q^{\frac{47}{24}} + q^{\frac{71}{24}} - 4q^{\frac{95}{24}} - 6q^{\frac{119}{24}} + 12q^{\frac{143}{24}} - \cdots. \tag{4.4}$$

The series $L(\tau)$ is easily seen to satisfy

$$L(\tau) = -12\mathcal{M}_{11,12}^{+}(\tau) + 12\mathcal{M}_{7,12}^{+}(\tau).$$

It is proven that

$$D(24\tau) - 12L(24\tau) - 12q^{-1}S(24\tau) = q^{-1} - 47q^{23} - 142q^{47} - 285q^{71} - 547q^{95} - \cdots$$

is a weight 3/2 weakly holomorphic modular form on $\Gamma_0(576)$ with Nebentypus $\left(\frac{12}{\bullet}\right)$, which is explicitly given in terms of a complicated expression involving Dedekind's eta-function.

Using the theory of p-adic modular forms, we have proved the following two theorems, which completely determine $s(n)$ modulo the troublesome primes 2 and 3.

Theorem 4.1 [FO08, Theorem 1.2]. *We have that $s(n)$ is odd if and only if $24n-1 = pm^2$, where m is an integer and $p \equiv 23 \pmod{24}$ is prime.*

If $p \geq 5$ is prime, then let

$$\delta(p) := (p^2-1)/24. \tag{4.5}$$

In what follows, we let $\left(\frac{\bullet}{\circ}\right)$ denote the Legendre symbol.

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Theorem 4.2 [FO08, Theorem 1.3]. *If $p \geq 5$ is prime, then for every non-negative integer n we have*

$$s(p^2n - \delta(p)) + \left(\frac{3-72n}{p}\right)s(n) + ps\left(\frac{n+\delta(p)}{p^2}\right) \equiv \left(\frac{3}{p}\right)(1+p)s(n) \pmod 3.$$

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