ON THE NATURAL ORDERING OF $\mathcal{J}$-CLASSES AND
OF IDEMPOTENTS IN A REGULAR SEMIGROUP

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1. Introduction and summary. In this paper we prove the following:

MAIN THEOREM. Let $S$ be a regular semigroup and $a, b$ any elements of $S$ such that $J_b \subseteq J_a$. Then, for each idempotent $e \in J_a$, there exists an idempotent $f \in J_b$ such that $f \leq e$.

This makes easy a conceptual proof of the result (see Theorem 6.39 [1], Theorem 3 [2] or Theorem 1 [4]) that a semigroup is primitive and regular if and only if it is a $0$-direct union of completely $0$-simple semigroups.

The above theorem can also be used in an obvious manner to simplify the statements of the results of Section 3, Chapter II of Lallement [3], in particular Theorem 2.17 [3].

We use wherever possible notations and conventions from Clifford and Preston [1].

2. The natural orderings. We shall prove in fact (see Theorem 1) a stronger result than the one above. The idea for the proof of this theorem came from the proof of the lemma in Warne’s paper [6]. In fact Warne effectively proves Theorem 1 for the special case of a regular semigroup with only two principal factors, both completely $0$-simple.

We prove first the following lemma.

LEMMA 1. Let $a, b$ be elements of a semigroup $S$ such that $J_b \subseteq J_a$ and such that the principal factor $J(b)/I(b)$ is $0$-simple or simple. Then $(a J_b) \cap J_b \neq \emptyset$.

Proof. Since $J(b)/I(b)$ is not a null semigroup, there exist elements $x, y \in J_b$ for which $xy \in J_b$. Now $J_x = J_b \subseteq J_a$, so there exist elements $u, v \in S^1$ such that $x = uav$; whence $xy = uavy \in J_b$. It follows that $vy, avy \in J(b)$ (because $av, vy \in J(b)$ and $av, vy \in I(b)$ would imply that $xy = uavy \in I(b)$).

The lemma may also be deduced from example 5, part (a), p. 36, [1] volume II.

THEOREM 1. Let $S$ be a semigroup and $a, b$ any elements of $S$ such that $J_b \subseteq J_a$ and such that each element of $J_b$ is regular. Then for each idempotent $e \in J_a$ (if such exists) there is an idempotent $f \in J_b$ such that $f \leq e$.

Proof. Take any idempotent $e \in J_a$. From Lemma 1, $(e J_b) \cap J_b \neq \emptyset$. Take any element $y \in (e J_b) \cap J_b$ and any inverse, $y'$ say, of $y$. By a routine calculation, we find that $y'e$ is an inverse of $y$; whence $yy'e = f \text{ (say)}$ is an idempotent, $f \in J_b$ and $f \leq e$.

REMARK 1. Proposition 3.1 of [5] is similar to Theorem 1. Its applications in common with Theorem 1 would include the two mentioned in paragraphs 3 and 4.

REMARK 2. It is clear that, for any idempotents $e, f$ in a semigroup $S$, $f \leq e$ implies that $J_f \subseteq J_e$. Theorem 1 and this converse remain true when $J$ is replaced by $L$ or $R$ (only
minor modifications of the proof are necessary to show this). It is then easy to show that the set of $L$- (or $R$- but not $J$-) classes of an inverse semigroup, under the natural ordering, is always a semilattice.

**Corollary 1.** (Due to Lallement and Petrich [2] Theorem 3, and Preston [4] Theorem 1. See also [1] Theorem 6.39.) A semigroup $S = S^0$ is primitive and regular if and only if it is a 0-direct union of completely 0-simple semigroups.

**Proof.** Suppose that $S$ is primitive and regular. Let $a, b$ be any non-zero elements of $S$ such that $J_a \neq J_b$.

Now, for any element $x \in I(a)$, we have $J_x < J_a$; whence, from Theorem 1, $0 \in J_x$ and $I(a) = \{0\}$. Therefore $S^1aS^1 = J_x \cup \{0\}$ for each non-zero element $a \in S$.

Also $J_{ab} \leq J_a$ and $J_{ab} \leq J_b$, whence either $J_{ab} < J_a$ or $J_{ab} < J_b$. In either case $ab = 0$, and so $S$ is the 0-direct union of the subsemigroups $\{J_x \cup \{0\} : x \in S \setminus \{0\}\}$. But for each $x \in S \setminus \{0\}$, $J_x \cup \{0\}$ is isomorphic to the principal factor $J(x)/I(x)$ and hence is 0-simple and thence completely 0-simple.

Conversely, since a completely 0-simple semigroup is primitive and regular, it is clear that a 0-direct union of completely 0-simple semigroups is also primitive and regular.

**References**


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