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Abstract

An affine symplectic singularity X with a good $\mathbf{C}^*$-action is called a conical symplectic variety. In this paper we prove the following theorem. For fixed positive integers N and d, there are only a finite number of conical symplectic varieties of dimension $2d$ with maximal weights N, up to an isomorphism. To prove the main theorem, we first relate a conical symplectic variety with a log Fano Kawamata log terminal (klt) pair, which has a contact structure. By the boundedness result for log Fano klt pairs with fixed Cartier index, we prove that conical symplectic varieties of a fixed dimension and with a fixed maximal weight form a bounded family. Next we prove the rigidity of conical symplectic varieties by using Poisson deformations.

1. Introduction

An affine variety X is conical if X can be written as $\mathrm{Spec}\, R$ with a finitely generated domain R over $\mathbf{C}$ which is positively graded: $R = \bigoplus_{i \geqslant 0} R_i$ where $R_0 = \mathbf{C}$. The grading determines a $\mathbf{C}^*$-action on X, and the origin $0 \in X$ defined by the maximal ideal $m := \bigoplus_{i>0} R_i$ is a unique fixed point of the $\mathbf{C}^*$-action. We often say that X has a good $\mathbf{C}^*$-action in such a situation.

A conical symplectic variety (X, ω) is a pair of a conical normal affine variety X and a holomorphic symplectic 2-form ω on the regular locus X_{reg}, where (i) ω extends to a holomorphic 2-form on a resolution $Z \to X$ of X, and (ii) ω is homogeneous with respect to the $\mathbf{C}^*$-action.

Conical symplectic varieties play an important role in algebraic geometry (cf. [Bea00, Nam13c]) and geometric representation theory (cf. [BPW12, BLPW14]). Examples are nilpotent orbit closures of a semisimple complex Lie algebra (cf. [CM93]), Slodowy slices to such nilpotent orbits [Slo80] and Nakajima quiver varieties [Nak94].

Two conical symplectic varieties (X_1, ω_1) and (X_2, ω_2) are called isomorphic if there is a $\mathbf{C}^*$-equivariant isomorphism $\varphi : X_1 \to X_2$ such that $\omega_1 = \varphi^* \omega_2$.

Take a set of minimal homogeneous generators $\{z_0, \ldots, z_n\}$ of the $\mathbf{C}$-algebra R. We may assume that $wt(z_0) \leqslant wt(z_1) \leqslant \cdots \leqslant wt(z_n)$ and further that their greatest common divisor is 1. We put $a_i := wt(z_i)$. Then the $(n+1)$-tuple $(a_0, \ldots, a_n)$ is uniquely determined by the graded algebra R. We call the number a_n the maximal weight of R. We state our main result.

Main Theorem. *For positive integers N and d, there are only a finite number of conical symplectic varieties of dimension $2d$ with maximal weights N, up to an isomorphism.*

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Example. We must bound the maximal weight of X for the Main Theorem to hold. In fact, A_{2n-1} surface singularities

$$f := x^{2n} + y^2 + z^2 = 0, \quad \omega := \mathrm{Res}(dx \wedge dy \wedge dz/f)$$

are conical symplectic varieties of dimension 2; their weights are $(1, n, n)$ and the maximal weights are not bounded above.

The proof of the Main Theorem consists of two parts. First, we shall relate a conical symplectic variety X of dimension $2d$ with a contact Fano orbifold $\mathbf{P}(X)^{\mathrm{orb}}$ of dimension $2d-1$. The underlying variety $\mathbf{P}(X)$ of the orbifold is equipped with a divisor Δ with standard coefficients, and $(\mathbf{P}(X), \Delta)$ is a log Fano variety with Kawamata log terminal (klt) singularities. If we fix the maximal weight N of X, then the Cartier index of $-K_{\mathbf{P}(X)} - \Delta$ is bounded above by a constant depending only on d and N. By a recent result of Hacon *et al.* [HMX14], the set of all such log Fano varieties forms a bounded family. This fact enables us to construct a flat family of conical symplectic varieties over a quasi-projective base so that any conical symplectic variety of dimension $2d$ with the maximal weight N appears somewhere in this family (Proposition 2.11).

Second, we shall prove that all fibres of the flat family on the same connected component are isomorphic as conical symplectic varieties (Proposition 3.3). Notice that a symplectic variety has a natural Poisson structure and the family constructed above can be regarded as a Poisson deformation of the symplectic variety. A conical symplectic variety X has a universal Poisson deformation over an affine space [Nam11]. The central fibre X of the universal family has a $\mathbf{C}^*$-action, but no nearby fibre does. This fact means that X is rigid under a Poisson deformation together with the $\mathbf{C}^*$-action (Corollary 3.2) and Proposition 3.3 follows. The Main Theorem is a corollary of Propositions 2.11 and 3.3.

2. Contact orbifolds

In this section (X, ω) is a conical symplectic variety of dimension $2d$ with the maximal weight N. By definition ω is homogeneous with respect to the $\mathbf{C}^*$-action. We denote by l the degree (weight) of ω. By [Nam13a, Lemma 2.2], we have $l > 0$.

By using the minimal homogeneous generators in the introduction we have a surjection from the polynomial ring to R,

$$\mathbf{C}[x_0, \dots, x_n] \to R,$$

which sends each x_i to z_i. Correspondingly, X is embedded in $\mathbf{C}^{n+1}$. The quotient variety $\mathbf{C}^{n+1} - \{0\}/\mathbf{C}^*$ by the $\mathbf{C}^*$-action $(x_0, \dots, x_n) \to (t^{a_0}x_0, \dots, t^{a_n}x_n)$ is the weighted projective space $\mathbf{P}(a_0, \dots, a_n)$. We put $\mathbf{P}(X) := X - \{0\}/\mathbf{C}^*$. By definition $\mathbf{P}(X)$ is a closed subvariety of $\mathbf{P}(a_0, \dots, a_n)$. Put $W_i := \{x_i = 1\} \subset \mathbf{C}^{n+1}$. Then the projection map $p : \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}(a_0, \dots, a_n)$ induces a map $p_i : W_i \to \mathbf{P}(a_0, \dots, a_n)$, which is a finite Galois covering of the image. The collection $\{p_i\}$ defines a smooth orbifold structure on $\mathbf{P}(a_0, \dots, a_n)$ in the sense of [Mum83, § 2]. More exactly, the following conditions are satisfied.

(i) For each i, W_i is a smooth variety, $p_i : W_i \to p_i(W_i)$ is a finite Galois covering,[1] and $\bigcup \mathrm{Im}(p_i) = \mathbf{P}(a_0, \dots, a_n)$.

(ii) Let $(W_i \times_{\mathbf{P}(a_0,\dots,a_n)} W_j)^n$ denote the normalization of the fibre product $W_i \times_{\mathbf{P}(a_0,\dots,a_n)} W_j$. Then the maps $(W_i \times_{\mathbf{P}(a_0,\dots,a_n)} W_j)^n \to W_i$ and $(W_i \times_{\mathbf{P}(a_0,\dots,a_n)} W_j)^n \to W_j$ are both étale maps.

[1] The precise definition of an orbifold only needs a slightly weaker condition: $p_i : W_i \to \mathbf{P}(a_0, \dots, a_n)$ factorizes as $W_i \xrightarrow{q_i} W_i/G_i \xrightarrow{r_i} \mathbf{P}(a_0, \dots, a_n)$ where G_i is a finite group and r_i is an étale map.

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The orbifold $\mathbf{P}(a_0, \ldots, a_n)$ admits an orbifold line bundle $O_{\mathbf{P}(a_0,\ldots,a_n)}(1)$. Put $D_i := \{x_i = 0\} \subset \mathbf{P}(a_0, \ldots, a_n)$ and $D := \bigcup D_i$. Since x_i are minimal generators, $\bar{D} := \mathbf{P}(X) \cap D$ is a divisor of $\mathbf{P}(X)$. Let

$$\bar{D} = \bigcup \bar{D}_\alpha$$

be the decomposition into irreducible components.[2]

The map $p : X - \{0\} \to \mathbf{P}(X)$ is a $\mathbf{C}^*$-fibre bundle over $\mathbf{P}(X) - \bar{D}$. But a fibre over a general point of $\bar{D}_\alpha$ may possibly be a multiple fibre. We denote its multiplicity by m_α.

By putting $U_i := X \cap W_i$ and $\pi_i := p_i|_{U_i}$, the collection $\{\pi_i : U_i \to \mathbf{P}(X)\}$ of covering maps induces a (not necessarily smooth) orbifold structure on $\mathbf{P}(X)$. Namely, we have the following properties.

(i) For each i, U_i is a normal variety, $\pi_i : U_i \to \pi_i(U_i)$ is a finite Galois covering, and $\bigcup \mathrm{Im}(\pi_i) = \mathbf{P}(X)$.

(ii) The maps $(U_i \times_{\mathbf{P}(X)} U_j)^n \to U_i$ and $(U_i \times_{\mathbf{P}(X)} U_j)^n \to U_j$ are both étale maps.

We put $\mathcal{L} := O_{\mathbf{P}(a_0,\ldots,a_n)}(1)|_{\mathbf{P}(X)}$, which is an orbifold line bundle on $\mathbf{P}(X)$. We call $\mathcal{L}$ the tautological line bundle. Then $X - \{0\} \to \mathbf{P}(X)$ can be regarded as an orbifold $\mathbf{C}^*$-bundle $(\mathcal{L}^{-1})^\times$. Notice that X has only rational Gorenstein singularities and, in particular, the log pair $(X, 0)$ of the X and the zero divisor has klt singularities. We define a $\mathbf{Q}$-divisor Δ by

$$\Delta := \sum (1 - 1/m_\alpha)\bar{D}_\alpha.$$

The following lemma [Nam13b, § 1, Lemma] will be a key step toward our main theorem.

Lemma 2.1. *The pair $(\mathbf{P}(X), \Delta)$ is a log Fano variety, that is, $(\mathbf{P}(X), \Delta)$ has klt singularities and $-(K_{\mathbf{P}(X)} + \Delta)$ is an ample $\mathbf{Q}$-divisor.*

Moreover, the symplectic structure on X induces a contact orbifold structure on $\mathbf{P}(X)$ [Nam13a, Theorem 4.4.1]. We shall briefly explain this. First of all, a contact structure on a complex manifold Z of dimension $2d - 1$ is an exact sequence of vector bundles

$$0 \to E \xrightarrow{j} \Theta_Z \xrightarrow{\theta} L \to 0,$$

with a vector bundle E of rank $2d - 2$ and a line bundle L. Here θ induces a pairing map

$$E \times E \to L \ (x, y) \to \theta([j(x), j(y)]),$$

and we require that it is non-degenerate. If Z admits such a contact structure, then we have $-K_Z \cong L^{\otimes d}$. The map θ can be regarded as a section of $\Omega^1_Z \otimes L$, and we call it the contact form. Moreover, L is called a contact line bundle.

We can slightly generalize this notion to a singular variety Z. Let us assume that Z is a normal variety of dimension $2d - 1$ and let L be a line bundle on Z. If Z_{reg} admits a contact structure with the contact line bundle $L|_{Z_{\mathrm{reg}}}$, then we call it a contact structure on Z. The twisted 1-form $\theta \in \Gamma(Z_{\mathrm{reg}}, \Omega^1_Z \otimes L|_{Z_{\mathrm{reg}}})$ is also called the contact form.

We now go back to our situation. As explained above, $\mathbf{P}(X)$ admits orbifold charts $U_i \to \mathbf{P}(X)$. The orbifold line bundle $O_{\mathbf{P}(a_0,\ldots,a_n)}(1)$ restricts to a line bundle L_i on U_i, and the collection $\{L_i\}$ determines an orbifold line bundle $\mathcal{L}$ on $\mathbf{P}(X)$. We then have a contact

[2] The index α is usually different from the original index i of D_i because $D_{i_1} \cap \cdots \cap D_{i_k} \cap \mathbf{P}(X)$ may possibly become an irreducible component of $\bar{D}$ or $D_i \cap \mathbf{P}(X)$ may split into more than two irreducible components of $\bar{D}$.

structure on each U_i with the contact line bundle $L_i^{\otimes l}$, where $l = wt(\omega)$. Let us denote the contact form by θ_i. Notice that θ_i is a section of $\Omega^1_{(U_i)_{\mathrm{reg}}} \otimes L_i^{\otimes l}|_{(U_i)_{\mathrm{reg}}}$. Consider the diagram

$$U_i \xleftarrow{p_i} (U_i \times_{\mathbf{P}(X)} U_j)^n \xrightarrow{p_j} U_j.$$

Then we have $p_i^*(\theta_i) = p_j^*(\theta_j)$ for all i and j. Thus $\theta := \{\theta_i\}$ can be regarded as a section of $\mathcal{H}\mathrm{om}(\Theta_{\mathbf{P}(X)^{\mathrm{orb}}}, \mathcal{L}^{\otimes l})$. We call the pair $(\theta, \mathcal{L}^{\otimes l})$ a contact orbifold structure on $\mathbf{P}(X)^{\mathrm{orb}}$, and the orbifold line bundle $\mathcal{L}^{\otimes l}$ is called its contact line bundle. Similarly to the ordinary case, we have an isomorphism $-K_{\mathbf{P}(X)^{\mathrm{orb}}} \cong \mathcal{L}^{\otimes ld}$ of orbifold line bundles.

By the construction of $\mathbf{P}(X)$, the orbifold line bundle $\mathcal{L}^{\otimes N!}$ is a usual line bundle on $\mathbf{P}(X)$ and so is $-K^{\otimes N!}_{\mathbf{P}(X)^{\mathrm{orb}}}$. Notice here that $K_{\mathbf{P}(X)^{\mathrm{orb}}} = p^*(K_{\mathbf{P}(X)} + \Delta)$, where $p : \mathbf{P}(X)^{\mathrm{orb}} \to \mathbf{P}(X)$ is the natural map. Therefore $N!(K_{\mathbf{P}(X)} + \Delta)$ is a Cartier divisor.

Theorem 2.2 (Hacon *et al.* [HMX14, Corollary 1.8]). *Let m and r be fixed positive integers. Let $\mathcal{D}$ be the set of klt log Fano pairs (Y, Δ) such that $\dim Y = m$ and $-r(K_Y + \Delta)$ are ample Cartier divisors. Then $\mathcal{D}$ forms a bounded family.*

In particular, the self-intersection number $(-K_Y - \Delta)^m$ is bounded above by some constant depending only on m and r.

We now apply Theorem 2.2 above by putting $r = N!$.

Lemma 2.3. *The weight l of ω is bounded above by some constant depending only on d and N.*

Proof. Since $-K_{\mathbf{P}(X)^{\mathrm{orb}}} \cong \mathcal{L}^{\otimes ld}$ and $\mathcal{L}^{\otimes N!} = p^*L$ for $L \in \mathrm{Pic}(\mathbf{P}(X))$, we have $-(K_{\mathbf{P}(X)} + \Delta) \sim_{\mathbf{Q}} ld/N! \cdot L$. Here $(-K_{\mathbf{P}(X)} - \Delta)^{2d-1}$ is bounded above by a constant depending on N and d. On the other hand, L^{2d-1} is a positive integer; this implies that l must be bounded above by a constant depending on N and d. □

Lemma 2.4. *The number of the minimal homogeneous generators of R is bounded above by some constant depending only on d and N.*

Proof. By Theorem 2.2 there is a positive integer q (which is a multiple of r) depending only on r and d such that $q(-K_{\mathbf{P}(X)} + \Delta)$ is a very ample Cartier divisor and $h^0(\mathbf{P}(X), q(-K_{\mathbf{P}(X)} - \Delta)) = h^0(\mathbf{P}(X), -K^{\otimes q}_{\mathbf{P}(X)^{\mathrm{orb}}})$ is bounded above by a constant depending on r and d.

There are only finitely many possibilities for a weight of R because the weight is less than or equal to N. Since $r = N!$, the integer q is a multiple of any possible weight.

Note that $-K^{\otimes q}_{\mathbf{P}(X)^{\mathrm{orb}}} \cong \mathcal{L}^{\otimes qld}$. Take an arbitrary weight, say a. Suppose that exactly s elements (say, $z_1, \ldots, z_s$) have the weight a among the minimal homogeneous generators. Note that these are elements of $H^0(\mathbf{P}(X), \mathcal{L}^{\otimes a})$. Write $q = q'a$. Then $(z_1)^{q'dl}$, $(z_1)^{q'dl-1}z_2$, $\ldots, (z_1)^{q'dl-1}z_s$ are linearly independent elements of $H^0(\mathbf{P}(X), \mathcal{L}^{\otimes qld})$. In fact, suppose to the contrary that there is a non-trivial relation $\lambda_1(z_1)^{q'dl} + \lambda_2(z_1)^{q'dl-1}z_2 + \cdots + \lambda_s(z_1)^{q'dl-1}z_s = 0$. Then we have an equality

$$(z_1)^{q'dl-1} \cdot (\lambda_1 z_1 + \cdots + \lambda_s z_s) = 0$$

in $R = \bigoplus_{i \geqslant 0} H^0(\mathbf{P}(X), \mathcal{L}^{\otimes i})$. Since R is a domain, we conclude that $z_1 = 0$ or $\Sigma \lambda_i z_i = 0$. But, by the assumption, both z_1 and $\Sigma \lambda_i z_i$ are non-zero, which is absurd; hence $(z_1)^{q'dl}$, $(z_1)^{q'dl-1}z_2$, $\ldots, (z_1)^{q'dl-1}z_s$ are linearly independent. This means that

$$s \leqslant h^0(\mathbf{P}(X), \mathcal{L}^{\otimes qld}) = h^0(\mathbf{P}(X), -K^{\otimes q}_{\mathbf{P}(X)^{\mathrm{orb}}}).$$

In particular, s is bounded above by a constant depending only on d and N. □

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The graded coordinate ring of a weighted projective space is called a weighted polynomial ring.

COROLLARY 2.5. *Fix positive integers d and N. Then there are finitely many weighted polynomial rings $S_1, \ldots, S_k$ such that any graded coordinate ring R of a conical symplectic variety of dimension $2d$ with the maximal weight N can be realized as a quotient of some S_i.*

In the corollary, we put $\mathbf{P}_i := \mathrm{Proj}(S_i)$. Then we have the following result.

COROLLARY 2.6. *There are flat families of closed subschemes of $\mathbf{P}_i (1 \leqslant i \leqslant k)$, $\mathcal{Y}_i \subset \mathbf{P}_i \times T_i$, parameterized by reduced quasi-projective schemes T_i such that, for any conical symplectic variety X of dimension $2d$ with the maximal weight N, there is a point $t \in T_i$ for some i and $\mathbf{P}(X) = \mathcal{Y}_{i,t}$.*

Proof. Let q be the least common multiple of all weights of the minimal homogeneous generators of all S_i. Then $O_{\mathbf{P}_i}(q)$ is an ample line bundle for every i. Take a conical symplectic symplectic variety X of dimension $2d$ with the maximal weight N. Then $\mathbf{P}(X)$ can be embedded in some $\mathbf{P}_i$. By Theorem 2.2 there are only finitely many possibilities of the Hilbert polynomial $\chi(\mathbf{P}(X), O_{\mathbf{P}(X)}(qn))$. Such closed subschemes of $\mathbf{P}_i$ form a bounded family. □

Let $\mathcal{Y} \subset \mathbf{P} \times T$ be one of the flat families in Corollary 2.6. Define a map $f : \mathcal{Y} \to T$ to be the composite $\mathcal{Y} \to \mathbf{P} \times T \stackrel{pr_2}{\to} T$. Let $\{W_i \to \mathbf{P}\}$ be the orbifold charts for the weighted projective space $\mathbf{P}$ constructed in the beginning of this section. Denote by G_i the Galois group for $W_i \to \mathbf{P}$. Then the collection $\{W_i \times T \to \mathbf{P} \times T\}$ also gives relative orbifold charts for $\mathbf{P} \times T/T$. By pulling back these charts by the inclusion map $\mathcal{Y} \to \mathbf{P} \times T$, we have relative orbifold charts $\{\mathcal{U}_i \stackrel{\pi_i}{\to} \mathcal{Y}\}$ for $\mathcal{Y}/T$. If necessary, we stratify T into a disjoint union of a finite number of locally closed sets by using the generic flatness property (cf. [Mum86, Lecture 8, Proposition]) repeatedly so that all $\mathcal{U}_i$ are flat over each stratum and replace T by the disjoint union of such subsets. Thus we may assume that $\mathcal{U}_i$ are all flat over T. Let $O^{\mathrm{orb}}_{\mathbf{P}}(1) := \{O_{W_i}(1)\}$ be the tautological orbifold line bundle on $\mathbf{P}$. Denote simply by $O_{W_i \times T}(1)$ the pullback of $O_{W_i}(1)$ by the projection $W_i \times T \to W_i$. Then $\{O_{W_i \times T}(1)|_{\mathcal{U}_i}\}$ gives a relative tautological orbifold line bundle $O^{\mathrm{orb}}_{\mathcal{Y}}(1)$ on $\mathcal{Y}$. For each $j \in \mathbf{Z}$, we define a usual sheaf $\mathcal{L}^j$ on $\mathcal{Y}$ by $\mathcal{L}^j := \{\pi_*^{G_i} O_{W_i \times T}(j)|_{\mathcal{U}_i}\}$. Notice that $\mathcal{L}^j$ is flat over T. On the other hand, for $t \in T$, one can consider the orbifold structure on $\mathcal{Y}_t$ induced by the embedding $\mathcal{Y}_t \subset \mathbf{P}$. We similarly define a tautological orbifold line bundle $O^{\mathrm{orb}}_{\mathcal{Y}_t}(1)$ and the usual sheaves $\mathcal{L}^j_t$ on $\mathcal{Y}_t$. We have $\mathcal{L}^j \otimes_{O_{\mathcal{Y}}} O_{\mathcal{Y}_t} \cong \mathcal{L}^j_t$.

We define

$$T' := \{t \in T \mid f_*\mathcal{L}^j \otimes_{O_T} k(t) \cong H^0(\mathcal{Y}_t, \mathcal{L}^j_t) \text{ and } f_*\mathcal{L}^j \text{ are locally free at } t \text{ for all } j \geqslant 0\}.$$

LEMMA 2.7. *The set T' is a non-empty Zariski open subset of T.*

Proof. First we show that there is a positive integer j_0 such that $H^1(\mathcal{Y}_t, \mathcal{L}^j_t) = 0$ for all $j \geqslant j_0$ and for all $t \in T$. Take a positive integer q so that $O_{\mathbf{P}}(q)$ is a very ample line bundle on $\mathbf{P}$. Notice that $\mathcal{L}^j \otimes O_{\mathbf{P}}(q) \cong \mathcal{L}^{j+q}$ for all j. We consider the sheaves $\mathcal{L}^j$ for j with $0 \leqslant j < q$. Notice that they are flat over T.

We shall prove that there is a positive integer n_j such that $H^1(\mathcal{Y}_t, \mathcal{L}^j_t(qn)) = 0$ for all $n \geqslant n_j$ and for all t. As T is of finite type over $\mathbf{C}$, we can take a positive integer n_j so that $R^p f_* \mathcal{L}^j(qn) = 0$ for all $p > 0$ and for all $n \geqslant n_j$ (Serre vanishing theorem). Notice that, for any point $t \in T$, one

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has $H^p(\mathcal{Y}_t, \mathcal{L}_t^j(qn)) = 0$ for $p > 2d-1$. Fix an integer n with $n \geqslant n_j$ and let us consider the base change map $\varphi^p(t) : R^p f_*\mathcal{L}^j(qn) \otimes k(t) \to H^p(\mathcal{Y}_t, \mathcal{L}_t^j(qn))$. By the base change theorem, when $\varphi^p(t)$ is surjective, the map $\varphi^{p-1}(t)$ is a surjection if and only if $R^p f_*\mathcal{L}^j(qn)$ is locally free at t. First of all, $\varphi^{2d}(t)$ is a surjection because $H^{2d}(\mathcal{Y}_t, \mathcal{L}_t^j(qn)) = 0$. Since $R^{2d} f_*\mathcal{L}^j(qn) = 0$, the map $\varphi^{2d-1}(t)$ is also a surjection. We can repeat this argument to conclude that $\varphi^1(t)$ is a surjection because $R^p f_*\mathcal{L}^j(qn) = 0$ for all $p > 0$. Since $R^1 f_*\mathcal{L}^j(qn) = 0$, we finally obtain that $H^1(\mathcal{Y}_t, \mathcal{L}_t^j(qn)) = 0$.

Put $\nu := \max\{n_0, \dots, n_{q-1}\}$. Then we have $H^1(\mathcal{Y}_t, \mathcal{L}_t^j) = 0$ for all $j \geqslant q\nu$ and for all $t \in T$. By the base change theorem, $f_*\mathcal{L}^j$ are locally free and $f_*\mathcal{L}^j \otimes_{O_T} k(t) \cong H^0(\mathcal{Y}_t, \mathcal{L}_t^j)$ for all $j \geqslant q\nu$ and for all $t \in T$.

We next consider the sheaves $\mathcal{L}^j$ for $j < q\nu$. For each such j, it is an open condition for T that $f_*\mathcal{L}^j$ is locally free at t and $f_*\mathcal{L}^j \otimes_{O_T} k(t) \cong H^0(\mathcal{Y}_t, \mathcal{L}_t^j)$ holds. Therefore T' is a non-empty Zariski open subset of T. □

Define

$$\mathcal{X} := (\mathbf{Spec}_T \oplus_{j \geqslant 0} f_*\mathcal{L}^j) \times_T T'.$$

As each direct summand $f_*\mathcal{L}^j$ is flat over T, the map $\mathcal{X} \to T'$ is flat. Let us return to Corollary 2.6. The construction above enables us to make a flat family $\mathcal{X}_i \to T'_i$ of affine schemes with good $\mathbf{C}^*$-actions on an open subset T'_i of each T_i.

Corollary 2.8. *There is a flat family of affine schemes with good $\mathbf{C}^*$-actions $\mathcal{X} \to T$ parameterized by reduced quasi-projective schemes T such that, for any conical symplectic variety X of dimension $2d$ with the maximal weight N, there is a point $t \in T$ and $X \cong \mathcal{X}_t$ as a $\mathbf{C}^*$-variety.*

Proof. The family $\mathcal{X} \to T$ is nothing but the disjoint union of $\{\mathcal{X}_i \to T'_i\}$. Let X be a conical symplectic variety of dim $2d$ with the maximal weight N. By Corollary 2.6 there is a point t of some T_i and $\mathbf{P}(X) = \mathcal{Y}_{i,t} \subset \mathbf{P}_i$. Since the coordinate ring R of X is normal, the natural maps $H^0(\mathbf{P}_i, O_{\mathbf{P}_i}(j)) \to H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(j))$ are surjective for all $j \geqslant 0$. This fact implies that $t \in T'_i$ and $\mathcal{X}_{i,t} = X$. □

Let $f : \mathcal{X} \to T$ be the flat family in Corollary 2.8. By Elkik [Elk78, Théorème 4], the set

$$\mathcal{X}^0 := \{x \in \mathcal{X} \mid \mathcal{X}_{f(x)} \text{ has rational singularities at } x\}$$

is a Zariski open subset of $\mathcal{X}$. By the $\mathbf{C}^*$-action we also see that

$$T^0 := \{t \in T \mid \mathcal{X}_t \text{ has rational singularities}\}$$

is a Zariski open subset of T. We take a resolution T_{rat} of T^0. Note that T_{rat} is the disjoint union of finitely many non-singular quasi-projective varieties. We put $\mathcal{X}_{\mathrm{rat}} := \mathcal{X} \times_T T_{\mathrm{rat}}$ and let $f_{\mathrm{rat}} : \mathcal{X}_{\mathrm{rat}} \to T_{\mathrm{rat}}$ be the induced flat family. Again by [Elk78], $\mathcal{X}_{\mathrm{rat}}$ has only rational singularities. In particular, it is normal. Notice that any conical symplectic variety of dimension $2d$ with the maximal weight N is realized as a fibre of this family.

We next stratify T_{rat} into the disjoint union of locally closed smooth subsets $T_{\mathrm{rat},i}$ so that $\mathcal{X}_{\mathrm{rat}} \times_{T_{\mathrm{rat}}} T_{\mathrm{rat},i} \to T_{\mathrm{rat},i}$ have $\mathbf{C}^*$-equivariant simultaneous resolutions. To obtain such a stratification, we first take a $\mathbf{C}^*$-equivariant resolution $\tilde{\mathcal{X}}_{\mathrm{rat}} \to \mathcal{X}_{\mathrm{rat}}$. By Bertini's theorem there is an open subset T^0_{rat} of T_{rat} such that this resolution gives simultaneous resolutions of fibres over T^0_{rat}. Next stratify the complement $T_{\mathrm{rat}} - T^0_{\mathrm{rat}}$ into locally closed smooth subsets, take maximal strata and repeat the same for the families over them. Thus we have proved the following result.

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PROPOSITION 2.9. *There is a flat family of affine varieties with good* $\mathbf{C}^*$*-actions* $\mathcal{X} \to T$ *parameterized by the disjoint union* T *of a finite number of quasi-projective non-singular varieties such that:*

(i) $\mathcal{X}_t$ *have only rational singularities for all* $t \in T$;

(ii) *there is a* $\mathbf{C}^*$*-equivariant simultaneous resolution* $\mathcal{Z} \to \mathcal{X}$ *of* $\mathcal{X}/T$ *(namely,* $\mathcal{Z}_t \to \mathcal{X}_t$ *are resolutions for all* $t \in T$*); and*

(iii) *for any conical symplectic variety* X *of dimension* $2d$ *with the maximal weight* N*, there is a point* $t \in T$ *and* $X \cong \mathcal{X}_t$ *as a* $\mathbf{C}^*$*-variety.*

Let $\mathcal{X} \xrightarrow{f} T$ and $\mathcal{Z} \xrightarrow{g} T$ be the families in Proposition 2.9. Let us consider the relative dualizing sheaf $\omega_{\mathcal{X}/T}$ of f. For $t \in T$, we have $\omega_{\mathcal{X}/T} \otimes_{O_{\mathcal{X}}} O_{\mathcal{X}_t} \cong \omega_{\mathcal{X}_t}$. The locus $T_{\mathrm{gor}} \subset T$ where $\omega_{\mathcal{X}_t}$ is invertible is an open subset of T. We put $\mathcal{X}_{\mathrm{gor}} := \mathcal{X} \times_T T_{\mathrm{gor}}$ and $\mathcal{Z}_{\mathrm{gor}} := \mathcal{Z} \times_T T_{\mathrm{gor}}$. Then f and g respectively induce maps $f_{\mathrm{gor}} : \mathcal{X}_{\mathrm{gor}} \to T_{\mathrm{gor}}$ and $g_{\mathrm{gor}} : \mathcal{Z}_{\mathrm{gor}} \to T_{\mathrm{gor}}$. Notice that any conical symplectic variety X of dimension $2d$ with the maximal weight N still appears in some fibre of f_{gor}.

PROPOSITION 2.10 (Base change theorem). *Let* $h : W \to S$ *be a morphism of quasi-projective schemes over* $\mathbf{C}$*. Assume that* W *is normal and* $\mathbf{C}^*$ *acts on* W *fibrewise with respect to* h*. Let* F *be a* $\mathbf{C}^*$*-linearized coherent* O_W*-module on* W*, which is flat over* S*. Then the higher direct image sheaves* $R^i h_* F$ *are naturally graded:* $R^i h_* F = \bigoplus_{j \in \mathbf{Z}} (R^i h_* F)(j)$*. Assume that* $(R^i h_* F)(j_0)$ $(i \geqslant 0)$ *are all coherent sheaves on* S *for* j_0*. Then the following properties hold.*

(a) *For each* i*, the function* $S \to \mathbf{Z}$ *defined by*

$$s \to \dim H^i(W_s, F_s)(j_0)$$

is upper-semicontinuous on S.

(b) *Assume that* S *is reduced and connected. If the function* $s \to \dim H^i(W_s, F_s)(j_0)$ *is constant, then* $(R^i h_* F)(j_0)$ *is a locally free sheaf on* S *and, for all* $s \in S$*, the natural map* $\phi^i_s : (R^i h_* F)(j_0) \otimes_{O_T} k(s) \to H^i(W_s, F_s)(j_0)$ *is an isomorphism.*

We can take $\mathbf{C}^*$-equivariant affine open coverings of W by the theorem of Sumihiro (cf. [KKMS73, ch. I, § 2]). Then the proof of Proposition 2.10 is similar to [Mum70, II, 5].

We apply this proposition to $g_{\mathrm{gor}} : \mathcal{Z}_{\mathrm{gor}} \to T_{\mathrm{gor}}$ and $\Omega^k_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}}$. Notice that $(R^i(g_{\mathrm{gor}})_* \Omega^k_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l)$ are all coherent sheaves on T_{gor} for any l. Let us consider the relative differential map

$$((g_{\mathrm{gor}})_* \Omega^2_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l) \xrightarrow{d} ((g_{\mathrm{gor}})_* \Omega^3_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l)$$

and put

$$\mathcal{F} := \mathrm{Ker}(d), \quad \mathcal{G} := \mathrm{Coker}(d).$$

Fix an integer l. Then one can find a non-empty Zariski open dense subset T_l of T_{gor} so that, if $t \in T_l$, then both $\mathcal{F}$ and $\mathcal{G}$ are free at t,

$$((g_{\mathrm{gor}})_* \Omega^2_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l) \otimes k(t) \cong H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^2_{\mathcal{Z}_{\mathrm{gor},t}})(l),$$

and

$$((g_{\mathrm{gor}})_* \Omega^3_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l) \otimes k(t) \cong H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^3_{\mathcal{Z}_{\mathrm{gor},t}})(l).$$

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A d-closed 2-form ω_0 of weight l on a fibre $\mathcal{Z}_{\mathrm{gor},t}$ $(t \in T_l)$ is an element of $\mathrm{Ker}[H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^2_{\mathcal{Z}_{\mathrm{gor},t}})(l) \xrightarrow{d} H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^3_{\mathcal{Z}_{\mathrm{gor},t}})(l)]$. By the exact sequence

$$\mathcal{F} \otimes k(t) \to H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^2_{\mathcal{Z}_{\mathrm{gor},t}})(l) \xrightarrow{d} H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^3_{\mathcal{Z}_{\mathrm{gor},t}})(l),$$

ω_0 comes from an element $\omega_0' \in \mathcal{F} \otimes k(t)$. Then ω_0' lifts to a local section ω of $\mathcal{F}$. If we regard ω as a local section of $((g_{\mathrm{gor}})_* \Omega^2_{\mathcal{Z}_{\mathrm{gor}}/T_{\mathrm{gor}}})(l)$, it is a d-closed relative 2-form extending the original ω_0.

Assume that $\mathcal{X}_t$ $(t \in T_l)$ is a conical symplectic variety and ω_0 is the extension of the symplectic 2-form on $\mathcal{X}_{t,\mathrm{reg}}$ to the resolution $\mathcal{Z}_t$. The wedge product $\wedge^d \omega_0$ is regarded as a section of the dualizing sheaf $\omega_{\mathcal{X}_{\mathrm{gor},t}}$ by the identification $H^0(\mathcal{Z}_{\mathrm{gor},t}, \Omega^{2d}_{\mathcal{Z}_{\mathrm{gor},t}}) \cong H^0(\mathcal{X}_{\mathrm{gor},t}, \omega_{\mathcal{X}_{\mathrm{gor},t}})$. Then $\wedge^d \omega_0$ generates the invertible sheaf $\omega_{\mathcal{X}_{\mathrm{gor},t}}$. We also see that $\wedge^d \omega$ generates $\omega_{\mathcal{X}_{\mathrm{gor}}/T_{\mathrm{gor}}}$ on near fibres of $\mathcal{X}_{\mathrm{gor},t}$.

The argument here shows that

$$T_{\mathrm{symp},l} := \{t \in T_l \mid \mathcal{X}_{\mathrm{gor},t} \text{ is a conical symplectic variety with a symplectic form of weight } l\}$$

is an open subset of T_l. We have fixed an integer l. But notice that the choice of such an l is finite by Lemma 2.3.

We put $\mathcal{X}_{\mathrm{symp},l} := \mathcal{X} \times_T T_{\mathrm{symp},l}$ and $\mathcal{Z}_{\mathrm{symp},l} := \mathcal{Z} \times_T T_{\mathrm{symp},l}$. Then $\mathcal{X}_{\mathrm{symp},l} \to T_{\mathrm{symp},l}$ is a flat family of conical symplectic varieties with symplectic forms of weight l and $\mathcal{Z}_{\mathrm{symp},l} \to T_{\mathrm{symp},l}$ is its simultaneous resolution.

To specify the symplectic form on each fibre of $\mathcal{X}_{\mathrm{symp},l} \to T_{\mathrm{symp},l}$, we consider the vector bundle $p : V_l := \mathbf{V}(\mathcal{F}^*|_{T_{\mathrm{symp},l}}) \to T_{\mathrm{symp},l}$. Each point $v \in V_l$ corresponds to a d-closed holomorphic 2-form ω_v on the regular part of $(\mathcal{X}_{\mathrm{symp},l})_{p(v)}$. Moreover, ω_v extends to a holomorphic 2-form on the resolution $(\mathcal{Z}_{\mathrm{symp},l})_{p(v)}$. Let V_l^0 be the non-empty Zariski open subset of V_l where ω_v is non-degenerate. Take the base change $\mathcal{X}_{\mathrm{symp},l} \times_{T_{\mathrm{symp},l}} V_l^0 \to V_l^0$. Then the regular locus of a fibre of this family is naturally equipped with a symplectic 2-form of weight l.

Stratify $T \backslash T_l$ into locally closed smooth subsets, take maximal strata and repeat the same for the families over them. Then we get the following proposition.

Proposition 2.11. *There is a flat family of the pairs of affine symplectic varieties with good $\mathbf{C}^*$-actions and symplectic forms: $(\mathcal{X}, \omega_{\mathcal{X}/T}) \to T$ parameterized by the disjoint union T of a finite number of quasi-projective non-singular varieties such that:*

(i) *for each connected component T_i of T, all fibres $(\mathcal{X}_t, \omega_t)$ over $t \in T_i$ are conical symplectic varieties admitting symplectic forms of a fixed weight $l_i > 0$;*

(ii) *there is a $\mathbf{C}^*$-equivariant simultaneous resolution $\mathcal{Z} \to \mathcal{X}$ of $\mathcal{X}/T$ (namely, $\mathcal{Z}_t \to \mathcal{X}_t$ are resolutions for all $t \in T$); and*

(iii) *for any conical symplectic variety (X, ω) of dimension $2d$ with the maximal weight N, there is a point $t \in T$ and $(X, \omega) \cong (\mathcal{X}_t, \omega_t)$ as a $\mathbf{C}^*$-symplectic variety.*

3. Rigidity of conical symplectic varieties

Let (X, ω) be a conical symplectic variety with a symplectic form ω of weight l. The symplectic form ω determines a Poisson structure on X_{reg}. By the normality of X, this Poisson structure uniquely extends to a Poisson structure $\{\,,\}$ on X. Here a Poisson structure on X precisely means a skew-symmetric $\mathbf{C}$-bilinear map $\{\,,\} : O_X \times O_X \to O_X$ which is a biderivation with respect to the first and second factors, and satisfies the Jacobi identity. We will consider a Poisson deformation of the Poisson variety. A T-scheme $\mathcal{X} \to T$ is called a Poisson T-scheme if there is a O_T-bilinear Poisson bracket $\{\,,\}_{\mathcal{X}} : O_{\mathcal{X}} \times O_{\mathcal{X}} \to O_{\mathcal{X}}$, which is a biderivation and satisfies the Jacobi identity. Let T be a scheme over $\mathbf{C}$ and let $0 \in T$ be a closed point.

A Poisson deformation of the Poisson variety X over T is a Poisson T-scheme $f : \mathcal{X} \to T$ together with an isomorphism $\varphi : \mathcal{X}_0 \cong X$ which satisfies the following conditions:

(i) f is a flat surjective morphism; and

(ii) $\{\,,\}_{\mathcal{X}}$ restricts to the original Poisson structure $\{\,,\}$ on X via the identification φ.

Two Poisson deformations $(\mathcal{X}/T, \varphi)$ and $(\mathcal{X}'/T, \varphi')$ with the same base are equivalent if there is a T-isomorphism $\mathcal{X} \cong \mathcal{X}'$ of Poisson schemes such that it induces the identity on the central fibre. For a local Artinian $\mathbf{C}$-algebra A with residue field $\mathbf{C}$, we define $\mathrm{PD}_X(A)$ to be the set of equivalence classes of Poisson deformations of X over $\mathrm{Spec}(A)$. Then it defines a functor

$$\mathrm{PD}_X : (\mathrm{Art})_{\mathbf{C}} \to (\mathrm{Set})$$

from the category of local Artinian $\mathbf{C}$-algebra with residue field $\mathbf{C}$ to the category of sets.

THEOREM 3.1 [Nam11, Theorem 5.5]. *There is a Poisson deformation $\mathcal{X}_{\mathrm{univ}} \to \mathbf{A}^m$ of X over an affine space $\mathbf{A}^m$ with $\mathcal{X}_{\mathrm{univ},0} = X$. This Poisson deformation has the following properties and is called the universal Poisson deformation of X.*

(i) *For any Poisson deformation $\mathcal{X} \to T$ of X over $T = \mathrm{Spec}(A)$ with $A \in (\mathrm{Art})_{\mathbf{C}}$, there is a unique morphism $\phi : T \to \mathbf{A}^m$ which sends the closed point of T to the centre $0 \in \mathbf{A}^m$ such that $\mathcal{X}/T$ and $\mathcal{X}_{\mathrm{univ}} \times_{\mathbf{A}^m} T/T$ are equivalent as Poisson deformations of X.*

(ii) *There are natural $\mathbf{C}^*$-actions on $\mathcal{X}_{\mathrm{univ}}$ and $\mathbf{A}^m$ induced from the $\mathbf{C}^*$-action on X such that the map $\mathcal{X}_{\mathrm{univ}} \to \mathbf{A}^m$ is $\mathbf{C}^*$-equivariant. Moreover the coordinate ring $\mathbf{C}[y_1, \ldots, y_m]$ of $\mathbf{A}^m$ is positively graded so that $wt(y_i) > 0$ for all i.*

COROLLARY 3.2. *Let (X, ω) be a conical symplectic variety and let $T := \mathrm{Spec}(A)$ be a non-singular affine curve with a base point $0 \in T$. Assume that $\mathcal{X} \to T$ is a Poisson deformation of X. Assume that $\mathbf{C}^*$ acts on $\mathcal{X}$ in such a way that:*

(i) *it induces a $\mathbf{C}^*$-action on each fibre of $\mathcal{X}/T$ and the $\mathbf{C}^*$-action on the central fibre coincides with the original $\mathbf{C}^*$-action on X; and*

(ii) *the Poisson bracket on each fibre is homogeneous with respect to this action.*

Then there is a $\mathbf{C}^*$-equivariant Poisson isomorphism $f : \mathcal{X} \times_T \hat{T} \cong X \times \hat{T}$ over $\hat{T} := \mathrm{Spec}(\hat{A})$, where $\hat{A}$ is the completion of A along the defining ideal m of 0.

Proof. Notice that $\hat{A} \cong \mathbf{C}[[t]]$. Put $T_n := \mathrm{Spec}(\mathbf{C}[[t]]/(t^{n+1}))$ and $X_n := \mathcal{X} \times_T T_n$. The formal Poisson deformation $\{X_n \to T_n\}$ determines a morphism $\phi : \mathrm{Spec}(\mathbf{C}[[t]]) \to \mathbf{A}^m$ by Theorem 3.1. By assumption (i), $\mathrm{Im}(\phi)$ is contained in the $\mathbf{C}^*$-fixed locus of $\mathbf{A}^m$. By the last property in (ii) of Theorem 3.1, this means that ϕ is the constant map to the origin of $\mathbf{A}^m$. We will now construct isomorphisms between formal Poisson deformations $\{X_n\}_{n\geqslant 0} \xrightarrow{\beta_n} \{X \times T_n\}_{n\geqslant 0}$, where the right-hand side is a trivial Poisson deformation of X. Assume that we already have β_{n-1}. Since ϕ is constant, we have an equivalence $X_n \overset{\beta'_n}{\cong} X \times T_n$ of Poisson deformations of X. We put $\beta'_{n-1} := \beta'_n|_{X_{n-1}}$. Then $\gamma_{n-1} := \beta_{n-1}^{-1} \circ \beta'_{n-1}$ is a Poisson automorphism of X_{n-1}. By (the proof of)

[Nam11, Corollary 2.5], γ_{n-1} lifts to a Poisson automorphism γ_n of X_n. Define $\beta_n := \beta'_n \circ \gamma_n^{-1}$. Then β_n is a Poisson isomorphism from X_n to $X \times T_n$ extending β_{n-1}.

Note that $\{X \times T_n\}_{n \geqslant 0}$ has a natural $\mathbf{C}^*$-action induced by the $\mathbf{C}^*$-action of X. By the isomorphisms $\{\beta_n\}_{n\geqslant 0}$ above, this $\mathbf{C}^*$-action induces a $\mathbf{C}^*$-action on $\{X_n\}_{n\geqslant 0}$. On the other hand, $\{X_n\}_{n\geqslant 0}$ has a $\mathbf{C}^*$-action inherited from $\mathcal{X}$. We will construct a Poisson automorphism $\{\psi_n\}_{n\geqslant 0}$ of $\{X_n\}_{n\geqslant 0}$ inductively so that these two $\mathbf{C}^*$-actions are compatible. At first we put $\psi_0 := \mathrm{id}$. Assume that we are given a Poisson automorphism ψ_n which makes two $\mathbf{C}^*$-actions compatible. By (the proof of) [Nam11, Corollary 2.5], ψ_n lifts to a Poisson automorphism ψ'_{n+1} of X_{n+1}. We denote by ζ_1, $\zeta_2 \in \Gamma(X, \Theta_{X_{n+1}/T_{n+1}})$ respectively the relative vector fields generating the first and second $\mathbf{C}^*$-actions. Let $\zeta \in \Gamma(X, \Theta_X)$ be the vector field (Euler vector field) generating the $\mathbf{C}^*$-action. Notice that $\zeta_1|_X = \zeta_2|_X = \zeta$. We write

$$(\psi'_{n+1})_*\zeta_1 - \zeta_2 = t^{n+1} \cdot \Sigma v_i,$$

with $v_i \in \Gamma(X, \Theta_X)(i)$. In other words, v_i is a homogeneous vector field of weight i, that is, $[\zeta, v_i] = iv_i$. The Lie derivative $L_{v_i}\zeta$ can be computed as

$$L_{v_i}\zeta = [v_i, \zeta] = -iv_i.$$

For the Poisson bivector θ, we have $L_{(\psi'_{n+1})_*\zeta_1}\theta = L_{\zeta_2}\theta = -l \cdot \theta$; and hence $L_{v_i}\theta = 0$.

By this observation, if we put

$$\psi_{n+1} := \psi'_{n+1} + t^{n+1}\Sigma_{i\neq 0}(1/i)v_i,$$

then ψ_{n+1} is still a Poisson automorphism of X_{n+1} (because $L_{v_i}\theta = 0$) and one can write

$$(\psi_{n+1})_*\zeta_1 - \zeta_2 = t^{n+1}v_0.$$

Finally, we show that $(\psi_{n+1})_*\zeta_1 = \zeta_2$ by using the fact that these are both integrated to $\mathbf{C}^*$-actions. Consider the two $\mathbf{C}^*$-actions on X_{n+1} generated by $(\psi_{n+1})_*\zeta_1$ and ζ_2. As both vector fields are $\mathbf{C}^*$-invariant (with respect to any one of the two $\mathbf{C}^*$-actions), these $\mathbf{C}^*$-actions mutually commute. We now prove that these $\mathbf{C}^*$-actions are the same by the induction on n (the index of T_n). The coordinate ring $\mathcal{R}$ of X_{n+1} is isomorphic to $R \otimes_{\mathbf{C}} \mathbf{C}[t]/(t^{n+2})$, where $X = \mathrm{Spec}(R)$. We may assume that one of the $\mathbf{C}^*$-actions corresponds to the usual grading $\bigoplus_{i\geqslant 0}(R_i \oplus tR_i \oplus \cdots \oplus t^{n+1}R_i)$. Let us consider the weight-i eigenspace V_i of another $\mathbf{C}^*$-action. Since two $\mathbf{C}^*$-actions are compatible, V_i decomposes as $V_i = \bigoplus_j V_{i,j}$ where $V_{i,j}$ is a subspace of $R_j \oplus tR_j \oplus \cdots \oplus t^{n+1}R_j$. By the induction hypothesis, we have $V_{i,j} \subset t^{n+1}R_j$ if $j \neq i$. But the weight-i eigenspace of $t^{n+1}R$ with respect to the second $\mathbf{C}^*$-action also coincides with $t^{n+1}R_i$ because $t^{n+1}R = t^{n+1}\mathcal{R} = (t^{n+1}) \otimes_{\mathbf{C}} R$. This means that $V_{i,j} = 0$ if $j \neq i$. Therefore $V_i = V_{i,i} \subset R_i \oplus tR_i \oplus \cdots \oplus t^{n+1}R_i$ and we conclude that the two $\mathbf{C}^*$-actions are the same.

Now the composite

$$\{X_n\}_{n\geqslant 0} \overset{\beta_n\circ\psi_n}{\to} \{X \times T_n\}_{n\geqslant 0}$$

is a $\mathbf{C}^*$-equivariant Poisson isomorphism. By using the $\mathbf{C}^*$-actions of both sides, we then get a desired $\mathbf{C}^*$-equivariant Poisson isomorphism $\mathcal{X} \times_T \hat{T} \cong X \times \hat{T}$ over $\mathrm{Spec}(\hat{A})$. □

Proposition 3.3. *Under the same assumption as in Corollary 3.2, all fibres are isomorphic as conical symplectic varieties.*

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Proof. Let us consider the two T-schemes $\mathcal{X}$ and $X \times T$ with $\mathbf{C}^*$-actions. We define a functor

$$\mathrm{Hom}_T^{\mathbf{C}^*}(\mathcal{X}, X \times T) : (T\text{-schemes}) \to (\mathrm{Set})$$

by $T' \to \mathrm{Hom}_{T'}^{\mathbf{C}^*}(\mathcal{X} \times_T T', X \times T')$. Then it is a functor of locally finite presentation. By Artin's approximation theorem [Art69], if we are given a $\mathbf{C}^*$-equivariant morphism $f : \mathcal{X} \times_T \hat{T} \to X \times \hat{T}$, then there is a pointed algebraic scheme $s_0 \in S$ (that is, a pointed scheme of finite type over $\mathbf{C}$) together with an étale map $h : (S, s_0) \to (T, 0)$ and a $\mathbf{C}^*$-equivariant morphism $g : \mathcal{X} \times_T S \to X \times S$ such that $g(s_0) : \mathcal{X} \times_T k(s_0) \to X \times k(s_0)$ coincides with $f(0) : \mathcal{X} \times_T k(0) \to X \times k(0)$. We apply this to the morphism f in Corollary 3.2. As f is an isomorphism, we may assume that g is also an isomorphism, if necessary, by suitably shrinking S. Then we have a $\mathbf{C}^*$-equivariant isomorphism $\mathcal{X} \times_T S \cong X \times S$. This implies that $\mathcal{X}_{h(s)}$ is isomorphic to X as a $\mathbf{C}^*$-variety for any closed point $s \in S$. By [Nam13a, Theorem 3.1], two conical symplectic varieties having symplectic 2-forms of the same weight are isomorphic if they are isomorphic as $\mathbf{C}^*$-varieties. □

Now, by Propositions 2.11 and 3.3, we obtain our Main Theorem.

Main Theorem. *For positive integers N and d, there are only a finite number of conical symplectic varieties of dimension $2d$ with maximal weights N, up to an isomorphism.*

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