MATRIX INVARIANTS AND COMPLETE INTERSECTIONS

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Consider the vector space of m-tuples of n by n matrices

$$X_{m,n} = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C}).$$

The linear group $\mathrm{GL}_n(\mathbb{C})$ acts on $X_{m,n}$ by simultaneous conjugation. The corresponding ring of polynomial invariants

$$\mathbb{C}[X_{m,n}]^{\mathrm{GL}_n(\mathbb{C})}$$

will be denoted by $C(n, m)$ and is called the ring of matrix invariants of m-tuples of n by n matrices. C. Procesi has shown in **[8]** that $C(n, m)$ is generated by traces of products of the corresponding generic matrices and, as such, coincides with the center of the trace ring of m generic n by n matrices $R(n, m)$, which is also the ring of equivariant maps from $X_{m,n}$ to $M_n(\mathbb{C})$.

Apart from this general result, very little is known about the explicit structure of $C(n, m)$. In **[7]**, it is shown that $C(2, 2)$ is the polynomial algebra

$$\mathbb{C}[\mathrm{Tr}(X_1), \mathrm{Tr}(X_2), \mathrm{Det}(X_1), \mathrm{Det}(X_2), \mathrm{Tr}(X_1X_2)].$$

The structure of $C(2, 3)$ was determined by Formanek **[0]**, see also **[3]** or **[10]**. Consider the polynomial algebra

$$\mathbb{C}[\mathrm{Tr}(X_1), \mathrm{Tr}(X_2), \mathrm{Tr}(X_3), D(X_1), D(X_2), D(X_3), \mathrm{Tr}(X_1X_2), \mathrm{Tr}(X_2X_3), \mathrm{Tr}(X_3X_1)];$$

then $C(2, 3)$ is a free module over this algebra of rank 2 generated by 1 and $\mathrm{Tr}(X_1X_2X_3)$. Moreover, $\mathrm{Tr}(X_1X_2X_3)$ satisfies the quadratic equation

$$X^2 - AX + B = O,$$

where

$$A = \mathrm{Tr}(X_1)\mathrm{Tr}(X_2X_3) + \mathrm{Tr}(X_2)\mathrm{Tr}(X_1X_3) + \mathrm{Tr}(X_3)\mathrm{Tr}(X_1X_2) - \mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\mathrm{Tr}(X_3),$$

$$\begin{aligned} B = {} & D(X_1)\mathrm{Tr}(X_2X_3)^2 + D(X_2)\mathrm{Tr}(X_1X_3)^2 + D(X_3)\mathrm{Tr}(X_1X_2)^2 \\ & - \mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\mathrm{Tr}(X_1X_2)D(X_3) - \mathrm{Tr}(X_1)\mathrm{Tr}(X_3)\mathrm{Tr}(X_1X_3)D(X_2) \\ & - \mathrm{Tr}(X_2)\mathrm{Tr}(X_3)\mathrm{Tr}(X_2X_3)D(X_1) \\ & + \mathrm{Tr}(X_1)^2D(X_2)D(X_3) + \mathrm{Tr}(X_2)^2D(X_1)D(X_3) \\ & + \mathrm{Tr}(X_3)^2D(X_1)D(X_2) \\ & - 4D(X_1)D(X_2)D(X_3) + \mathrm{Tr}(X_1X_2)\mathrm{Tr}(X_1X_2)\mathrm{Tr}(X_2X_3). \end{aligned}$$

In general, $C(2, m)$ is a polynomial algebra in m variables over the center of the generic Clifford algebra for m-ary quadratic forms of degree at most 4, see **[2]**, and it can be expressed in terms of $\mathrm{SO}_3(\mathbb{C})$-invariants, see **[9]**.

The structure of $C(3, 2)$ was determined in **[10]** and implicitly in **[4]**. Consider the polynomial algebra

$$\mathbb{C}[\mathrm{Tr}(X_1), \mathrm{Tr}(X_1^2), \mathrm{Tr}(X_1^3), \mathrm{Tr}(X_2), \mathrm{Tr}(X_2^2), \mathrm{Tr}(X_2^3), \mathrm{Tr}(X_1X_2), \mathrm{Tr}(X_1X_2^2), \mathrm{Tr}(X_1^2X_2), \mathrm{Tr}(X_1^2X_2^2)];$$

$C(3, 2)$ is a free module of rank 2 over this algebra generated by 1 and $\mathrm{Tr}(X_1X_2X_1^2X_2^2)$. Apart from these results only $C(4, 2)$ is known, see [**10**].

What is known about the homological properties of $C(n, m)$? In view of the Hochster–Roberts results, $C(n, m)$ is a Cohen–Macaulay algebra and, because it is the ring of invariants of the simple group $\mathrm{PGL}_n(\mathbb{C})$, it is a unique factorization domain and hence Gorenstein, see for example [**1**]. In [**5**], it is shown that $C(n, m)$ is never regular except when $(m, n) = (2, 2)$, which is, as we have seen above, a polynomial algebra. Recall that an algebra $\mathbb{C}[X_1, \ldots, X_k]/I$ is said to be a complete intersection if the height of I coincides with the minimal number of generators of I. It follows from the above explicit descriptions that $C(2, 3)$ and $C(3, 2)$ are hypersurfaces and hence complete intersections. The main result of this note will assert that there are no others.

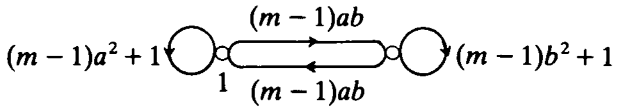
Before we come to the proof, we recall some results of [**5**]. Let $V_{m,n}$ be the variety corresponding to $C(n, m)$; then it is well known that $V_{m,n}$ parametrizes the isomorphism classes of semi-simple n-dimensional representations of $\mathbb{C}\langle X_1, \ldots, X_m\rangle$. A point $\xi \in V_{m,n}$ is said to be of representation type $\tau = (e_1, k_1; \ldots; e_r, k_r)$ if the corresponding isomorphism class of semi-simple representations is built from r distinct simple components of dimensions k_i occurring with multiplicity e_i. If τ is such a representation type then $V_{m,n}(\tau)$ is defined to be the subset of $V_{m,n}$ consisting of all points of representation type τ. In [**5**], it is shown that the sets $V_{m,n}(\tau)$ form a finite stratification into locally closed smooth subvarieties. Moreover $V_{m,n}(\tau)$ lies in the closure of $V_{m,n}(\tau')$ if and only if τ is a degeneration of τ'. If $\xi \in V_{m,n}$ is of representation type $(e_1, k_1; \ldots; l_r, k_r)$ then one forms the quiver Δ_ξ consisting of r vertices $(x_1, \ldots, x_r)$ and $(m-1)k_i^2 + 1$ loops in vertex x_1 and $(m-1)k_ik_j$ directed edges from x_i to x_j. Let d_ξ be the dimension vector $(e_1, \ldots, e_r)$. Then it is proved in [**5**] that the étale locale structure of $V_{m,n}$ near ξ is that of the variety $V(\Delta_\xi, d_\xi)$ of semi-simple representations of the quiver Δ_ξ of dimension vector d_ξ near the origin. Moreover, the coordinate ring of $V(\Delta_\xi, d_\xi)$ is generated by traces of oriented cycles in the quiver Δ_ξ, see [**6**].

Let V be the variety corresponding to the algebra $\mathbb{C}[x_1, \ldots, x_k]/I$; then V is said to be locally a complete intersection in v if $I\mathbb{C}[x_1, \ldots, x_k]_m$ is generated by a regular sequence of length the height of I for all maximal ideals m lying over v. It is clear that the subset of all points $v \in V$ such that V is locally a complete intersection in v forms an open subvariety $V^{\mathrm{c.i}}$. We are now in a position to state the main result.

THEOREM. *If* $(m, n) \neq (2, 2), (2, 3)$ *or* $(3, 2)$ *then the locally complete intersection locus* $V_{m,n}^{\mathrm{c.i}}$ *coincides with the open subvariety of regular points* $V_{m,n}^{\mathrm{reg}} = V_{m,n}[(1, n)]$.

Proof. Since being locally a complete intersection can be expressed in homological terms, it is preserved under étale extensions; so we only need to study the étale local structure of $V_{m,n}$ near a point ξ. Suppose $V_{m,n}^{\mathrm{c.i}}$ is strictly larger than $V_{m,n}[(1, n)]$, which is precisely the nonsingular locus when $(m, n) \neq (2, 2)$, see [**5**, Theorem II.3.4]. Then, by the stratification result mentioned before, $V_{m,n}^{\mathrm{c.i}}$, must contain a point ξ corresponding to a semi-simple representation having two distinct simple components, i.e. ξ is of type

$(1, a; 1, b)$ with $a + b = n$. The corresponding quiver Δ_ξ is



and the dimension vector $d_\xi = (1, 1)$. The coordinate ring of $V(\Delta_\xi, d_\xi)$ is then a polynomial algebra in $(m-1)(a^2+b^2)+2$ variables over

$$\mathbb{C}[W] = \mathbb{C}[t_{ij} : 1 \leq i, j \leq (m-1)ab]/I_2,$$

where I_2 is the ideal generated by the determinants of all 2 by 2 minors of the generic matrix $(t_{ij})_{i,j}$. It is well known that W is a complete intersection in the origin if and only if $(m-1)ab = 2$. By étale decent, this finishes the proof.

In view of the explicit descriptions of $C(2, 2)$, $C(2, 3)$ and $C(3, 2)$ given before, and the above theorem, we obtain the following corollary immediately.

COROLLARY. *The ring of matrix invariants* $C(m, n)$ *is a complete intersection if and only if* $(m, n) = (2, 2)$, $(2, 3)$ *or* $(3, 2)$. *In particular, it is a complete intersection if and only if the corresponding ring of equivariant maps* $R(n, m)$ *is of finite global dimension.*

The last statement follows from [**5**].

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