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# INCONGRUENT EMBEDDINGS OF A BOUQUET INTO SURFACES

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**Two 2-cell embeddings $i, j$ of a graph $G$ into surfaces $\mathbb{S}$ and $\mathbb{S}'$ are said to be congruent with respect to a subgroup $\Gamma$ of $\mathrm{Aut}(G)$ if there are a homeomorphism $h : \mathbb{S} \to \mathbb{S}'$ and an automorphism $\gamma \in \Gamma$ such that $h \circ i = j \circ \gamma$. In this paper, we compute the total number of congruence classes of 2-cell embeddings of any bouquet of circles into surfaces with respect to a group consisting of graph automorphisms of a bouquet.**

## 1. INTRODUCTION AND PRELIMINARIES

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathrm{Aut}(G)$ denote the automorphism group of $G$. Any graph $G$ can be regarded as a topological space in the following sense: By regarding the vertices of $G$ as 0-cells and the edges of $G$ as 1-cells, the graph $G$ can be identified with a finite 1-dimensional CW-complex in the Euclidean 3-space $\mathbb{R}^3$. Subdivision does not change the homeomorphic type of a graph as a topological space and any graph can be simple by a subdivision. The graphs $G$, $H$ are homeomorphic if and only if they have respective subdivisions $G'$ and $H'$ such that $G'$ and $H'$ are isomorphic graphs. Throughout this paper, all surfaces are compact and connected 2−dimensional manifolds is without boundary.

An *embedding* of a graph $G$ into a surface $\mathbb{S}$ is a topological embedding $i : G \to \mathbb{S}$. If every component of $\mathbb{S} - i(G)$, called a *region*, is homeomorphic to an open disk, then the embedding $i : G \to \mathbb{S}$ is called a *2-cell embedding*. Every embedding treated in this paper is a 2-cell embedding. Two 2-cell embeddings $i : G \to \mathbb{S}$ and $j : G \to \mathbb{S}'$ of a graph $G$ into surfaces are said to be *congruent with respect to a subgroup* $\Gamma$ of $\mathrm{Aut}(G)$ if there are a homeomorphism $h : \mathbb{S} \to \mathbb{S}'$ and an automorphism $\gamma \in \Gamma$ such that $h \circ i = j \circ \gamma$. If two surfaces $\mathbb{S}$ and $\mathbb{S}'$ are homeomorphic we identify $\mathbb{S}$ with $\mathbb{S}'$ in this note. If two embeddings are congruent with respect to $\mathrm{Aut}(G)$, we say that they are *congruent*. If the surfaces are oriented and the surface homeomorphism $h$ preserves orientations, we call it *oriented congruence*. Mull, Rieper and White [**2**] enumerated the orientable congruence classes of a graph $G$ into oriented surfaces with respect to the full automorphism group of $G$. Kwak and Lee [**1**] gave some algebraic characterisations and formulas for enumerating the congruence classes of 2-cell embeddings of a graph $G$ into a surface, and the congruence classes of 2-cell embeddings of complete graphs were enumerated.

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In this paper, we enumerate the number of congruence classes of 2-cell embeddings of bouquets of circles into surfaces which are orientable or nonorientable. Bouquets are those of fundamental graphs in building blocks and covering constructions in topological graph theory.

Let $|\mathcal{C}_\Omega(H)|$ denote the number of congruence classes of 2-cell embeddings of a graph $H$ into surfaces with respect to a subgroup $\Omega$ of $\mathrm{Aut}(H)$. In order to calculate the number $|\mathcal{C}_\Omega(H)|$ easily, we shall adopt a simple graph $G$ homeomorphic to $H$ and consider $|\mathcal{C}_\Gamma(G)|$ for an appropriate subgroup $\Gamma$ of $\mathrm{Aut}(G)$.

From now on, all graphs $G$ are simple graphs and we let $N(v)$ denote the neighbourhood of a vertex $v \in V(G)$, that is, the set of all vertices adjacent to $v$. An *embedding scheme* $(\rho, \lambda)$ for $G$ consists of a *rotation scheme* $\rho$ which assigns a cyclic permutation $\rho_v$ on $N(v)$ to each $v \in V(G)$ and a *voltage map* $\lambda$ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{1, -1\}$ to each $e \in E(G)$. Stahl [**3**] showed that every embedding scheme for a graph $G$ determines a 2-cell embedding of the graph into a surface, and every 2-cell embedding of the graph $G$ into a surface is determined by such a scheme. Kwak and Lee [**1**] gave the following:

**Lemma 1.1.** *Let $(\rho, \lambda)$ and $(\tau, \mu)$ be two embedding schemes for a graph $G$ with the corresponding embeddings $i : G \to \mathbb{S}$ and $j : G \to \mathbb{S}$ respectively, and let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$. *Then these two embeddings $i, j$ are congruent with respect to $\Gamma$ if and only if there are $\gamma \in \Gamma$ and a function $f : V(G) \to \mathbb{Z}_2$ such that $\tau_{\gamma(v)} = \gamma \circ (\rho_v)^{f(v)} \circ \gamma^{-1}$ and $\mu(\gamma(e)) = f(u)\lambda(e)f(v)$ for all $e = uv \in E(G)$.*

Now we introduce some notation. Suppose that any graph automorphism $\gamma$ of a graph $G$ is given. Then the subgroup $\langle\gamma\rangle$ of $\mathrm{Aut}(G)$ generated by $\gamma$ acts on the vertex set $V(G)$ by $(\gamma, v) \mapsto \gamma(v)$ and on the edge set $E(G)$ of $G$ by $(\gamma, e) \mapsto \gamma(e)$. Let $V_\gamma$ be a complete set of orbit representatives under the action of $\langle\gamma\rangle$ on $V(G)$ and $E_\gamma$ a complete set of orbit representatives under the action of $\langle\gamma\rangle$ on $E(G)$. Let $|v|$ denote the cardinality of the orbit of $v$ under the $\langle\gamma\rangle$-action on $V(G)$.

Let $P_{(v;\gamma)}$ denote the set of all cycle permutations $\sigma$ on $N(v)$ such that

$$\gamma^{|v|}\big|_{N(v)} \circ \sigma \circ \gamma^{-|v|}\big|_{N(v)} = \sigma$$

and $I_{(v;\gamma)}$ the set of all cycle permutations $\sigma$ on $N(v)$ such that

$$\gamma^{|v|}\big|_{N(v)} \circ \sigma \circ \gamma^{-|v|}\big|_{N(v)} = \sigma^{-1}.$$

Let $\mathrm{j}(\sigma) = (\mathrm{j}_1, \cdots, \mathrm{j}_n)$ denote the cycle type of a permutation $\sigma$ of $\{1, 2, \cdots, n\}$ where $\mathrm{j}_k$ is the number of cycles of length $k$ in a factorisation of $\sigma$ into disjoint cycles. As usual, $\phi(n)$ represents the value of a natural number $n$ under the Euler phi-function $\phi$.

According to the cycle type of $\gamma \in \mathrm{Aut}(G)$, $|P_{(v;\gamma)}|$ and $|I_{(v;\gamma)}|$ are given as follows [**1, 2**], where $|X|$ denotes the cardinality of a set $X$.

**Lemma 1.2.** *Let $\gamma \in \mathrm{Aut}(G)$, $v \in V(G)$ and $|N(v)| = n$. Then*

$$|P_{(v;\gamma)}| = \begin{cases} \phi(d)\left(\frac{n}{d}-1\right)!\, d^{n/d-1} & \text{if } \ \mathrm{j}\left(\gamma^{|v|}|_{N(v)}\right) = (0,\cdots,0,\mathrm{j}_d = \frac{n}{d},0,\cdots,0) \\ 0 & \text{otherwise.} \end{cases}$$

*and*

$$|I_{(v;\gamma)}| = \begin{cases} \left(\frac{n-1}{2}\right)!\, 2^{(n-1)/2} & \text{if } n \text{ is odd and } \ \mathrm{j}(\gamma^{|v|}|_{N(v)}) = \left(1,\frac{n-1}{2},0,\cdots,0\right), \\ \left(\frac{n}{2}\right)!\, 2^{n/2-1} & \text{if } n \text{ is even and } \ \mathrm{j}(\gamma^{|v|}|_{N(v)}) = \left(0,\frac{n}{2},0,\cdots,0\right), \\ \left(\frac{n-2}{2}\right)!\, 2^{n/2-1} & \text{if } n \text{ is even and } \ \mathrm{j}(\gamma^{|v|}|_{N(v)}) = \left(2,\frac{n-2}{2},0,\cdots,0\right), \\ 0 & \text{otherwise.} \end{cases}$$

## 2. Enumeration theorems for the embeddings of a bouquet

For each natural number $n$, let $B_n$ denote the bouquet of $n$ circles, which is the graph with one vertex and $n$ self-loops. We note that bouquets are those of fundamental graphs in building blocks and covering constructions in topological graph theory. Any connected graph can be reduced to a bouquet by contracting a spanning tree to a vertex and Cayley graphs and many other regular graphs are covering graphs of bouquets.

Clearly, when $n = 1$, $|\mathcal{C}_{\mathrm{Aut}(B_1)}(B_1)| = |\mathcal{C}_{\{I\}}(B_1)| = 2$ where $I$ is the identity automorphism of $B_1$; one is the embedding of $B_1$ into the sphere and the other is the embedding into the projective plane. So we assume that $n \geqslant 2$. Now, in order to enumerate the number of congruence classes of 2-cell embeddings of $B_n$, we introduce a simple graph $H_n$ which is homeomorphic to $B_n$. Let $v_0$ be the vertex of $B_n$ and $\{c_1, c_2, ..., c_n\}$ the set of $n$ self-loops of $B_n$. For each $i \in \{1, \cdots, n\}$, we insert two vertices $v_i$ and $v_{-i}$ in the interior of $c_i$. The resulting graph is $H_n$. For $i \in \{1, \cdots, n\}$, let $e_i$ be the edge joining $v_i$ to $v_{-i}$. Let $T_n$ be the spanning tree of $H_n$ with $2n$ edges joining $v_0$ to $v_j$ for $j = -n, \cdots, -1, 1, \cdots, n$.

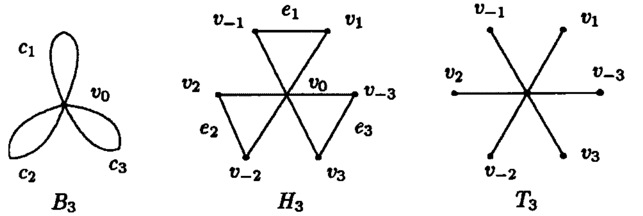


Figure 1. The graphs $B_3$, $H_3$ and $T_3$.

Note that every automorphism of $H_n$ fixes $T_n$. Let $\Lambda$ denote the subgroup of $\mathrm{Aut}(H_n)$ generated by the flips of triangles in $H_n$, which is isomorphic to the direct sum of $n$ copies of $\mathbb{Z}_2$'s and will be identified with the set of functions from $\{1, \cdots, n\}$ to $\mathbb{Z}_2$ with pointwise multiplication.

Let $S_n$ be the symmetric group on $\{1, \cdots, n\}$. Define an $S_n$ action on $\Lambda$ by $(\sigma f)(i) = f(\sigma^{-1}(i))$ for all $\sigma \in S_n$, $f \in \Lambda$ and $i \in \{1, \cdots, n\}$. The automorphism group $\mathrm{Aut}(H_n)$ of $H_n$ can be regarded as the semidirect product of $S_n$ and $\Lambda$ with respect to this action where its group operation is given by $(\varsigma, g)(\sigma, f) = (\varsigma\sigma, (\sigma^{-1}g)f)$. Moreover, $\mathrm{Aut}(H_n)$ is regarded as a subgroup of the symmetric group $S_{2n}$ on $\{ v_{-n}, \cdots, v_{-1}, v_1, \cdots, , v_n \}$ via

$$(\sigma, f)(v_j) = v_{(j/|j|)f(|j|)\sigma(|j|)}$$

for each $j \in \{-n, \cdots, -1, 1, \cdots, n\}$.

Let $\mathcal{E}_0(H_n)$ denote the set of all embedding schemes $(\rho, \lambda)$ for $H_n$ with $\lambda(e) = 1$ for all $e \in E(T_n)$.

For an embedding scheme $(\rho, \lambda)$ for $H_n$, we construct an embedding scheme $(\rho_0, \lambda_0)$ as follows: let $f(v_0) = 1$ and $f(v_j) = \lambda(v_0 v_j)$ for $j = -n, \cdots, -1, 1, \cdots, n$ and define $\rho_0 = \rho$ and $\lambda_0(e) = f(i_e)\lambda(e)f(t_e)$, where $i_e, t_e$ are the vertices incident to the edge $e \in H_n$. Then $(\rho_0, \lambda_0) \in \mathcal{E}_0(H_n)$.

Let the group $\Gamma \times \mathbb{Z}_2$ act on $\mathcal{E}_0(H_n)$ by

$$(\gamma, \alpha)(\rho, \lambda) = \big((\gamma, \alpha)\rho, (\gamma, \alpha)\lambda\big)$$

for any $(\gamma, \alpha) \in \Gamma \times \mathbb{Z}_2$ and $(\rho, \lambda) \in \mathcal{E}_0(H_n)$, where

$$\big[(\gamma, \alpha)\rho\big]_v = \gamma \circ \big(\rho_{\gamma^{-1}(v)}\big)^{\alpha} \circ \gamma^{-1},$$
$$\big[(\gamma, \alpha)\lambda\big](e) = \lambda\big(\gamma^{-1}(e)\big)$$

for any $v \in V(H_n)$ and $e \in E(H_n)$.

Then from Lemma 1.1, we can drive the following.

**Lemma 2.1.** *If $(\rho, \lambda)$ and $(\tau, \mu)$ are two embedding scheme for $G$ with the corresponding embeddings $i : H_n \to \mathbb{S}$ and $j : H_n \to \mathbb{S}$ respectively, then these two embeddings $i, j$ are congruent with respect to a subgroup $\Gamma$ of $\mathrm{Aut}(H_n)$ if and only if $(\gamma, \alpha)(\rho_0, \lambda_0) = (\tau_0, \mu_0)$ for some $(\gamma, \alpha) \in \Gamma \times \mathbb{Z}_2$.*

By using this lemma and Burnside's lemma we have the following theorem.

**Theorem 2.2.** *Let $\Omega$ be a subgroup of $\mathrm{Aut}(G)$. Then*

$$\big|\mathcal{C}_\Omega(B_n)\big| = \big|\mathcal{C}_\Gamma(H_n)\big| = \frac{1}{2|\Gamma|} \sum_{(\gamma,\alpha)\in\Gamma\times\mathbb{Z}_2} \big|F(\gamma, \alpha)\big|,$$

*where $\Gamma$ is the semidirect product of $\Omega$ and $\Lambda$ under the above action and $F(\gamma, \alpha) = \big\{(\rho, \lambda) \in \mathcal{E}_0(H_n) \mid (\gamma, \alpha)(\rho, \lambda) = (\rho, \lambda)\big\}$.*

Let $(\rho, \lambda) \in \mathrm{F}(\gamma, \alpha)$. We first observe that for any natural number $n$, $\lambda\big(\gamma^n(e)\big) = 1 = \lambda(e)$ for $e \in E(T_n)$.

Hence $\lambda$ is determined by the values of $\lambda$ on $E(H_n - T_n) = \{e_1, e_2, \cdots, e_n\}$. Since $\lambda\big(\gamma^n(e_i)\big) = \lambda(e_i)$ for $e_i \in E(H_n - T_n)$, $\lambda(e_i) = \lambda(e_j)$ if $e_i$ and $e_j$ are in the same orbit.

Hence $\lambda$ is completely determined by the value $\lambda(e_k)$ on the orbit representatives $e_k$ of the classes in $\{e_1, e_2, \cdots, e_n\}/\langle\gamma\rangle$.

And now observe that $\rho_{v_j}$ is the identity on $N(v_j)$ for $j = -n, \cdots, -1, 1, \cdots, n$ and $\gamma^n(v_0) = v_0$ for any natural number $n$ and so $|v_0| = 1$. Thus

$$\rho_{v_0} = \rho_{\gamma^{|v_0|}(v_0)} = \gamma^{|v_0|} \circ (\rho_{v_0})^{\alpha^{|v_0|}} \circ \gamma^{-|v_0|}.$$

Hence $\rho$ is determined by the value $\rho_{v_0}$ at $v_0$. Moreover, $\rho_{v_0} \in P_{(v_0;\gamma)}$ if and only if $\alpha^{|v_0|} = 1$ and $\rho_{v_0} \in I_{(v_0;\gamma)}$ if and only if $\alpha^{|v_0|} = -1$. Therefore we have the following.

**Theorem 2.3.** *Let $\gamma \in \mathrm{Aut}(H_n)$. Then*

$$|F(\gamma, \alpha)| = 2^{|E'_\gamma|}|P_{(v_0;\gamma)}||I_{(v_0;\gamma)}|,$$

*where $E'_\gamma$ denotes the set of orbit representatives for $\langle\gamma\rangle$ acting on $E(H_n - T_n) = \{e_1, e_2, \cdots, e_n\}$.*

## 3. Enumerating incongruent embeddings of a bouquet

In this section, we compute the number of congruence classes of 2-cell embeddings of a bouquet of $n$ circles $B_n$ into surfaces with respect to the full automorphism group $\mathrm{Aut}(B_n)$ and the trivial group $\{I\}$.

We first observe that the flip of each self-loop in $B_n$ gives the identity automorphism of $B_n$, so that the automorphism group $\mathrm{Aut}(B_n)$ is the symmetric group on the set of $n$ self-loops $\{c_1, \cdots, c_n\}$ of $B_n$, which corresponds to permuting the $n$ self-loops in $B_n$. Furthermore, we see that $|\mathcal{C}_{\mathrm{Aut}(B_n)}(B_n)| = |\mathcal{C}_{\mathrm{Aut}(H_n)}(H_n)|$ and $|\mathcal{C}_{\{I\}}(B_n)| = |\mathcal{C}_{\Lambda}(H_n)|$.

If the cycle type of $\gamma \in \mathrm{Aut}(H_n)$ is not one of the types $(2, n-1, 0, \cdots, 0)$, $(0, \cdots, 0, \mathrm{j}_d = 2n/d, 0, \cdots, 0)$, where $d|2n$ (that is, $d$ is a divisor of $2n$), then $|\mathrm{F}(\gamma, \alpha)| = 0$, for any $\alpha \in \mathbb{Z}_2$. Hence, in order to enumerate the congruence classes of 2-cell embeddings of a bouquet of $n$ circles $B_n$ into a surface, it suffices to consider the following subsets $A_d$ of $\mathrm{Aut}(H_n)$, where $d = 0$ or $d|2n$:

$$A_0 = \{\gamma \in \mathrm{Aut}(H_n) \mid \mathrm{j}(\gamma) = (2, n-1, 0, \cdots, 0)\}$$

and

$$A_d = \{\gamma \in \mathrm{Aut}(H_n) \mid \mathrm{j}(\gamma) = (0, \cdots, 0, \mathrm{j}_d = 2n/d, 0, \cdots, 0)\}.$$

Now, let

$$X_{0;k} = \{\sigma \in S_n \mid \mathrm{j}(\sigma) = (n-2k, k, 0, \cdots, 0)\}$$

for $0 \leqslant k \leqslant [(n-1)/2]$,

$$X_{d;k} = \{\sigma \in S_n \mid \mathrm{j}(\sigma) = (0, \cdots, 0, \mathrm{j}_{d/2} = 2n/d - 2k, 0, \cdots, 0, \mathrm{j}_d = k, 0, \cdots, 0)\}$$

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for even $d$ with $d|2n$ and $0 \leqslant k \leqslant [n/d]$,

$$X_{d;-1} = \{\sigma \in S_n \mid \mathrm{j}(\sigma) = (0, \cdots, 0, \mathrm{j}_d = n/d, 0, \cdots, 0)\}$$

for odd $d$ with $d|n$. Here $[a]$ is the largest integer which does not exceed the real number $a$.

Then every element in $A_d$ can be described by an element of $X_{d;k}$, and each element $\sigma$ in $X_{d;k}$ induces exactly $m$ elements of $A_d$, where

$$m = \begin{cases} 2^k(n-2k) & \text{if } d = 0 \text{ and } 0 \leqslant k \leqslant [(n-1)/2], \\ 2^{n+k-2n/d} & \text{if } d \text{ is even with } d|2n \text{ and } 0 \leqslant k \leqslant [n/d], \\ 2^{n-n/d} & \text{if } d \text{ is odd with } d|n. \end{cases}$$

Since the number of permutations in $S_n$ of cycle type $(\mathrm{j}_1, \cdots, \mathrm{j}_n)$ is

$$\frac{n!}{\prod_{k=1}^{n} \mathrm{j}_k!} k^{\mathrm{j}_k},$$

we have

$$\begin{aligned} |A_0| &= \sum_{k=0}^{[(n-1)/2]} \frac{n!}{(n-2k)!2^k k!} 2^k(n-2k) \\ &= \sum_{k=0}^{[(n-1)/2]} \frac{n!}{(n-2k-1)!k!}, \\ |A_d| &= \frac{n!}{(n/d)!} d^{-n/d} 2^{n-n/d} \end{aligned}$$

for odd $d$ with $d|2n$, and

$$\begin{aligned} |A_d| &= \sum_{k=0}^{[n/d]} \frac{n!}{(2n/d-2k)!(d/2)^{2n/d-2k} k! d^k} 2^{n+k-2n/d} \\ &= \sum_{k=0}^{[n/d]} \frac{n!}{(2n/d-2k)!k!} d^{k-2n/d} 2^{n-k} \end{aligned}$$

for even $d$ with $d|2n$.

For any $\gamma \in A_d$ with $d = 0$ or $d|2n$, the number $|E'_\gamma|$ is given by:

$$|E'_\gamma| = \begin{cases} n-k & \text{if } \gamma \text{ is induced from an element of } X_{0;k}, \\ \dfrac{2n}{d} - k & \text{if } \gamma \text{ is induced from an element of } X_{d;k}, \\ \dfrac{n}{d} & \text{if } \gamma \text{ is induced from an element of } X_{d;-1}. \end{cases}$$

Now, using the Lemmas and Theorems from Sections 1 and 2, we calculate $\sum_{\gamma\in A_d} |\mathrm{F}(\gamma,\alpha)|$ where $d=0$ or $d|2n$, and $\alpha = 1$ or $-1$.

If $d = 0$,

$$\sum_{\gamma\in A_0} |\mathrm{F}(\gamma,1)| = 0,$$

$$\sum_{\gamma\in A_0} |\mathrm{F}(\gamma,-1)| = \sum_{k=0}^{[(n-1)/2]} \frac{n!(n-1)!}{(n-2k-1)!k!} 2^{2n-k-1}.$$

If $d=1$ or $d \geqslant 3$, then clearly $\sum_{\gamma\in A_d} |\mathrm{F}(\gamma,-1)| = 0$.

For odd $d$ with $d|2n$,

$$\sum_{\gamma\in A_d} |\mathrm{F}(\gamma,1)| = \frac{n!(2n/d-1)!}{(n/d)!} d^{n/d-1}\phi(d)2^n.$$

For even $d$ with $d|2n$,

$$\sum_{\gamma\in A_d} |\mathrm{F}(\gamma,1)| = n!(\frac{2n}{d}-1)!\phi(d)2^{2n/d+n} \sum_{k=0}^{[n/d]} \frac{d^{k-1}}{(2n/d-2k)!k!2^{2k}},$$

$$\sum_{\gamma\in A_2} |\mathrm{F}(\gamma,-1)| = (n!)^2 2^{2n-1} \sum_{k=0}^{[n/2]} \frac{1}{(n-2k)!k!2^k}.$$

We summarise our discussions to get the following theorems.

**THEOREM 3.1.** *The number of congruence classes of embeddings of a bouquet of $n$ circles $B_n$ is*

$$\begin{aligned}
|\mathcal{C}_{\mathrm{Aut}(B_n)}(B_n)| &= \frac{1}{2}\sum_{d|n,\ d=odd} \frac{(2n/d-1)!}{(n/d)!}\phi(d)d^{n/d-1} \\
&+\frac{1}{4}\sum_{d|n}\left(\sum_{k=0}^{[n/(2d)]} \frac{d^{k-1}2^{-k}}{(n/d-2k)!k!}\right)\phi(2d)(\frac{n}{d}-1)!2^{n/d} \\
&+n!2^{n-2}\sum_{k=0}^{[n/2]} \frac{1}{(n-2k)!k!2^k} \\
&+(n-1)!2^{n-2}\sum_{k=0}^{[(n-1)/2]} \frac{1}{(n-2k-1)!k!2^k}.
\end{aligned}$$

The subgroup $\Lambda$ of $\mathrm{Aut}(H_n)$ generated by the flips of triangles in $H_n$, consists of the elements that can be induced from the identity permutation of $S_n$. If the cycle type of $\gamma \in \Lambda$ is not one of 3 types $(2n,0,\cdots,0)$, $(0,n,0,\cdots,0)$ or $(2,n-1,0,\cdots,0)$, then

$|F(\gamma, \alpha)| = 0$ for any $\alpha \in \mathbb{Z}_2$. On the other hand, the numbers of elements of $\Lambda$ with cycle type $(2n, 0, \cdots, 0)$, $(0, n, 0, \cdots, 0)$ and $(2, n-1, 0, \cdots, 0)$ are 1, 1 and $n$ respectively. Since $|\mathcal{C}_{\{I\}}(B_n)| = |\mathcal{C}_\Lambda(H_n)|$, the following result can be derived from the previous discussion.

**Theorem 3.2.** *The number of congruence classes of 2-cell embeddings of a bouquet of n circles $B_n$ into a surface with respect to the trivial group $\{I\}$ is*

$$|\mathcal{C}_{\{I\}}(B_n)| = (2n-1)!\,2^{-1} + (n-1)!\,2^{n-2} + n!\,2^{n-1}.$$

The following table 1 shows the number of congruence classes of 2-cell embeddings of $B_n$ for small numbers n, as calculated from Theorem 3.1.

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{C}_{\mathrm{Aut}(B_n)}(B_n)\rvert$ | 2 | 6 | 26 | 173 | 1844 | 29570 | 628680 | $\cdots$ |

Table 1.

Remark 3.3. There exist 6 non congruent 2-cell embeddings of a bouquet of two circles with respect to $\mathrm{Aut}(B_2)$; one embedding into the sphere, one into the torus, two into the projective plane and two into the Klein bottle, as shown in Figure 2.

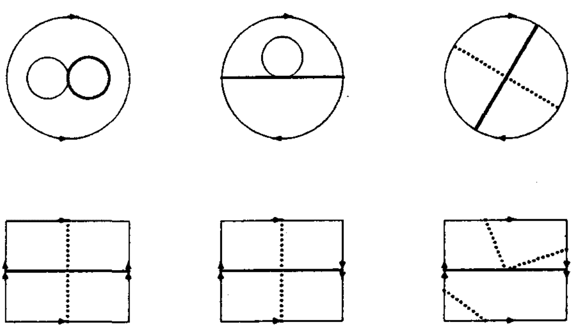

Figure 2. 6 incongruent 2-cell embeddings of $B_2$.

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