Conclusion

The numbers \( n \) and \( m \) constructed in both cases above are the closest solutions to the previous triangle-square pair. The recursive method will capture all triangular numbers that are also square by starting with 1, the first number that is both triangular and square. The tree structure below shows triangle-square pairs \((n, m)\) satisfying \( \frac{1}{2}n(n + 1) = m^2 \) starting at the top with the root \((1, 1)\). A branch to the left or right indicates that case 1 or case 2 was used.

\[(1, 1) \quad (8, 6) \quad (49, 35) \]
\[(288, 204) \quad (1681, 1189) \quad (9800, 6930) \quad (57121, 40391) \]

The sequence of square-triangular numbers is \( 1 = 1^2 \), \( 36 = 6^2 \), \( 1225 = 35^2 \), \( 41616 = 204^2 \), \( 1413721 = 1189^2 \), \( 49024900 = 6930^2 \), \( 1631432881 = 40391^2 \), ....

References

JINGCHENG TONG and PETER A. BRAZA
Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

87.56 Conjugates of Pythagorean triples

Introduction

Let \( x, y, z \) be three positive integers. If \( x^2 + y^2 = z^2 \), then we say that \((x, y, z)\) is a Pythagorean triple. If furthermore, \( \gcd(x, y, z) = 1 \), \((x, y, z)\) is called primitive. In many textbooks on number theory, one can find the following result.
Theorem 1: The positive integers \( x, y, z \) form a primitive Pythagorean triple, with \( y \) even, if, and only if, there are relatively prime positive integers \( m \) and \( n, \) \( m > n, \) with opposite parity, such that \( x = m^2 - n^2, \) \( y = 2mn, \) \( z = m^2 + n^2. \)

If \( m \) and \( n \) are exchanged, \( y, z \) do not change, but \( x \) will be a negative integer. Is there a new expression for \( x, y, z \) that involves two parameters \( p, q \) and is symmetric, i.e. if \( p, q \) are exchanged, \( (x, y, z) \) is still a Pythagorean triple?

In this note we give such a symmetric expression, and from this expression we obtain a structure of conjugacy on Pythagorean triples, which is similar to the conjugacy of complex numbers.

Symmetric expression

Let \( q = m - n, p = n \) in Theorem 1. Then

\[
\begin{align*}
x &= q(q + 2p), \\
y &= 2p(p + q) \\
z &= (p + q)^2 + p^2.
\end{align*}
\]

In the above expression, \( p, q \) are coprime positive integers and \( q \) is always odd.

The significance of the new expression is that it leads a new triple \( (x^*, y^*, z^*) \) by exchanging \( p, q \)

\[
\begin{align*}
x^* &= p(p + 2q) \\
y^* &= 2q(p + q) \\
z^* &= (p + q)^2 + q^2.
\end{align*}
\]

And hence

\[
\begin{align*}
x^* &= \frac{1}{2}(x + 2y - z), \\
y^* &= x - y + z, \\
z^* &= \frac{1}{2}(x - 2y + 3z).
\end{align*}
\]

Main results

Before we go any further, we must have a convention. In Theorem 1, \( y \) has to be an even number. Here we hold it as a rule:

Definition 1. A Pythagorean triple \( (x, y, z) \) is said to be normalised if \( y \) is even.

Definition 2. Let \( (x, y, z) \) be a normalised Pythagorean triple, then \( (x^*, y^*, z^*) \) obtained by transformation (1) is called the conjugate of \( (x, y, z) \).

Definition 3. A normalised Pythagorean triple is called self-conjugated if its conjugate is itself.
The Definition 2 of conjugate is shown to be well-defined by the following Propositions 1 to 3.

**Proposition 1.** If \((x, y, z)\) is a normalised Pythagorean triple, so is \((x^*, y^*, z^*)\).

**Proof.** Since \(y^* = (x + z) - y\), \(y^*\) is always even because \(x, z\) are both even or both odd, and \(y\) is even. It is easily checked that

\[
(x^*)^2 + (y^*)^2 = (z^*)^2 = \frac{1}{4}(x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz).
\]

**Proposition 2.** If \((x, y, z)\) is a normalised Pythagorean triple with conjugate \((x^*, y^*, z^*)\) and \(k\) is a positive integer, then the Pythagorean triple \((kx, ky, kz)\) has conjugate \((kx^*, ky^*, kz^*)\).

**Proof:** Since transformation (1) is linear, the result is trivial.

**Proposition 3.** If the conjugate of \((x, y, z)\) is \((x^*, y^*, z^*)\), then the conjugate of \((x^*, y^*, z^*)\) is \((x, y, z)\).

**Proof:** From (1), we have

\[
\begin{align*}
x + 2y - z &= 2x^*, \\
x - y + z &= y^*, \\
x - 2y + 3z &= 2z^*.
\end{align*}
\]

Solving this system of equations, we have

\[
\begin{align*}
x &= \frac{1}{2}(x^* + 2y^* - z^*), \\
y &= x^* - y^* + z^*, \\
z &= \frac{1}{2}(x^* - 2y^* + 3z^*).
\end{align*}
\]

**Proposition 4.** A normalised Pythagorean triple \((x, y, z)\) is self-conjugated if, and only if, \(\frac{x}{y} = \frac{3}{4}\).

**Proof:** From transformation (1), \(x = x^* = \frac{1}{2}(x + 2y - z)\), hence \(z = 2y - x\), so \(z^2 = x^2 + y^2 = 4y^2 - 4xy + x^2\), i.e. \(y(3y - 4x) = 0\). Therefore \(\frac{x}{y} = \frac{3}{4}\).

Proposition 4 establishes the unique character of the normalised Pythagorean triple \((3, 4, 5)\): every self-conjugated normalised Pythagorean triple is an integral multiple of the \((3, 4, 5)\).

Now we discuss the similarity of conjugacies between the set \(\mathcal{P}\) of all normalised Pythagorean triples and the set of complex numbers.

Let \(\alpha\) be a normalised Pythagorean triple, and \(\alpha^*\) be its conjugate. Then we have

(i) \(\alpha \in \mathcal{P} \rightarrow \alpha^* \in \mathcal{P}\);

(ii) \((\alpha^*)^* = \alpha\);

(iii) \(\alpha = \alpha^*\) if and only if \(\alpha\) is a multiple of \((3, 4, 5)\);

(iv) \((k\alpha)^* = k\alpha^*\) for any positive integer \(k\).
This is the reason we think of the terminology ‘conjugate’. The conjugate is good only for normalised Pythagorean triples since transformation (1) could produce fractions for non-normalised ones like (4, 3, 5).

JINGCHENG TONG

Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

87.57 Using Pythagorean triples to generate square roots of $I_2$

A. J. B. Ward [1] provides a procedure for iteratively solving non-linear matrix equations, and points out that these equations can have multiple solutions. For example, he notes that, surprisingly, $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}$ is a square root of $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (making the former matrix its own inverse). It is interesting to pursue the issue of the square roots of $I_2$, and we find an unexpected connection to Pythagorean triples: positive integers $x, y, z$ such that $x^2 + y^2 = z^2$ so that $x, y, z$ form the sides of an integer right triangle.

It turns out that there are an infinite number of symmetric as well as an infinite number of non-symmetric square roots of $I_2$, and we focus on the symmetric ones. All of these can be found from $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implying

$$a^2 + b^2 = 1$$

$$b(a + c) = 0$$

$$b^2 + c^2 = 1.$$ 

If $b = 0$ then $a = \pm 1$ and $c = \pm 1$ giving us the diagonal square root matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

If $b \neq 0$, then $(a + c) = 0$ so $c = -a$, and $a^2 + b^2 = 1$ so $b = \pm \sqrt{1 - a^2}$. Hence all the non-diagonal symmetric square roots of $I_2$ are given by $\begin{pmatrix} a & \pm \sqrt{1 - a^2} \\ \pm \sqrt{1 - a^2} & -a \end{pmatrix}$ with $a \neq \pm 1$. Focusing on real solutions we require $a^2 < 1$, and further focusing on rational solutions we require that $a$ and $\sqrt{1 - a^2}$ be rational.

For rational solutions we can write, in terms of a common denominator, $a = \pm x/z$ and $\sqrt{1 - a^2} = y/z$, where $x$ is a non-negative integer and $y$ and $z$ are positive integers, and where $(y/z)^2 = 1 - (x/z)^2$ implying $x^2 + y^2 = z^2$. So we have found this result: