ON THE NATURAL ORDERING OF \( \mathcal{J} \)-CLASSES AND OF IDEMPOTENTS IN A REGULAR SEMIGROUP

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1. Introduction and summary. In this paper we prove the following:

MAIN THEOREM. Let \( S \) be a regular semigroup and \( a, b \) any elements of \( S \) such that \( J_b \preceq J_a \). Then, for each idempotent \( e \in J_a \), there exists an idempotent \( f \in J_b \) such that \( f \preceq e \).

This makes easy a conceptual proof of the result (see Theorem 6.39 [1], Theorem 3 [2] or Theorem 1 [4]) that a semigroup is primitive and regular if and only if it is a \( 0 \)-direct union of completely \( 0 \)-simple semigroups.

The above theorem can also be used in an obvious manner to simplify the statements of the results of Section 3, Chapter II of Lallement [3], in particular Theorem 2.17 [3].

We use wherever possible notations and conventions from Clifford and Preston [1].

2. The natural orderings. We shall prove in fact (see Theorem 1) a stronger result than the one above. The idea for the proof of this theorem came from the proof of the lemma in Warne's paper [6]. In fact Warne effectively proves Theorem 1 for the special case of a regular semigroup with only two principal factors, both completely \( 0 \)-simple.

We prove first the following lemma.

LEMMA 1. Let \( a, b \) be elements of a semigroup \( S \) such that \( J_b \preceq J_a \) and such that each element of \( J_b \) is regular. Then for each idempotent \( e \in J_a \) (if such exists) there is an idempotent \( f \in J_b \) such that \( f \preceq e \).

Proof. Take any idempotent \( e \in J_a \). From Lemma 1, \((e J_b) \cap J_b \neq \emptyset \). Take any element \( y \in (e J_b) \cap J_b \) and any inverse, \( y' \) say, of \( y \). By a routine calculation, we find that \( y'y' = f \) (say) is an idempotent, \( f \in J_b \) and \( f \preceq e \).

REMARK 1. Proposition 3.1 of [5] is similar to Theorem 1. Its applications in common with Theorem 1 would include the two mentioned in paragraphs 3 and 4.

REMARK 2. It is clear that, for any idempotents \( e, f \) in a semigroup \( S, f \preceq e \) implies that \( J_f \preceq J_e \). Theorem 1 and this converse remain true when \( J \) is replaced by \( L \) or \( R \) (only
minor modifications of the proof are necessary to show this. It is then easy to show that the set of $L$- (or $R$- but not $J$-) classes of an inverse semigroup, under the natural ordering, is always a semilattice.

**Corollary 1.** (Due to Lallement and Petrich [2] Theorem 3, and Preston [4] Theorem 1. See also [1] Theorem 6.39.) A semigroup $S = S^0$ is primitive and regular if and only if it is a 0-direct union of completely 0-simple semigroups.

**Proof.** Suppose that $S$ is primitive and regular. Let $a, b$ be any non-zero elements of $S$ such that $J_a \neq J_b$.

Now, for any element $x \in I(a)$, we have $J_x < J_a$; whence, from Theorem 1, $0 \in J_x$ and $I(a) = \{0\}$. Therefore $S^1 a S^1 = J_x \cup \{0\}$ for each non-zero element $a \in S$.

Also $J_{ab} \leq J_a$ and $J_{ab} \leq J_b$, whence either $J_{ab} < J_a$ or $J_{ab} < J_b$. In either case $ab = 0$, and so $S$ is the 0-direct union of the subsemigroups $\{J_x \cup \{0\} : x \in S \setminus \{0\}\}$. But for each $x \in S \setminus \{0\}$, $J_x \cup \{0\}$ is isomorphic to the principal factor $J(x)/I(x)$ and hence is 0-simple and thence completely 0-simple.

Conversely, since a completely 0-simple semigroup is primitive and regular, it is clear that a 0-direct union of completely 0-simple semigroups is also primitive and regular.

**References**


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