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HERMITE-FEJÉR TYPE INTERPOLATION AND KOROVKIN'S THEOREM

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In this note we consider Hermite-Fejér interpolation at the zeros of Jacobi polynomials and with additional boundary conditions. For the associated Hermite-Fejér type operators $F_{mr}^{(\alpha+r,\beta)}$ and special values of α,β it was proved by the first author in recent papers that one has uniform convergence on the whole interval $[-1,1]$. The second author could show by introducing the concept of asymptotic positivity how to get the known convergence results for the classical Hermite-Fejér interpolation operators. In the present paper we show, using a slightly modified Bohman-Korovkin theorem for asymptotically positive functionals, that the Hermite-Fejér type interpolation polynomials $F_{mr}^{(\alpha+r,\beta)}f$, $f \in C([-1,1])$, converge pointwise to f for arbitrary $\alpha,\beta > -1$. The convergence is uniform on $[-1+\delta, 1-\delta]$.

1. INTRODUCTION

Let $X = \{x_{km}\}$ with

$$-1 \leqslant x_{mm} < x_{m-1,m} < \ldots < x_{1m} < 1$$

be a set of nodes, $I = [-1,1]$, and $f \in C(I)$. As it is well known, there exists the uniquely determined Hermite-Fejér interpolation polynomial $F_m f \in \Pi_{2m-1}, m \in \mathbb{N}$, such that

$$F_m f(x_k) = f(x_k), \quad (F_m f)'(x_k) = 0, \quad 1 \leqslant k \leqslant m,$$

where $x_k = x_{km}$. Now let x_{km} be the roots of the m-th Jacobi polynomial $P_m^{(\alpha,\beta)}, \alpha,\beta > -1$. Szegö [6] proved that for the corresponding Hermite-Fejér interpolation polynomials the convergence

$$\lim_{m\to\infty} \|F_m^{(\alpha,\beta)} f - f\| = 0 \tag{1.1}$$

holds for any $f \in C(I)$, if $-1 < \alpha,\beta < 0$. Here $\|\cdot\|$ is the usual maximum norm on I. If $\alpha \geqslant 0$, (1.1) is no longer valid for some $f \in C(I)$. In order to have convergence we can prescribe some additional interpolation conditions at $+1$.

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Let $r \in \mathbb{N}$ be given; then the Hermite-Fejér type operator

$$F_{m,r} : C(I) \to \Pi_{2m+r-1}$$

is defined by the conditions

$$F_{m,r}f(x_k) = f(x_k), \quad 1 \leqslant k \leqslant m, \tag{1.2}$$

$$(F_{m,r}f)'(x_k) = 0, \quad 1 \leqslant k \leqslant m, \tag{1.3}$$

$$F_{m,r}f(1) = f(1), \tag{1.4}$$

$$(F_{m,r}f)^{(j)}(1) = 0, \quad 1 \leqslant j \leqslant r-1. \tag{1.5}$$

For $s \in \mathbb{N}$ we can prescribe conditions at -1, too:

$$F_{m,r,s}f(-1) = f(-1), \quad (F_{m,r,s}f)^{(l)}(-1) = 0, \quad 1 \leqslant l \leqslant s-1.$$

If $(r,s) = (1,0)$ the corresponding operators $F_{m,1,0} = F_{m,1}$ are referred to as almost Hermite-Fejér interpolation operators; if $(r,s) = (1,1)$ then we obtain the so-called quasi Hermite-Fejér polynomials (see for example the references in the bibliography [**1**]). Now let $s = 0$, that is, no conditions at -1, as the behaviour of the operators $F_{m,r,s}$ is symmetric with respect to α and β, r and $s, 1$ and -1.

For the roots x_k of the Jacobi polynomial $P_m^{(\alpha,\beta)}$ we have

$$\lim_{m\to\infty} \|F_{m,r}^{(\alpha,\beta)}f - f\| = 0$$

for each $f \in C(I)$, if $-1 < \beta < 0$, $r - 3/2 \leqslant \alpha < r$, $\alpha > -1$, and $|(\alpha - r) - \beta| \leqslant 1$ (see [**2**]).

In [4] and [**5**] it is shown that the convergence behaviour of the Hermite-Fejér interpolation operators $F_m^{(\alpha,\beta)} = F_{m,o}^{(\alpha,\beta)}$ may be treated by Korovkin's theorem in all Jacobi cases with $\alpha, \beta > -1$, that is, also for operators which are not positive in the usual sense. This was possible by introducing the concept of asymptotic positivity.

The aim of this paper is to show that the convergence results for the Hermite-Fejér type operators $F_{m,r}^{(\alpha,\beta)}$ may be proved via the Bohman-Korovkin theorem, too.

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2. Hermite-Fejér Type Interpolation at Jacobi Zeros

For $f \in C(I)$ the interpolation polynomial $F_{m,r}f$ has the representation

$$F_{m,r}f(x) = \sum_{k=1}^{m} \left(\frac{1-x}{1-x_k}\right)^r v_k(x) l_k^2(x) f(x_k) + L_{oo}(x) f(1)$$

with
$$v_k(x) = 1 + \left(\frac{r}{1-x_k} - \frac{\omega_m''(x_k)}{\omega_m'(x_k)}\right)(x - x_k),$$

$$L_{oo}(x) = \left(\frac{\omega_m(x)}{\omega_m(1)}\right)^2 \sum_{\rho=o}^{r-1} B_{\rho omr}(1-x)^\rho,$$

$$\omega_m(x) = \prod_{k=1}^{m} (x - x_k),$$

and
$$l_k(x) = \frac{\omega_m(x)}{\omega_m'(x_k)(x - x_k)}.$$

Thereby the coefficients $B_{\rho omr} \in \mathbb{R}$ are uniquely determined by the interpolation conditions (see [2] and [7]). The associated Hermite operator

$$H_{m,r} : C^{r-1}(I) \to \Pi_{2m+r-1}$$

then has the representation

$$H_{m,r}f(x) = F_{m,r}f(x) + \sum_{k=1}^{m} \left(\frac{1-x}{1-x_k}\right)^r (x - x_k) l_k^2(x) f'(x_k) + \sum_{i=1}^{r-1} L_{oi}(x) f^{(i)}(1)$$

with
$$L_{oi}(x) = \left(\frac{\omega_m(x)}{\omega_m(1)}\right)^2 \sum_{\rho=i}^{r-1} B_{\rho imr}(1-x)^\rho$$

and suitable coefficients $B_{\rho imr} \in \mathbb{R}$. We have additionally to (1.2) and (1.4) the conditions

$$(H_{m,r}f)'(x_k) = f'(x_k), \quad 1 \leqslant k \leqslant m, \text{ and} \tag{2.1}$$

$$(H_{m,r}f)^{(j)}(1) = f^{(j)}(1), \quad 1 \leqslant j \leqslant r-1. \tag{2.2}$$

Choosing the nodes x_k arbitrarily in $[-1,1[$ we obtain by Rolle's theorem that $L_{oo}(x) \geqslant 0$ for all $x \in I$. For $r = 0$ in [4] and [5] it is shown that for every $f \in C(I)$ the sequence of Hermite-Fejér interpolation polynomials to the zeros of the Jacobi polynomial $P_m^{(\alpha,\beta)}$ converges to f. The convergence is pointwise for all x with $|x| < 1$ and uniform in every closed subinterval provided $\alpha, \beta > -1$. Locher observed that the Hermite-Fejér functionals (for fixed $|x| < 1$) are asymptotically positive.

DEFINITION: We call a sequence of functionals $L_m : C(I) \to \mathbb{R}$ *asymptotically positive*, if and only if there exist positive functionals $Q_m : C(I) \to \mathbb{R}$ and continuous functionals $R_m : C(I) \to \mathbb{R}$ with

$$L_m = Q_m + R_m \text{ for each } m \in \mathbb{N}$$

and

$$\lim_{m \to \infty} \|R_m\| = 0 \tag{2.3}$$

Here $\|\cdot\|$ is the operator norm induced by the maximum norm on $C(I)$.

We obtain the

THEOREM. (Bohman-Korovkin) *Let $(L_m)_{m \geqslant 1}$ be asymptotically positive; then for each $f \in C(I), x \in I$ and $\delta_m > 0$ there holds*

$$|L_m f - f(x)| \leqslant |f(x)||Q_m e_o - e_o(x)| + \left(Q_m e_o + \delta_m^{-2} Q_m g_x\right) \cdot \omega(f, \delta_m; I) + |R_m f|.$$

Here $\omega(f, \cdot; I)$ denotes the modulus of continuity of f, g_x denotes the function with $g_x(t) = (x-t)^2$ and e_o denotes the constant function with $e_o(t) = 1$ for each $t \in I$.

For fixed $|x| < 1$ the functional $F_{m,r,x} : C(I) \to \mathbb{R}$ is defined by

$$F_{m,r,x} f := F_{m,r} f(x).$$

Because $L_{oo}(x) \geqslant 0$ it is

$$\begin{aligned} F_{m,r,x} f &= \left[\sum_{\substack{k=1 \\ v_k(x) \geqslant o}}^{m} \left(\frac{1-x}{1-x_k} \right)^r v_k(x) l_k^2(x) f(x_k) + L_{oo}(x) f(1) \right] \\ &\quad + \sum_{\substack{k=1 \\ v_k(x) < o}}^{m} \left(\frac{1-x}{1-x_k} \right)^r v_k(x) l_k^2(x) f(x_k) \\ &=: Q_{m,x} f + R_{m,x} f. \end{aligned}$$

Then $Q_{m,x} : C(I) \to \mathbb{R}$ is positive and $R_{m,x} : C(I) \to \mathbb{R}$ is continuous with

$$\|R_{m,x}\| = \sup_{\|f\| \leqslant 1} |R_{m,x} f| = \sum_{\substack{k=1 \\ v_k(x) < o}}^{m} \left(\frac{1-x}{1-x_k} \right)^r |v_k(x)| l_k^2(x).$$

Because $F_{m,r,x} e_o = 1$ it follows that

$$Q_{m,x} e_o = F_{m,r,x} e_o - R_{m,x} e_o = 1 + \|R_{m,x}\|$$

and $$Q_{m,x} g_x = F_{m,r,x} g_x - R_{m,x} g_x \leqslant |F_{m,r,x} g_x| + 4 \cdot \|R_{m,x}\|.$$

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We have to compute $F_{m,r,x}g_x$; because $g_x = H_{m,r}g_x$ for $m \geqslant 2$ it follows that

$$\begin{aligned} g_x(t) &= H_{m,r}g_x(t) \\ &= F_{m,r}g_x(t) + \sum_{k=1}^{m} \left(\frac{1-t}{1-x_k}\right)^r (t-x_k)l_k^2(t)g_x'(x_k) + \sum_{i=1}^{\min(2,r-1)} L_{oi}(t)g_x^{(i)}(1) \\ &=: F_{m,r}g_x(t) - 2\sum_{k=1}^{m} \left(\frac{1-t}{1-x_k}\right)^r \left(\frac{\omega_m(t)}{\omega_m'(x_k)}\right)^2 \frac{x-x_k}{t-x_k} + S_1(t) + S_2(t) \end{aligned}$$

with

$$S_1(t) = 2\left(\frac{\omega_m(t)}{\omega_m(1)}\right)^2 \sum_{\rho=1}^{r-1} B_{\rho 1 m r}(1-t)^\rho(1-x)$$

and

$$S_2(t) = 2\left(\frac{\omega_m(t)}{\omega_m(1)}\right)^2 \sum_{\rho=2}^{r-1} B_{\rho 2 m r}(1-t)^\rho.$$

$g_x(x) = 0$ implies

$$F_{m,r,x}g_x = 2\omega_m^2(x)(1-x)^r \sum_{k=1}^{m} \frac{1}{(1-x_k)^r} \frac{1}{(\omega_m'(x_k))^2} - (S_1(x) + S_2(x)).$$

3. Generalised Gauss-Lobatto Formulae

Now we consider formulae of the form

$$\int_{-1}^{1} (1-t)^\alpha(1+t)^\beta P(t)dt = \sum_{k=1}^{m} \lambda_{kmr}^{(\alpha,\beta)} P(x_{km}) + \sum_{i=o}^{r-1} \mu_{imr}^{(\alpha,\beta)} P^{(i)}(1) + Q_{mr}^{(\alpha,\beta)}(P)$$

for $\alpha,\beta > -1$ with $Q_{mr}^{(\alpha,\beta)}(P) = 0$ for all $P \in \Pi_{2m+r-1}$. Let $x_k = x_{km}^{(\alpha+r,\beta)}$ be the roots of the Jacobi polynomial $P_m = P_m^{(\alpha+r,\beta)}$, and $H_{m,r}^{(\alpha+r,\beta)}$ the corresponding Hermite operator. Then for $P \in \Pi_{2m+r-1}$ we have

$$\begin{aligned} &\int_{-1}^{1} (1-t)^\alpha(1+t)^\beta P(t)dt \\ &\quad= \int_{-1}^{1} (1-t)^\alpha(1+t)^\beta F_{m,r}^{(\alpha+r,\beta)} P(t)dt \\ &\quad+ \sum_{k=1}^{m} \left(\int_{-1}^{1} (1-t)^{\alpha+r}(1+t)^\beta l_k(t)\frac{P_m(t)}{(1-x_k)^r P_m'(x_k)}dt\right) P'(x_k) \\ &\quad+ \sum_{i=1}^{r-1} \left(\int_{-1}^{1} (1-t)^\alpha(1+t)^\beta L_{oi}(t)dt\right) P^{(i)}(1) \\ &\quad=: \sum_{k=1}^{m} \lambda_{kmr}^{(\alpha,\beta)} P(x_k) + \sum_{k=1}^{m} \tilde{\lambda}_{kmr}^{(\alpha,\beta)} P'(x_k) + \sum_{i=o}^{r-1} \mu_{imr}^{(\alpha,\beta)} P^{(i)}(1). \end{aligned}$$

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Because $l_k \in \Pi_{m-1}$ the orthogonality relation yields $\widetilde{\lambda}_{kmr}^{(\alpha,\beta)} = 0$. Furthermore we get

$$\lambda_{kmr}^{(\alpha,\beta)} = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \left(\frac{1-t}{1-x_k}\right)^r l_k^2(t)dt.$$

With these coefficients $\lambda_{kmr}^{(\alpha,\beta)}$ and $\mu_{imr}^{(\alpha,\beta)}$ we obtain a solution of the problem stated above. It is (see [3])

$$\lambda_{kmr}^{(\alpha,\beta)} = c_{m,r}^{(\alpha,\beta)} \frac{1}{(1-x_k)^r(1-x_k^2)\left(P_m^{(\alpha+r,\beta)'}(x_k)\right)^2}$$

with

$$c_{m,r}^{(\alpha,\beta)} = 2^{\alpha+r+\beta+1}\frac{\Gamma(m+\alpha+r+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+r+\beta+1)}$$

and $\mu_{omr}^{(\alpha,\beta)} \geqslant 0$ because $L_{oo} \geqslant 0$.

4. Convergence of the Operators $F_{m,r,x}^{(\alpha+r,\beta)}$

Lemma 1. *For arbitrary $\alpha, \beta > -1$ and fixed $|x| < 1$ it follows*

$$|F_{m,r,x}^{(\alpha+r,\beta)} g_x| = \mathcal{O}\left(\left(P_m^{(\alpha+r,\beta)}(x)\right)^2\right).$$

Proof: We have

$$F_{m,r,x}^{(\alpha+r,\beta)} g_x = 2\left(P_m^{(\alpha+r,\beta)}(x)\right)^2 (1-x)^r \cdot \frac{1}{c_{m,r}^{(\alpha,\beta)}} \sum_{k=1}^{m} \lambda_{kmr}^{(\alpha,\beta)}(1-x_k^2) - (S_1(x)+S_2(x)).$$

Because $P_m^{(\alpha+r,\beta)}(1) = \binom{m+\alpha+r}{m}$ and $|B_{\rho imr}| = \mathcal{O}\left(m^{2(\rho-i)}\right)$ for $\rho \geqslant i$ (see [2] and [7]), we obtain

$$\begin{aligned}|S_i(x)| &= \left(P_m^{(\alpha+r,\beta)}(x)\right)^2 \mathcal{O}\left(m^{-2(\alpha+r)}\right) \sum_{\rho=i}^{r-1} m^{2(\rho-i)} \\ &= \left(P_m^{(\alpha+r,\beta)}(x)\right)^2 \mathcal{O}\left(m^{-2(i+\alpha+1)}\right).\end{aligned}$$

Further, we have

$$\begin{aligned}\sum_{k=1}^{m} \lambda_{kmr}^{(\alpha,\beta)}(1-x_k^2) &\leqslant \sum_{k=1}^{m} \lambda_{kmr}^{(\alpha,\beta)} + \mu_{omr}^{(\alpha,\beta)} \\ &= \int_{-1}^{1} (1-t)^\alpha(1+t)^\beta dt = d^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.\end{aligned}$$

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Now we consider $\|R_{m,x}\|$. The differential equation for the Jacobi polynomials yields

$$v_k(x) = v_{km}^{(\alpha+r,\beta)}(x) = 1 - \frac{\alpha - \beta + (\alpha+\beta+2)x_k}{1 - x_k^2}(x - x_k).$$

Hint: Here $x_k = x_{km}^{(\alpha+r,\beta)}$ is a root of $P_m^{(\alpha+r,\beta)}$.

LEMMA 2. (See [4]) *If* $|x| < 1$ *and* $v_{km}^{(\alpha+r,\beta)}(x) < 0$, *there exists an* $\varepsilon = \varepsilon(x,\alpha,\beta) > 0$ *with*

$$|x - x_k| \geqslant \varepsilon \quad \text{and} \quad |\pm 1 - x_k| \geqslant \varepsilon$$

for all $m \geqslant m_o(\varepsilon)$.

Now we can prove the following.

LEMMA 3. *For fixed* $|x| < 1$ *we obtain with* $R_{mx} = R_{mx}^{(\alpha+r,\beta)}$:

$$\|R_{mx}\| = \mathcal{O}\left(\left(P_m^{(\alpha+r,\beta)}(x)\right)^2\right).$$

PROOF: Lemma 2 implies, with $P_m = P_m^{(\alpha+r,\beta)}$,

$$\begin{aligned}\|R_{mx}\| &= -(1-x)^r (P_m(x))^2 \cdot \sum_{\substack{k=1 \\ v_k(x)<o}}^{m} \frac{1}{(1-x_k)^r(1-x_k^2)(P_m'(x_k))^2(x-x_k)^2} \\ &\quad \cdot \left[1 - x(\alpha-\beta+(\alpha+\beta+2)x_k) + (\alpha-\beta)x_k + (\alpha+\beta+1)x_k^2\right] \\ &\leqslant (P_m(x))^2 \cdot \frac{\rho}{\varepsilon^{2+r}} \sum_{k=1}^{m} \frac{1}{(1-x_k^2)(P_m'(x_k))^2},\end{aligned}$$

where ρ is an upper bound of

$$|1 - x(\alpha-\beta+(\alpha+\beta+2)x_k) + (\alpha-\beta)x_k + (\alpha+\beta+1)x_k^2|,$$

for example $\rho = 1 + 2|\alpha-\beta| + (\alpha+\beta+2) + |\alpha+\beta+1|$.

Now we have

$$\begin{aligned}\sum_{k=1}^{m} \frac{1}{(1-x_k^2)(P_m'(x_k))^2} &= \frac{1}{c_{m,r}^{(\alpha,\beta)}} \sum_{k=1}^{m} \lambda_{kmo}^{(\alpha+r,\beta)} \\ &= \frac{1}{c_{m,r}^{(\alpha,\beta)}} \int_{-1}^{1} (1-t)^{\alpha+r}(1+t)^\beta dt = \mathcal{O}(1) \quad \text{for} \quad m \to \infty.\end{aligned}$$

□

Thus we get the following.

Theorem. *For fixed* $|x| < 1$ *we set*

$$\delta_m^2(x) := |F_{m,r,x}^{(\alpha+r,\beta)} g_x| + 4 \cdot \|R_{m,x}^{(\alpha+r,\beta)}\|.$$

Then it follows for each $f \in C(I)$*:*

$$|F_{m,r}^{(\alpha+r,\beta)} f(x) - f(x)| \leqslant 2 \cdot \|f\| \cdot \|R_{mx}^{(\alpha+r,\beta)}\| + \left(2 + \|R_{mx}^{(\alpha+r,\beta)}\|\right) \omega(f, |\delta_m(x)|;\ I)$$
$$= \mathcal{O}\left(\left(P_m^{(\alpha+r,\beta)}(x)\right)^2\right) + \mathcal{O}\left(\omega\left(f, |P_m^{(\alpha+r,\beta)}(x)|\ ;\ I\right)\right).$$

Remark. If $|x| \leqslant 1 - \delta$ with $0 < \delta < 1$ and $v_{km}^{(\alpha+r,\beta)}(x) < 0$, there exists an $\varepsilon = \varepsilon(\alpha,\beta) > 0$, independently of x, with $|x - x_k| \geqslant \varepsilon$ and $|\pm 1 - x_k| \geqslant \varepsilon$ for all $m \geqslant m_0(\varepsilon)$. Because $|P_m^{(\alpha+r,\beta)}(x)| = o(1)$ uniformly for $|x| \leqslant 1 - \delta$ for all $\alpha, \beta > -1$ (see [6]) we get

$$\lim_{m\to\infty} \|F_{m,r}^{(\alpha+r,\beta)} f - f\|_{[-1+\delta, 1-\delta]} = 0$$

for every $f \in C(I)$. Here $\|\cdot\|_{[-1+\delta,1-\delta]}$ is the maximum norm on $[-1+\delta, 1-\delta]$.

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