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A stability theorem for minimal foliations on a torus

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Abstract. This paper is concerned with minimal foliations; these are foliations whose leaves are extremals of a prescribed variational problem, as for example foliations consisting of minimal surfaces. Such a minimal foliation is called stable if for any small perturbation of the variational problem there exists a minimal foliation conjugate under a smooth diffeomorphism to the original foliation. In this paper the stability of special foliations of codimension 1 on a higher-dimensional torus is established. This result requires small divisor assumptions similar to those encountered in dynamical systems. This theorem can be viewed as a generalization of the perturbation theory of invariant tori for Hamiltonian systems to elliptic partial differential equations for which one obtains quasi-periodic solutions.

1. *Introduction*

(a) In this paper we consider foliations of codimension 1 on a higher-dimensional torus T^d whose leaves are extremals of a variational problem. A special case is given by a foliation whose leaves are minimal surfaces with respect to a given metric. While one usually considers compact minimal surfaces, we will be led to foliations with non-compact leaves. We will call such a foliation *stable* if under small perturbations of the variational problem (resp. of the metric) there exists a new foliation which is conjugate to the given one under a diffeomorphism close to the identity. The purpose of this paper is to establish the stability of such foliations under certain hypotheses. In particular, these assumptions require each leaf to be dense on the torus. From an analytic point of view our result leads to the existence of quasi-periodic solutions of non-linear partial differential equations generalizing such statements for Hamiltonian systems.

Before formulating the result, we illustrate it with the example of foliations of minimal surfaces on a flat torus. We consider the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$, denote the coordinates on $\mathbb{R}^d$ by $x_1, x_2, \ldots, x_d$ and the flat metric by

$$ds_0^2 = \sum_{\nu=1}^{d} dx_\nu^2.$$

Then for any vector $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \neq 0$ we obtain a foliation of minimal surfaces by the parallel hypersurfaces

$$\sum_{\nu=1}^{d} \alpha_\nu x_\nu = \text{const.}$$

Not every such foliation will be stable under perturbations of the metric. We have to require that these leaves are dense on the torus. This is equivalent to the condition that the normal line $\{\lambda\bar{\alpha}\}$, $\lambda \in \mathbb{R}$ does not meet any lattice point of $\mathbb{Z}^d$ except the origin. Actually, we have to impose a stronger restriction; namely, that the distance of this normal line from a lattice point $\bar{j} \in \mathbb{Z}^d\backslash(0)$ is 'not too small'. We assume that there exist positive constants γ, τ for which

$$\sum_{\nu,\mu=1}^{d} (\alpha_\nu j_\mu - \alpha_\mu j_\nu)^2 \geq \gamma\left(\sum_{\nu=1}^{d} j_\nu^2\right)^{-\tau} \tag{1.1}$$

for all $\bar{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d\backslash(0)$ holds. We shall show that under this Diophantine condition (1.1) the above foliation is indeed stable under smooth perturbations of the metric. On the other hand, under larger perturbations of the metric generally no such minimal foliations exist, as was shown by Bangert [1]. As the perturbation increases, the foliations lose their smoothness and disintegrate to 'laminations' [**16**]. Also, if the condition (1.1) is violated, such a foliation can disintegrate under arbitrarily small perturbations; in other words, one may not have stability if the condition (1.1) is violated. For $d = 2$ the necessity of (1.1) for stability follows from [**13**].

(b) To describe the set-up more generally, we will describe the leaves of the foliation in non-parametric form. With $n = d - 1$ we set $x = (x_1, x_2, \ldots, x_n)$ and write the hypersurface of codimension 1 in the form

$$x_{n+1} = u(x), \qquad d = n + 1. \tag{1.2}$$

In the following we will write $\bar{x} = (x_1, \ldots, x_{n+1})$ for $(n+1)$-vectors. The variational problem will be written in the form

$$\int F(x, u, u_x)\, dx, \tag{1.3}$$

where $dx = dx_1\, dx_2 \ldots dx_n$ and $F = F(\bar{x}, p)$ is a smooth function of period 1 in the first $d = n + 1$ variables, while p varies in an open subset of $\mathbb{R}^n$. Thus

$$F \in C^\infty(\Omega),$$

where Ω is an open domain in $T^{n+1} \times \mathbb{R}^n$ with the property that $\pi(\Omega) = T^{n+1}$, π being the projection $\pi(\bar{x}, p) = \bar{x}$. We will require that F satisfies the Legendre condition

$$\sum_{\nu,\mu=1}^{n} F_{p_\nu p_\mu}(\bar{x}, p)\xi_\nu \xi_\mu \geq \lambda |\xi|^2 \tag{1.4}$$

for all $\xi \in \mathbb{R}^n$, $(\bar{x}, p) \in \Omega$. The positive constant λ could be normalized to be 1.

The functions u representing the leaves of the foliation are required to satisfy the Euler equation

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}(F_{p_\nu}(x, u, u_x)) = F_u(x, u, u_x), \tag{1.5}$$

which is a non-linear elliptic differential equation. The solutions of (1.5) will be called extremals.

To define a minimal foliation on a torus, we consider its lift on $\mathbb{R}^{n+1}$, taking account of the $\mathbb{Z}^{n+1}$-action.

Definition. For a given variational problem (1.3) and $0 \le r \le \infty$ we define a $\mathbb{Z}^{n+1}$-invariant C^r-minimal foliation as a function $u \in C^r(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, taking $(x, \lambda) \to u = u(x, \lambda)$ such that:

(i) For each fixed $\lambda \in \mathbb{R}$ the function $u(x, \lambda)$ is an extremal of (1.5).

(ii) For fixed $x \in \mathbb{R}^n$ the mapping is a C^r-homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ with $u(x, \lambda) < u(x, \lambda')$ for $\lambda < \lambda'$, and $\partial u / \partial \lambda > 0$ if $r \ge 1$.

(iii) The foliation given by the leaves $x_{n+1} = u(x, \lambda)$, $\lambda \in \mathbb{R}^{n+1}$, is invariant under the $\mathbb{Z}^{n+1}$-action.

We will be mainly concerned with C^∞-foliations, corresponding to the case $r = \infty$. However, we note that even for $F \in C^\infty$ a minimal foliation may not be differentiable (see [**17**]).

The extremals representing the leaves of such a minimal foliation are special solutions of the Euler equations. They minimize the functional (1.3) taken over any large ball, compared to any other admissible functions with the same boundary values. This follows from the fact that the leaves of such a foliation can be viewed as a 'field of extremals' in the sense of calculus of variations. It is well known that every extremal which can be embedded into such a field of extremals is minimal provided (1.4) holds. In other words, a field of extremals is always a field of minimals. Moreover, the leaves of a minimal foliation obviously have no self-intersections on the torus T^{n+1}. In other words, for the leaves of a minimal foliation (in non-parametric form) only minimal solutions of (1.5) without self-intersections qualify.

These minimal solutions of (1.5) without self-intersections have been studied in [**16**], [**17**] and [**2**] under additional growth condition of F. From the theory developed there it follows that to a given $\mathbb{Z}^{n+1}$-invariant minimal foliation one can associate a unique 'normal vector' $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n, -1)$ such that for every leaf

$$\sup_x |u(x) - (\alpha, x)| < \infty, \qquad \alpha = (\alpha_1, \ldots, \alpha_n).$$

Moreover, there exists a function $U = U(x, \theta)$

$$U(x, \theta) - \theta \in C^r(T^d), \qquad \partial_\theta U(x, \theta) > 0, \qquad \text{if } r \ge 1,$$

such that the leaves of the foliation take the form

$$x_{n+1} = U(x, (\alpha, x) + \beta), \qquad \beta = \text{const}. \tag{1.6}$$

In the following we will therefore consider only smooth foliations of this form.

Geometrically one can interpret the representation (1.6) as follows. The foliation (1.6) is conjugate to the foliation of parallel hypersurfaces

$$x_{n+1} = (\alpha, x) + \beta, \qquad \beta = \text{const},$$

under the C^r-homeomorphism

$$\bar{x} \to (x, U(\bar{x})),$$

which induces a C^r-homeomorphism on the torus. In particular, two smooth foliations belonging to the same vector α are conjugate.

Thus a minimal foliation (1.6) is characterized by a vector $\alpha \in \mathbb{R}^n$ and a function $U = U(\bar{x})$ which has to satisfy the conditions

$$\begin{aligned} &\text{(i)} \quad \mathfrak{L}_\alpha(F, U) = \sum_{\nu=1}^{n} D_\nu F_{p_\nu}(x, U, DU) - F_u(x, U, DU) = 0, \qquad D_\nu = \partial_{x_\nu} + \alpha_\nu \partial_{x_{n+1}}, \\ &\text{(ii)} \quad U - x_{n+1} \in C^r(T^{n+1}), \\ &\text{(iii)} \quad \partial_{x_{n+1}} U > 0. \end{aligned} \tag{1.7}$$

Equation (1.7) (i) is no longer elliptic but degenerate. It is the Euler-Lagrange equation to the degenerate variational problem

$$\int_{T^{n+1}} F(x, U, DU)\, d\bar{x}, \tag{1.8}$$

which depends implicitly on α. An important feature of this functional is its invariance under the translation $x_{n+1} \to x_{n+1} + \text{const.}$

The quest for minimal foliations of (1.3) is therefore reduced to solving the partial differential equation (1.7). The condition (ii) can be viewed as a boundary condition. The stability problem reduces to the perturbation problem for equation (1.7). Assume that for a given $\alpha \in \mathbb{R}^n$ and a given integrand $F^* \in C^\infty(\Omega)$ a smooth solution $U = U^*$ of (1.7),

$$\mathfrak{L}(F^*, U^*) = 0, \qquad U^* - x_{n+1} \in C^\infty(T^{n+1}), \qquad \partial_{x_{n+1}} U^* > 0,$$

is given such that (x, U^*, DU^*) belongs to Ω, the domain of definition of F^*. If F is in the C^r-topology close to F^*, we ask for a solution of (1.7) for the same α and with $|U - U^*|_{C^1}$ sufficiently small in order that $\partial_{x_{n+1}} U > 0$ and (x, U, DU) remains in Ω. Then the solution U represents via (1.6) a foliation belonging to the same vector α and therefore is conjugate to the unperturbed foliation.

(c) To formulate our main result, we impose a Diophantine condition on the vector α. We require that there exist positive constants τ, γ such that

$$\sum_{\nu=1}^{n} (\alpha_\nu j_{n+1} + j_\nu)^2 \geq \gamma (1 + j_{n+1}^2)^{-\tau} \tag{1.9}$$

holds for all $\bar{j} = (j_1, \ldots, j_n, j_{n+1}) \in \mathbb{Z}^{n+1} \setminus (0)$. This condition looks similar to (1.1); in fact, both conditions are for $\alpha_{n+1} = -1$ and $|\alpha| \leq c$ equivalent with different constants γ (see [17]). For $\tau > 1/n$ and for almost all α there exists a constant γ such that (1.9) holds (see e.g. [22]). For simplicity we will choose τ as an integer.

The following theorem asserts the existence of a solution of (1.7) if an approximate solution U^* is known. For such an approximate solution U^* we will require that with some positive integer a and some positive constant M

$$\begin{aligned} &U^*(\bar{x}) - x_{n+1} \in C^a(T^{n+1}), \\ &|U^* - x_{n+1}|_{C^a} \leq M, \qquad \partial_{x_{n+1}} U^* > M^{-1}, \\ &(x, U^*(\bar{x}), DU^*(\bar{x})) \in \Omega \qquad \text{for all } \bar{x} \in T^{n+1} \end{aligned} \tag{1.10}$$

holds and that $|\mathfrak{L}(F, U^*)|_{C^\tau}$ is small.

THEOREM 1. *Let $F \in C^\infty(\Omega)$ and $\alpha \in \mathbb{R}^n$ satisfying (1.9) be given. We will determine positive integers $a = a(n, \tau)$, $b = b(n, \tau)$ with the following property. For any $\varepsilon > 0$,*

$M>0$ there exists δ depending on n, τ, γ, ε, M and upper bounds for $|\alpha|$ and $|F|_{C^b(\Omega)}$ such that, if U^* satisfies (1.10) *and*

$$|\mathfrak{L}(F, U^*)|_{C^\tau}<\delta, \tag{1.11}$$

then there exists an exact solution U of (1.7) *with*

$$|U-U^*|_{C^2}<\varepsilon, \qquad U^*-x_{n+1}\in C^\infty(T^{n+1}). \tag{1.12}$$

The above formulation shows that it is not necessary to assume that U^* is the solution of an equation $\mathfrak{L}(F^*, U^*)=0$ for some F^*, but it is only required that U^* is an 'approximate solution' of (1.7). Moreover, it is not necessary to assume that U^* is in C^∞. We point out also that the smallness condition (1.11) depends only on a finite number of derivatives of F, U^*, and still we can guarantee that $U\in C^\infty$. With a uniqueness theorem (see § 5) this can be used to establish a regularity theorem for (1.7) by taking for U^* the solution itself.

We postpone specifying the values of a, b to a later theorem, where the corresponding conditions are expressed in terms of Sobolev norms. As far as the differentiability class is concerned our results are very crude. To obtain optimal results would require a refined choice of norms which would become rather involved. It is also not necessary to assume that $F\in C^\infty(\Omega)$ and it would suffice to require $F\in C^l(\Omega)$ for some large l.

(d) As typical application we mention:

Example 1.1. For the integrand

$$F(\bar{x}, p)=\tfrac{1}{2}|p|^2+\lambda V(\bar{x}), \qquad V\in C^\infty(T^{n+1}),$$

the Euler equation becomes the periodic partial differential equation

$$\Delta u=\lambda V_u(x, u).$$

Then theorem 1 guarantees the existence of quasi-periodic solutions

$$u(x)=U(x, (\alpha, x)+\beta)$$

for any α satisfying (1.9) and $|\lambda|$ sufficiently small. Indeed, as an approximate solution we choose $U^*(\bar{x})=x_{n+1}$, which is the exact solution of (1.7) for $\lambda=0$, so that

$$\mathfrak{L}(F, U^*)=\sum_{\nu=1}^{n}(\partial_{x_\nu}+\alpha_\nu\partial_{x_{n+1}})^2U^*-\lambda V(\bar{x})=-\lambda V(\bar{x})$$

can be made small by choice of λ. One has to keep in mind that in the above formulation of the smallness condition, $|\lambda|<\lambda^*$, λ^* depends on an upper bound for $|\alpha|$. However, in this special case one finds quasi-periodic solutions also for large $|\alpha|$.

Example 1.2. We mention that our proof below also gives quasi-periodic solutions of the type (1.6) without any smallness condition on the potential. We assume again that

$$F(\bar{x}, p)=\tfrac{1}{2}|p|^2+V(\bar{x}), \tag{1.13}$$

where $V\in C^\infty(T^{n+1})$. Then for any α satisfying (1.9) and $|\alpha|$ *sufficiently large* there exists a solution U of (1.7).

The point is that for integrands of the type (1.13) the constant δ can be chosen independently of an upper bound on $|\alpha|$ since DU does not enter in the non-linearity

$V = V(x, U)$. The proof of this extension of theorem 1 will not be carried out but is straightforward.

To obtain an approximate solution for $\mathfrak{L}(F, U^*) \sim 0$ in this case, we replace equation (1.7) by

$$|\alpha|^2 \partial^2_{x_{n+1}} U^* = V_u(\bar{x}).$$

We may without change of the Euler equation add a function of x alone to V and achieve the result $V = \partial_{x_{n+1}} Z(\bar{x})$, $Z \in C^\infty(T^{n+1})$. Then

$$U^* = x_{n+1} + |\alpha|^{-2} Z(\bar{x})$$

clearly satisfies

$$\mathfrak{L}(F, U^*) = \left(\sum_{\nu=1}^{n} D_\nu^2 - |\alpha|^2 \partial^2_{x_{n+1}} \right) U^* - (V_u(x, x_{n+1} + |\alpha|^{-2} Z) - V_u(\bar{x})) = O(|\alpha|^{-1}).$$

Extended to this situation, the theorem yields a solution U with

$$|U - x_{n+1}|_{C^2} = O(|\alpha|^{-1}).$$

Example 1.3. Foliations whose leaves are minimal hypersurfaces with respect to a metric $g = (g_{\nu\mu}(\bar{x})) \in C^\infty$ near a flat metric g^*. In this case we set

$$F = \left(\sum_{\nu,\mu=1}^{n+1} g^{\nu\mu}(\bar{x}) p_\nu p_\mu \right)^{1/2} (\det g)^{1/2}, \qquad p_{n+1} = -1, \qquad (g^{\nu\mu}) = (g_{\nu\mu})^{-1}.$$

For a flat metric g^* independent of $\bar{x}$ we can choose $U^* = x_{n+1}$ for any α. Thus our theorem guarantees a smooth minimal foliation for any α satisfying (1.9) and a C^∞-metric sufficiently close to the flat metric g^*. Since the condition (1.9), for $|\alpha| \leq \text{const}$, is equivalent with (1.1), this proves the stability statement given at the beginning of the Introduction, say for $|\alpha| \leq 1$. But by change of coordinates we can always reduce ourselves to this situation.

We point out that this integrand does not have quadratic growth for $|p| \to \infty$ which is relevant for the global theory [16]. However, for our theorem no growth conditions are required, since F has to be known only in a domain

$$\Omega = T^{n+1} \times B_r(\alpha),$$

where $B_r(\alpha)$ is the ball in $\mathbb{R}^n$ of radius $r > 0$ about α, so that

$$(x, U^*, DU^*) = (\bar{x}, \alpha)$$

belongs to Ω.

(e) There is an extensive literature on the theory of foliations and we wish to relate our result to the problems developed there. One of the central questions has been to decide when a foliation is 'taut'; that is, for which foliations a Riemannian metric can be found such that all leaves of the foliations are minimal. Basic contributions to this problem are due to Rummler [**20**] and Sullivan [**23**], in particular for the case of compact foliations. These authors also derived a criterion by which a metric given on the leaves of a foliation can be extended to the ambient space in such a way that the leaves are minimal submanifolds. Haefliger [**5**] cast their criterion into a different form which involves the transverse holonomy group only, which in our case is a finitely generated group of commuting circle diffeomorphisms. In fact, for the codimension 1 foliation given by parallel hypersurfaces on T^d at the beginning of this paper the criterion assures that *any* C^∞-metric given on the leaves can be extended to a metric on T^d such that the leaves are minimal hypersurfaces precisely if the Diophantine condition (1.1) holds. In [**6**] other circle actions have also been studied.

In those studies the foliation is prescribed and a metric is sought such that the leaves are minimal with respect to this metric. In this paper, however, we are concerned with the converse question: to determine a minimal foliation in a certain conjugacy class for a given metric. In contrast to those studies which deal with foliations of arbitrary codimension q and arbitrary manifolds M, we restrict ourselves to a very special situation, namely $q = 1$, $M = T^d$, and, moreover, consider only the perturbation problem for small perturbations of the metric. From an analytic point of view our problem is a non-linear one, while the above extension problem is a linear one. But it should be pointed out that there is a close connection between these problems since the solvability of the linearized equation of our problem is closely related to the extension problem of a metric given on the leaves. One can expect that a codimension 1 minimal foliation on a compact manifold M is stable whenever it is has the extension property that any metric on the leaves admits an extension to M such that the leaves are minimal. For the case of the torus T^d this is indeed the case; we hope to return to this question in the future.

It is unfortunate that the term 'stability of a foliation' may lead to confusion since it was used by Rummler [**20**] in a different sense, where it refers, however, only to compact foliations. We were guided by the concepts of structural stability for dynamical systems, i.e. stability of a minimal foliation means here that under a small perturbation of the metric one has conservation of a foliation in the same equivalence class.

(f) Theorem 1 can be seen in the context of the previous paper [**16**], where *generalized solutions* of the degenerate elliptic partial differential equation were constructed under certain quadratic growth restrictions on the function F. These generalized solutions are usually discontinuous; Bangert [**1**] constructed explicit examples illustrating this discontinuous behaviour. In this context the above result can be viewed as a *regularity theorem* for these solutions under additional assumptions, namely the Diophantine condition (1.9) and the smallness condition (1.11). For a survey of this circle of problems and its connections with Hamiltonian mechanics, in particular the theory of Aubry and Mather, we refer to [**3**], [**17**] and [**12**].

This theorem is a genuine generalization of the perturbation theory of invariant tori for Hamiltonian systems to elliptic partial differential equations. Indeed, for $n = 1$ it agrees with the theorem of preservation of invariant tori on a fixed energy surface for systems with two degrees of freedom, the foliation corresponding to the two-dimensional torus

$$(x, \theta) \rightarrow (x, U(x, \theta), DU(x, \theta))$$

in the three-dimensional phase space.

We want to point out that the traditional proof of these results is based on transformation theory, i.e. on the use of canonical transformations. Since for partial differential equations canonical transformations are essentially given by extension of point transformations [19], one is forced to avoid this technique. This is indeed possible and the following proof, which depends on studying the Euler equations in the configuration space (instead of the Hamiltonian equations in the phase space), is in a way considerably simpler than the earlier proofs, at least if applied to ordinary differential equations. The basic ideas of applying rapid convergent iteration techniques is, however, the same. For Hamiltonian systems of more than two degrees of freedom one encounters an additional difficulty due to the non-commutativity of matrices. Recently, Salamon and Zehnder [21] found an elegant way to circumvent this difficulty and thus to obtain a simple proof of the theorem on invariant tori for Hamiltonian systems of n degrees of freedom.

For partial differential equations one also reduces the problem to solving linear but degenerate differential equations. The difficulty one encounters in solving such equations involving small denominators can be overcome with a trick employed by S. M. Kozlov in his study of a linear eigenvalue problem [10]. It allows the differential equations to be put into a form in which they contain only the differential operators $D_1, D_2, \ldots, D_n$ of (1.7), leading to rather simple *a priori* L^2-estimates. In short, after these preparations one can establish the proof of theorem 1 by a familiar iteration technique by operating with Sobolev norms including negative norms, rather than with Hölder norms. Then we follow the approach developed in [15] even though it yields rather crude results. In [15] Sobolev estimates are also employed and the necessary estimates for the norms presented. However, one can see from our exposition that the result could be derived from an abstract implicit function theorem, as it was derived by Zehnder [24].

In the next section we will describe a regularized version of (1.7) and formulate a generalization of theorem 1 to that case. In § 3 we derive the estimates for the linearized equation. For the reader familiar with these methods it will be clear how to proceed from the results of § 3. For completeness we carry out the details of the proofs in § 4.

2. *The regularized variational problem*

(a) The variational problem (1.8) is, as mentioned before, degenerate and it is convenient to replace it by a non-degenerate problem

$$J_\alpha^\nu(U) = \int_{T^{n+1}} G(x, U, \bar{D}U)\, d\bar{x}, \tag{2.1}$$

where

$$\begin{gathered} G(\bar{x}, \bar{p}) = \tfrac{1}{2}\nu a_0(\bar{x}) p_{n+1}^2 + F(\bar{x}, p), \\ \bar{D} = (D_1, D_2, \ldots, D_{n+1}), \qquad D_\nu = \partial_{x_\nu} + \alpha_\nu \partial_{x_{n+1}}, \qquad D_{n+1} = \partial_{x_{n+1}}. \end{gathered} \tag{2.2}$$

If $\nu > 0$ and $a_0(\bar{x}) \geq 1$, then the Legendre condition

$$\sum_{\mu,\lambda=1}^{n+1} G_{p_\mu p_\lambda} \xi_\mu \xi_\lambda \geq \sum_{\mu=1}^{n} \xi_\mu^2 + \nu \xi_{n+1}^2 \tag{2.3}$$

is satisfied. The variational problem can be used for the constructions of minimals

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$U = U(\bar{x}, \nu)$ which depend on ν. Our goal is to establish estimates for the solutions of the Euler equations

$$\nu D_{n+1}(a_0(x, U)D_{n+1}U) - (\nu/2)a_{0,x_{n+1}}(D_{n+1}U)^2 + \mathfrak{L}_\alpha(F, U) = 0 \tag{2.4}$$

which are independent of $\nu \in (0, 1]$. Although this regularization is not really necessary for the following proof, it is convenient and, moreover, gives a stronger result. But mainly this approach reduces the problem to establishing ν-independent estimates and separates it from the existence question.

We point out that the variational problem (2.1) is also invariant under the translation $x_{n+1} \to x_{n+1} + \text{const}$, so that as well as $U(\bar{x})$, $U(\bar{x} + \lambda e_{n+1})$ is also a solution of (2.4). On the other hand, up to this translation the solution of (2.4) with the condition that $U - x_{n+1}$ has period 1 in all variables is unique:

THEOREM 2. *If U_1, U_2 are two solutions of* (2.4) *with $\nu > 0$, satisfying*

$$U_i(\bar{x}) - x_{n+1} \in C^2(T^{n+1}), \qquad i = 1, 2,$$

then there exists a $\lambda^* \in \mathbb{R}$ with

$$U_2(\bar{x}) = U_1(\bar{x} + \lambda^* e_{n+1}).$$

Proof. This result follows from the maximum principle for elliptic equations. Since

$$U_1(\bar{x} + \lambda e_{n+1}) = U_1(\bar{x}) + \lambda \qquad \text{if } \lambda \text{ integer},$$

it is clear that the continuous function

$$f(\lambda) = \min_{\bar{x} \in T^{n+1}} (U_1(\bar{x} + \lambda e_{n+1}) - U_2(\bar{x}))$$

satisfies $f(+\infty) = +\infty$, $f(-\infty) = -\infty$ and therefore we can find a λ^* with $f(\lambda^*) = 0$. The function

$$Z(\bar{x}) = U_1(\bar{x} + \lambda^* e_{n+1}) - U_2(\bar{x}) \geq 0$$

takes on the value 0 and is the solution of an elliptic differential equation obtained from (2.4) by taking the difference for $U = U_1(\bar{x} + \lambda^* e_{n+1})$ and $U = U_2(\bar{x})$. Since $Z(\bar{x}) \geq 0$ takes on its minimum 0 in the interior, it follows that $Z \equiv 0$.

(b) *Transformation property.* As indicated in § 1, a function V, $V(\bar{x}) - x_{n+1} \in C^1(T^{n+1})$ with $\partial_{x_{n+1}} V > 0$, gives rise to a diffeomorphism

$$\bar{x} \to (x, V(\bar{x}))$$

of the torus T^{n+1} onto itself. We use this remark to transform $U = U(\bar{x})$ into $V(x, U(\bar{x}))$ (rather than transforming the independent variables). Then the functional J^ν_α is mapped into another one with the integrand $G \circ \phi_V$, where

$$\phi_V \colon (x, x_{n+1}, p, p_{n+1}) \to (x, V(\bar{x}), V_x + V_{x_{n+1}} p, V_{x_{n+1}} p_{n+1}) \tag{2.5}$$

is the prolonged mapping. These mappings clearly form a group with

$$\phi_W \circ \phi_V = \phi_{W * V}, \qquad W * V = W(x, V(\bar{x})).$$

The unit element corresponds to the function $V(\bar{x}) = x_{n+1}$.

The Euler–Lagrange expression for J^ν_α will be denoted by

$$\begin{aligned}\bar{\mathfrak{L}}_\alpha(G, U) &= \sum_{\mu=1}^{n+1} D_\mu G_{p_\mu}(x, U, \bar{D}U) - G_u(x, U, \bar{D}U)\\ &= \nu D_{n+1}(a_0(x, U)D_{n+1}U) - (\nu/2)a_{0,x_{n+1}}(D_{n+1}U)^2 + \mathfrak{L}_\alpha(F, U).\end{aligned} \tag{2.6}$$

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Then the transformation (2.5) maps the Euler–Lagrange expression

$$\mathfrak{L}_\alpha(G, U) \to \mathfrak{L}_\alpha(G, V * U)V_{x_{n+1}} = \mathfrak{L}_\alpha(G \circ \phi_V, U).$$

This follows readily by considering $\mathfrak{L}(G, U)$ as the first variation of (2.1). This transformation preserves the class of integrands (2.2) taking $G = G(\bar{x}, \bar{p})$ into

$$\begin{aligned} \tilde{G}(\bar{x}, \bar{p}) &= G \circ \phi_V = \tfrac{1}{2}\nu\tilde{a}_0(\bar{x})p_{n+1}^2 + F \circ \phi_V, \\ \tilde{a}_0(\bar{x}) &= a_0(x, V(\bar{x}))V_{x_{n+1}}^2, \\ \tilde{G}_{p_\mu p_\lambda}(\bar{x}, \bar{p}) &= (G_{p_\mu p_\lambda} \circ \phi_V)V_{x_{n+1}}^2. \end{aligned} \tag{2.7}$$

This transformation can be used to replace the approximate solution U^* of $\mathfrak{L}_\alpha(G, U^*) \sim 0$ by $U^* = x_{n+1}$ by setting $V = U^*$. This reduction is not essential for the following proof, but the above transformation property will be helpful for the understanding of the construction in the next section, in particular for the identity (3.2).

(c) The fact that functional J_α^ν is invariant under the translation $x_{n+1} \to x_{n+1} + \varepsilon$ has the consequence that for any U with $U - x_{n+1} \in C^1(T^{n+1})$ one has the identity

$$\int_{T^{n+1}} \mathfrak{L}_\alpha(G, U)U_{x_{n+1}}\, d\bar{x} = 0. \tag{2.8}$$

This follows by differentiation of J_α^ν with respect to ε. It can also be derived from the identity

$$\mathfrak{L}_\alpha(G, U)U_{x_{n+1}} = \sum_{\mu=1}^{n+1} D_\mu(G_{p_\mu}(x, U, \bar{D}U)U_{x_{n+1}}) - D_{n+1}G(x, U, \bar{D}U),$$

which shows that the right-hand side is a divergence expression. Therefore one obtains (2.8) by integration, using the periodicity condition $U - x_{n+1} \in C^1(T^{n+1})$.

(d) *To formulate the generalized version of theorem 1 for the regularized variational* problem, we make use of Sobolev spaces $H^s(T^d)$. For a smooth function ϕ on the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d = n+1)$ we define the Sobolev norm $\|\phi\|_r$ with the help of the Fourier representation

$$\phi = \sum_{j \in \mathbb{Z}^d} \hat{\phi}_j e^{2\pi i(j,x)}$$

by

$$\|\phi\|_r^2 = \sum_{j \in \mathbb{Z}^d} (1 + |j|^2)^r |\hat{\phi}_j|^2, \qquad |j|^2 = \sum_{\mu=1}^{d} j_\mu^2$$

for any real r. The closure of $C^\infty(T^d)$ under this norm defines the Sobolev space $H^r(T^d)$, where also negative norms, as they were considered by P. D. Lax, are admitted. With H_0^r we denote the subspace of those $\phi \in H^r$ for which

$$\hat{\phi}_0 = \int_{T^d} \phi\, dx = 0.$$

We will use the standard results about these spaces, in particular that for $s > t$ we have $H^s \subset H^t$ and

$$\|\phi\|_t \le \|\phi\|_s \qquad \text{for all } \phi \in H^s.$$

Moreover, the embedding $H^s \to H^t$ is compact.

Single bars will be reserved for uniform norms, i.e. for integers $r \geq 0$ we abbreviate

$$|\phi|_r = |\phi|_{C^r} \qquad \text{for } \phi \in C^r(T^d). \tag{2.9}$$

Then one has for $\phi \in C^r(T^d)$

$$\|\phi\|_r \leq c_r |\phi|_r.$$

In the opposite direction one has for any r and $t > d/2$

$$|\phi|_r \leq c_{r,t} \|\phi\|_{r+t} \qquad \text{for } \phi \in H^{r+t}, \tag{2.10}$$

which is the simplest Sobolev inequality.

(e) To formulate our result about the solution of (2.4), we consider a function $G = G(\bar{x}, \bar{p}, \nu)$ of the form (2.2), where $F \in C^\infty(\Omega)$ satisfies the Legendre condition (1.4) with $\lambda = 1$. Moreover, we assume

$$a_0 \in C^\infty(T^{n+1}), \qquad a_0 \geq 1,$$

so that G satisfies the Legendre condition

$$\sum_{\mu,\nu=1}^{n+1} G_{p_\mu p_\lambda} \xi_\mu \xi_\lambda \geq \nu \xi_{n+1}^2 + \sum_{\mu=1}^{n} \xi_\mu^2 \tag{2.11}$$

for $\nu > 0$.

Our aim is to solve the Euler equation for (2.1). We will consider the function G fixed and introduce the functional (2.6):

$$E(U) = \mathfrak{L}_\alpha(G, U) = \nu D_{n+1}(a_0(x, U) D_{n+1} U) - \frac{\nu}{2} a_{0,x_{n+1}} U_{x_{n+1}}^2$$

$$+ \sum_{\mu=1}^{n} D_\mu F_{p_\mu}(x, U, DU) - F_u(x, U, DU). \tag{2.12}$$

Similarly, as in theorem 1, we assume that U^* is an approximate solution of $E(U) = 0$ for a fixed $\nu \in (0, 1]$ in the following sense. With some positive integer $a = a(\tau, n) > d/2 + 1$ and some constant M we require that

$$\begin{aligned} &U^*(\bar{x}) - x_{n+1} \in H^a(T^{n+1}), \\ &\|U^* - x_{n+1}\|_a < M, \qquad \partial_{x_{n+1}} U^* > M^{-1}, \\ &(x, U^*(\bar{x}), DU^*(\bar{x})) \in \Omega \qquad \text{for all } \bar{x} \in T^{n+1}, \end{aligned} \tag{2.13}$$

and that for some fixed $\nu \in (0, 1]$ the expression $\|E(U^*)\|_\tau$ is small.

THEOREM 3. *Let $F \in C^\infty(\Omega)$, $a_0 \in C^\infty(T^{n+1})$, $a_0 \geq 1$, and assume that $\alpha \in \mathbb{R}^n$ satisfies* (1.9). *Let τ be an integer $> \frac{1}{2}(n+1)$ and set*

$$a = 9\tau + 10, \qquad b = 17\tau + 19.$$

Then, given $\varepsilon > 0$, $M > 0$, there exists a positive number δ depending on n, τ, γ, ε, M and upper bounds for $|\alpha|$ and $|F|_{C^b(\Omega)}$, $|a_0|_{C^b}$ with the property: if for some $\nu \in (0, 1]$ there exists an approximate solution satisfying (2.13) *and*

$$\|E(U^*)\|_\tau < \delta, \tag{2.14}$$

then there exists an exact solution U of $E(U) = 0$ with $U - x_{n+1} \in C^\infty$,

$$\|U - U^*\|_{5\tau+6} < \varepsilon, \qquad \|U - x_{n+1}\|_r < C_r \tag{2.15}$$

for all integers $r \geq 1$, where the constants C_r depend on F, a_0 but not on ν.

For $\nu > 0$ the equation $E(U) = 0$ is an elliptic partial differential equation for which even a global existence theory is available. The point of this theorem is that the estimates (2.14), (2.15) are independent of the choice of ν and therefore give rise to a solution of the degenerate equation for $\nu = 0$. In fact, in this way we derive theorem 1 as a consequence of theorem 3.

Let U^* be an approximate solution in the sense of (1.10), (1.11). Then the conditions (2.13) are satisfied and

$$\|E(U^*)\|_\tau \le \delta + \nu c$$

with a constant c depending on M, since $\tau + 2 < a$. Thus we have

$$\|E(U^*)\|_\tau < 2\delta \qquad \text{for } 0 < \nu < c^{-1}\delta,$$

i.e. an approximate solution with δ replaced by 2δ. By theorem 3 there exists an exact solution $U = U(x, \nu)$ of $E(U) = 0$ for all $\nu \in (0, c^{-1}\delta)$ satisfying (2.15). By theorem 2 these solutions are unique up to a phase shift which can be normalized by the condition

$$\int_{T^{n+1}} (U(\bar{x}) - x_{n+1})\, dx = 0.$$

For $\nu \to 0$ we obtain a solution of the equation $\mathfrak{L}(F, U) = 0$ with $U - x_{n+1} \in C^\infty(T^{n+1})$. Moreover, because $5\tau + 6 > d/2 + 2$, we conclude from (2.15) and (2.10) that $|U - U^*|_{C^2} < c\varepsilon$. We can assume that ε is chosen so small that $(x, U(\bar{x}), DU(\bar{x})) \in \Omega$. Thus, replacing $c\varepsilon$ by ε, we see that theorem 1 is a consequence of theorem 3.

(f) Before turning to the proof of theorem 3 in the next two sections, we collect some standard estimates for Sobolev norms, which are needed below.

The Sobolev norm $\|\phi\|_r$ defined above is a logarithmically convex function of r. For each $\phi \in H^m$ and $r < s \le m$ one has

$$\|\phi\|_{\lambda r + (1-\lambda)s} \le \|\phi\|_r^\lambda \|\phi\|_s^{1-\lambda} \qquad \text{for } \lambda \in (0, 1). \tag{2.16}$$

Sometimes it is preferable to write this non-linear inequality in an equivalent linear form. Since for any positive numbers ε, u, v one has

$$u^\lambda v^{1-\lambda} \le \varepsilon^{-(1-\lambda)/\lambda} u + \varepsilon v,$$

(2.16) implies

$$\|\phi\|_{\lambda r + (1-\lambda)s} \le \varepsilon^{-(1-\lambda)/\lambda} \|\phi\|_r + \varepsilon \|\phi\|_s \tag{2.17}$$

for all $\varepsilon > 0$. Actually, this relation is up to a constant equivalent to (2.16). If we set $t = \lambda r + (1-\lambda)s \in (r, s)$, this inequality takes the form

$$\|\phi\|_t \le \varepsilon^{-(t-r)/(s-t)} \|\phi\|_r + \varepsilon \|\phi\|_s, \qquad r < t < s$$

for all $\varepsilon > 0$, $\phi \in H^s$.

For the non-linear operations we need the following estimates. If $\phi, \psi \in H^r \cap C^0$, where r is a positive integer, then there exists a constant c_r such that

$$\|\phi\psi\|_r \le c_r(|\phi|_0 \|\psi\|_r + |\psi|_0 \|\phi\|_r) \tag{2.18}$$

holds. This inequality can be derived from the estimates

$$\left(\int_{T^d} |\partial^\rho \phi|^{2r/\rho}\, dx\right)^{\rho/2r} \le c_r |\phi|_0^{1-\rho/r} \|\phi\|_r^{\rho/r} \tag{2.19}$$

for all derivatives ∂^ρ of order $|\rho| \le r$. This is a special case of an inequality by Gagliardo and Nirenberg (see [4] and [**18**]).

Combining (2.18) with (2.10), we obtain for $t > d/2$

$$\|\phi\psi\|_r \le c_{t,r}(\|\phi\|_t \|\psi\|_r + \|\psi\|_t \|\phi\|_r)$$

and for $r = t > d/2$

$$\|\phi\psi\|_t \le c_t \|\phi\|_t \|\psi\|_t, \qquad \phi, \psi \in H^t. \tag{2.20}$$

Therefore for $t > d/2$ the space H^t is a Banach algebra, sometimes referred to as a Schauder ring.

Incidentally, applying (2.20) to powers ϕ^p with $p = 2^j$, we conclude that

$$\|\phi^p\|_0 \le \|\phi^p\|_t \le c_t^p \|\phi\|_t^p,$$

showing that for $t > d/2$

$$\sup|\phi| = \lim_{p \to \infty} \|\phi^p\|_0^{1/p} \le c_t \|\phi\|_t,$$

which gives us back (2.10).

For negative norms one has for $\phi \in H^{-t}$, $\psi \in H^t$ with $t > d/2$

$$\|\phi\psi\|_{-t} \le c_t \|\phi\|_{-t} \|\psi\|_t. \tag{2.21}$$

This follows from the characterization of the negative norm

$$\|\phi\|_{-t} = \sup_{\|\zeta\|_t \le 1} (\phi, \zeta),$$

where $(\,,)$ denotes the extension of the inner product in H^0. Indeed, by (2.20) we have for all $\zeta \in H^t$

$$|(\phi\psi, \zeta)| = |(\phi, \psi\zeta)| \le \|\phi\|_{-t} \|\psi\zeta\|_t \le \|\phi\|_{-t} c_t \|\psi\|_t \|\zeta\|_t,$$

proving (2.21).

Finally, we need an estimate for the composition of functions. Assume that

$$f \in C^r(T^d \times \Omega), \qquad \phi \in C^0(T^n, \Omega) \cap H^r(T^n),$$

where Ω is bounded. Then the composition $f(x, \phi(x))$ belong to H^r and for integers $r \ge 0$ one has

$$\|f(x, \phi)\|_r \le c_r |f|_{C^r}(1 + \|\phi\|_r), \tag{2.22}$$

where the constant c_r depends on r and the diameter of Ω. Notice that it is implicit that $|\phi|_0$ is bounded by the diameter of Ω. This inequality follows from (2.19) (see [**15**]).

For $t > d/2$, $\phi, \psi \in H^t$ and $f = f(x, y) \in C^{t+1}$ we have the inequality

$$\|f(x, \phi) - f(x, \psi)\|_t \le c_t |f|_{C^{t+1}}(1 + \|\phi\|_t + \|\psi\|_t)\|\phi - \psi\|_t. \tag{2.23}$$

This follows from

$$f(x, \phi) - f(x, \psi) = \int_0^1 \frac{d}{d\lambda} f(x, \phi + \lambda(\psi - \phi))\, d\lambda = \int_0^1 f_y(x, \phi + \lambda(\psi - \phi))\, d\lambda \cdot (\psi - \phi),$$

and from (2.20):

$$\|f(x, \phi) - f(x, \psi)\|_\tau \le \sup_{\lambda \in [0,1]} \|f_y(x, \phi + \lambda(\psi - \phi))\|_t \|\psi - \phi\|_t.$$

Using (2.22), we obtain (2.23).

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We also need the simple approximation properties of ϕ by trigonometrical polynomials. For $N \geq 0$ and $\phi \in \bigcup_r H^r$ we set

$$S_N\phi = \sum_{|j| \leq N} \hat{\phi}_j e^{2\pi i\langle j,x\rangle} \in \bigcap_r H^r = H^\infty.$$

Then one has obviously

$$\begin{aligned} \|S_N\phi\|_s &\leq (1+N^2)^{t/2}\|\phi\|_r, \qquad t = \max(s-r, 0), \\ \|(I-S_N)\phi\|_r &\leq (1+N^2)^{-(s-r)/2}\|\phi\|_s \qquad \text{for } r<s \end{aligned} \tag{2.24}$$

for all $\phi \in H^\infty(T^d) = C^\infty(T^d)$.

3. *H^r-estimates for the linearized equation*

(a) The proof of theorem 3 is based on an iterative construction of the solution providing at the same time ν-independent estimates for them. Since the problem has been cast in the form of an implicit function theorem, the work of Zehnder [**24**] is most appropriate for such a construction. We also refer to the elegant presentations of Hörmander ([**7**], [**8**]) based on Nash's approach. However, these papers do not seem to be directly applicable since the estimates for the linearized equations are somewhat weaker than required there. We also refer to a formulation (without proof) of an implicit function theorem, based on Hörmander's work, in Iwasaki's paper [**9**], in which only an approximate solution of the linearized equation is postulated. In the following we adapt the method developed in [**15**], which consists in applying alternatingly Newton's method and a smoothing process.

All these approaches are based on finding an approximate solution of the linear equation

$$E'(U)V + E(U) = 0, \tag{3.1}$$

where $E'(U)$ is the Frechet derivative of the functional $E(U)$ defined in (2.12). We want to point out that it is not necessary and not possible to find such an approximate solution for the equation $E'(U)V + g = 0$ for an arbitrary function g. This is due to the fact that, if U is a solution of $E(U) = 0$, then differentiation with respect to x_{n+1} yields $V = U_{x_{n+1}}$ as a solution of the homogeneous equation

$$E'(U)V = 0.$$

Therefore the solvability of the inhomogeneous equation requires a compatibility condition for g.

To describe this compatibility condition, we introduce, following Kozlov [**10**], the function $W = V \cdot U_{x_{n+1}}^{-1}$, using that $U_{x_{n+1}} > 0$. Then by this transformation $V = U_{x_{n+1}}$ is transformed into $W = 1$.

This situation is expressed by the following identity:

$$U'\{E'(U)(U'W) - W\partial_{x_{n+1}}E(U)\} = \sum_{\mu,\lambda=1}^{n+1} D_\mu(U'^2 G_{p_\mu p_\lambda} D_\lambda W) = -L(W), \tag{3.2}$$

where $U' = \partial_{x_{n+1}} U$ and the differential operator L is defined by this formula. We will verify it presently by a direct calculation; it follows also from the transformation formula (2.7) of the last section. If $E(U) = 0$, one sees that (3.2) represents a transformation of the differential operator $E'(U)$ into the operator L, possessing the constant as null solution; however, the identity is valid generally, i.e. even if $E(U)$ does not vanish.

To prove (3.2), we write

$$E'(U)V=\sum_{\mu,\lambda=1}^{n+1} D_\mu(a_{\mu\lambda}D_\lambda V)-bV,$$

$$E'(U)U'=\sum_{\mu,\lambda=1}^{n+1} D_\mu(a_{\mu\lambda}D_\lambda U')-bU',$$

where

$$a_{\mu\lambda}=G_{p_\mu p_\lambda}(x,U,\bar{D}U),\qquad b=\sum_{\mu=1}^{n+1} D_\mu G_{p_\mu u}-G_{uu}.$$

Setting $V=U'W$, we find

$$\begin{aligned}E'(U)U'W-W\partial_{\bar{x}_{n+1}}E(U)&=\sum_{\mu,\lambda}\{D_\mu(a_{\mu\lambda}U'D_\lambda W)+D_\mu(a_{\mu\lambda}WD_\lambda U')\\&\qquad -WD_\mu(a_{\mu\lambda}D_\lambda U')\}\\&=\sum_{\mu,\lambda}\{D_\mu(a_{\mu\lambda}U'D_\lambda W)+a_{\mu\lambda}(D_\mu W)(D_\mu U')\}.\end{aligned}$$

Note that this expression agrees with

$$(U')^{-1}\sum D_\mu(a_{\mu\lambda}U'^2D_\lambda W),$$

which proves (3.2).

In trying to find an approximate solution for (3.1) we can, with the help of this formula and by dropping the quadratically small term $W\partial_{x_{n+1}}E$, replace equation (3.1) by

$$LW=U'E(U),\qquad V=U'W,\qquad U'=\partial_{x_{n+1}}U. \tag{3.3}$$

This inhomogeneous equation automatically satisfies the compatibility condition. Indeed, by (2.8) we have

$$\int_{T^d} U_{x_{n+1}}E(U)\,d\bar{x}=0,$$

which allows us to solve (3.3) for a function W of period 1 in all variables.

For the following it will be important to derive ν-independent estimates for the solutions of the equation

$$L\phi=g, \tag{3.4}$$

where

$$L\phi=-\left\{\sum_{\nu,\lambda=1}^{n} D_\mu(a_{\mu\lambda}(\bar{x})D_\lambda\phi)+\nu D_{n+1}(a_{n+1}D_{n+1}\phi)\right\}. \tag{3.5}$$

In our application we will use (with a change of notation)

$$a_{\mu\lambda}(\bar{x})=U'^2G_{p_\mu p_\lambda},\qquad a_{n+1}=U'^2a_0, \tag{3.6}$$

but for the moment we consider the coefficients as known functions in $C^\infty(T^d)$ which satisfy

$$\sum_{\mu,\lambda=1}^{n} a_{\mu\lambda}\xi_\mu\xi_\lambda+\nu a_{n+1}\xi_{n+1}^2\ge\sum_{\mu=1}^{n}\xi_\mu^2+\nu\xi_{n+1}^2. \tag{3.7}$$

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Such an estimate with a factor, say, $\frac{1}{2}$ follows from (2.11) if $U'^2 > \frac{1}{2}$. In the following we will ignore such a factor, e.g. by replacing G by $2G$.

(b) For $\nu > 0$ the operator L of (3.5) is elliptic and maps the Sobolev space

$$H_0^r = \left\{ \phi \in H^r \,\middle|\, \int_{T^d} \phi \, d\bar{x} = 0 \right\}$$

one-to-one onto H_0^{r-2} if $r \geq 2$. In order to obtain estimates which are independent of ν, we need the following:

LEMMA 3.1. *For $\phi \in C^\infty(T^d)$ we have*

$$\sum_{\mu=1}^{n} \|D_\mu \phi\|_0^2 + \nu \|D_{n+1}\phi\|_0^2 \leq (L\phi, \phi). \tag{3.8}$$

Moreover, if α satisfies (1.9), *one has*

$$\sum_{\mu=1}^{n} \|D_\mu \phi\|_r^2 \geq \gamma \|\phi\|_{-\tau+r}^2 \qquad \text{if } (\phi, 1) = 0 \tag{3.9}$$

for all real r; here $(\,,)$ denotes the inner product

$$(\phi, \psi) = \int_{T^d} \phi\psi \, d\bar{x}.$$

The proof of (3.8) is a straightforward consequence of the definitions of (3.5) and (3.7). The inequality (3.9) follows directly by the Fourier representation of ϕ from (1.9). This inequality reflects the loss of τ derivatives due to the 'small divisors'; actually, according to (1.9), one has only a loss of τ derivatives in the x_{n+1}-directions. By replacing j_{n+1}^2 by $|\bar{j}|^2$, we do not take account of this fact, which would have to be considered if one wanted to get better differentiability results.

COROLLARY 3.2. (of lemma 3.1). *For $\nu > 0$ the mapping*

$$L: H_0^1 \to H_0^{-1}$$

has a bounded inverse. Moreover, if $g \in H_0^\infty = \bigcap_r H_0^r$, then the unique solution $\phi \in H_0^1$ of $L\phi = g$ belongs to H_0^∞.

Indeed, the first statement follows from (3.8) since for $\phi \in H_0^1$ the left-hand side dominates $\|\phi\|_1$, hence

$$\|\phi\|_1 \leq c(\nu)(L\phi, \phi) \leq c(\nu)\|L\phi\|_{-1}\|\phi\|_1.$$

From this the existence and boundedness of L^{-1} follows; the second statement is also standard since L is an elliptic operator for $\nu > 0$ with smooth coefficients. However, the norm $\|L^{-1}\|$ is dependent on ν. To obtain ν-independent estimates, we show that L^{-1} when viewed as a mapping from H_0^τ to $H_0^{-\tau}$ has a norm bounded by γ^{-1}.

COROLLARY 3.3. *For $0 \leq \nu \leq 1$ and $\phi \in H_0^\infty$ one has the inequality*

$$\gamma \|\phi\|_{-\tau} \leq \|L\phi\|_\tau. \tag{3.10}$$

This follows from (3.9) for $r = 0$ and (3.8):

$$\gamma \|\phi\|_{-\tau}^2 \leq (L\phi, \phi) \leq \|L\phi\|_\tau \|\phi\|_{-\tau}.$$

(c) We need similar estimates for $\|\phi\|_{-\tau+r}$ for large positive r. For this purpose it is important to control the constants in their dependence on the coefficients $a_{\mu\lambda}$. We assume that $A>1$ is chosen so that for a given positive integer r

$$\sum_{\mu,\lambda=1}^{n} \|a_{\mu\lambda}\|_r + \|a_{n+1}\|_r \le A. \tag{3.11}$$

In addition, we require that with a constant c_0

$$\sum_{\mu,\lambda=1}^{n} |a_{\mu\lambda}|_{C^1} + |a_{n+1}|_{C^1} \le c_0. \tag{3.12}$$

LEMMA 3.4. *Under the assumptions* (3.11), (3.12) *one has for all $\phi\in H_0^\infty$ the estimate*

$$\sum_{\mu=1}^{n} \|D_\mu\phi\|_r^2 + \nu\|\partial_{x_{n+1}}\phi\|_r^2 \le c_r\left\{(L\phi,\phi)_r + A^2\left(\sum_{\mu=1}^{n} |D_\mu\phi|_0^2 + \nu|\partial_{x_{n+1}}\phi|_0^2\right)\right\}, \tag{3.13}$$

where c_r is a constant depending on c_0 and r but not on ν or A, and $(\,,)_r$ denotes the inner product in H_0^r.

The proof of (3.13) uses the representation

$$(L\phi,\phi)_r = ((1-(2\pi)^{-2}\Delta)^r L\phi, \phi)$$

and requires the estimate of commutators of differential operators of order $2r$ and L. We forego the calculation, since this has been carried out, though less explicitly, by Kozlov [**10**]. It is essential for the following to control the dependence on the coefficients of L via the constant A.

(d) We put the estimate (3.13) into more explicit form. Using (2.10) and (2.17), we have for $r>t>(n+1)/2$

$$\sum_{\mu=1}^{n} |D_\mu\phi|_0^2 + \nu|\partial_{x_{n+1}}\phi|_0^2 \le \sum_{\mu=1}^{n} \|D_\mu\phi\|_t^2 + \nu\|\partial_{x_{n+1}}\phi\|_t^2$$

$$\le \varepsilon^{-t/(r-t)}\left(\sum_{\mu=1}^{n} \|D_\mu\phi\|_0^2 + \nu\|\partial_{x_{n+1}}\phi\|_0^2\right) + \varepsilon\left\{\sum_{\mu=1}^{n} \|D_\mu\phi\|_r^2 + \nu\|\partial_{x_{n+1}}\phi\|_r^2\right\}.$$

Setting $\varepsilon=(2c_rA^2)^{-1}$, we can rewrite equation (3.13) as

$$\sum_{\mu=1}^{n} \|D_\mu\phi\|_r^2 + \nu\|\partial_{x_{n+1}}\phi\|_r^2 \le c_r'\{(L\phi,\phi)_r + A^{2r/(r-t)}(\sum \|D_\mu\phi\|_0^2 + \nu\|\partial_{x_{n+1}}\phi\|_0^2)\}.$$

Finally, using (3.8), we can replace the last term on the right by $A^{2r/(r-t)}(L\phi,\phi)_0$ and obtain for $r>t>(n+1)/2$

$$\sum_{\mu=1}^{n} \|D_\mu\phi\|_r^2 + \nu\|\partial_{x_{n+1}}\phi\|_r^2 \le c_r'\{(L\phi,\phi)_r + A^{2r/(r-t)}(L\phi,\phi)_0\}.$$

By (3.9) the left-hand side is larger than $\gamma\|\phi\|_{-\tau+r}^2$ and, since

$$(L\phi,\phi)_r \le \|L\phi\|_{\tau+r}\|\phi\|_{-\tau+r},$$

we find

$$\gamma\|\phi\|_{-\tau+r}^2 \le c_r''\{\|L\phi\|_{\tau+r}^2 + A^{2r/(r-t)}(L\phi,\phi)_0\}.$$

Finally, with (3.10) one has

$$(L\phi, \phi)_0 \le \|L\phi\|_\tau \|\phi\|_{-\tau} \le \gamma^{-1}\|L\phi\|_\tau^2.$$

Therefore we have for all $\phi \in H_0^\infty$, $\nu \in (0,1)$ and any positive integer r the inequalities

$$\|\phi\|_{-\tau} \le \gamma^{-1}\|L\phi\|_\tau, \qquad \|\phi\|_{-\tau+r} \le c_r\{\|L\phi\|_{\tau+r} + A^{r/(r-t)}\|L\phi\|_\tau\}, \tag{3.14}$$

where c_r is a constant depending on γ, c_0 but not on ν or A.

LEMMA 3.5. *Let L be the operator given by* (3.5) *and assume that* (3.7), (3.11), (3.12) *hold. Then for $\nu \in (0,1)$ and $g \in H_0^\infty(T^d)$ the equation*

$$L\phi = g$$

has a unique solution $\phi \in H_0^\infty(T^d)$ which satisfies the ν-independent estimates

$$\|\phi\|_{-\tau} \le \gamma^{-1}\|g\|_\tau, \qquad \|\phi\|_{-\tau+r} \le c_r\{\|g\|_{\tau+r} + A^{r/(r-t)}\|g\|_\tau\}.$$

(e) With the help of this lemma we can prove theorem 3 by constructing a solution of $E(U)=0$ from an approximate solution in a ν-independent neighbourhood. The basic step is to construct from an approximate solution $U \in C^\infty$, with $\|E(U)\|_\tau$ small, an improved approximate solution $U+V$, where V is defined as follows. Let $W \in H_0^\infty(T^d)$ be the unique solution of

$$LW = U_{x_{n+1}}E(U) \tag{3.15}$$

with L as defined in (3.2), and set

$$V = S_N(U_{x_{n+1}}W), \tag{3.16}$$

where S_N is the truncation described at the end of § 2 with an appropriately chosen large N.

To see that $U+V$ is an improved approximation, one has to show that $E(U+V)$ is smaller than $E(U)$ in appropriate norms. This will be done in detail in § 4, but we give the underlying reasoning. The expression

$$E(U+V) - E(U) - E'(U)V$$

is quadratically small in V, hence in $E(U)$, and we have to show that

$$E(U) + E'(U)V = E(U) + E'(U)(U_{x_{n+1}}W) + E'(U)(I-S_N)(U_{x_{n+1}}W)$$

is small. For this purpose we use the identity (3.2) and (3.15) to get

$$E'(U)(U_{x_{n+1}}W) = W(\partial_{x_{n+1}}E) - U_{x_{n+1}}^{-1}LW = W(\partial_{x_{n+1}}E) - E(U)$$

so that

$$E(U) + E'(U)V = W(\partial_{x_{n+1}}E) + E'(U)(I-S_N)(U_{x_{n+1}}W). \tag{3.17}$$

The first term on the right-hand side depends quadratically on E since W can be estimated linearly by E. The second term will be made small by the choice of N.

The smoothing operator S_N is needed because of the loss of differentiability in this process. If l, τ are integers, l sufficiently large, then

$$E: H^{l-\tau} \to H^{l-\tau-2} \tag{3.18}$$

since E is a differential operator of order 2. Next we consider L^{-1} as an operator from $H_0^{\tau+r}$ to $H_0^{-\tau+r}$ since it is bounded independently of $\nu \in (0,1)$ (see lemma 3.5); hence with $r = l - 2\tau - 2$

$$L^{-1}U_{x_{n+1}}E(U): H^{l-\tau} \to H^{l-q-\tau}, \tag{3.19}$$

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where $q=2\tau+2$ represents the loss of derivatives from U to $(U_{x_{n+1}}W)$. By inserting S_N and defining V by (3.16), we recover that $V\in H^\infty(T^d)$ but have to estimate the error. This will be done in the next section.

4. *Proof of theorem* 3

(a) For the proof of theorem 3 we use the construction and the estimates of the previous section. First we replace the given approximation U^* with $U^*-x_{n+1}\in H^a(T^d)$ by the smooth function U^0 defined by

$$U^0-x_{n+1}=S_N(U^*-x_{n+1})\in H^\infty(T^d), \tag{4.1}$$

where N will be chosen appropriately. Then we construct a sequence U^s $(s\geq 1)$ of improved approximate solutions by the recursion formula

$$U^{s+1}-U^s=S_{N_s}((\partial_{x_{n+1}}U^s)W), \qquad W=L_s^{-1}(\partial_{x_{n+1}}U^s)E(U^s), \tag{4.2}$$

where N_s will be chosen later (see (4.7)) and L_s is the differential operator of (3.2) for $U=U^s$. In the following we will show that

$$(x, U^s(\bar{x}), DU^s(\bar{x}))\in\Omega \qquad \text{for all } \bar{x}\in T^d, \tag{4.3}$$

so that $F(x, U^s, DU^s)$ is defined and the sequence U^s converges to an exact solution U of $E(U)=0$ for which the claimed ν-independent estimates hold and for which $U-x_{n+1}\in C^\infty(T^d)$.

For the first step we recall that

$$(x, U^*(\bar{x}), DU^*(\bar{x}))\in\Omega \qquad \text{for all } \bar{x}\in T^d$$

and that Ω was assumed to be open. Therefore there exists a number $R>0$ such that the ball of radius R

$$B_R(x, U^*(\bar{x}), DU^*(\bar{x}))\in\Omega \qquad \text{for all } \bar{x}\in T^d.$$

To prove (4.3), it suffices to check that

$$|U^*-U^s|_0+|DU^*-DU^s|_0<R.$$

Similarly, to verify that $\partial_{x_{n+1}}U^s>(2M)^{-1}$, it suffices to check that $|\partial_{x_{n+1}}(U^*-U^s)|<(2M)^{-1}$. In other words, with a positive number $0<\eta\leq\min((2M)^{-1}, c^{-1}R)$ it is sufficient to verify

$$|U^*-U|_{C^1}<\eta \qquad \text{for } U=U^s. \tag{4.4}$$

We will also require $\eta\leq\varepsilon$, where ε is the given number in theorem 3.

(b) To facilitate the estimates, we fix the norms. For simplicity we require τ to be an integer with

$$\tau>(n+1)/2 \tag{4.5}$$

so that $\|\ \|_\tau$ dominates the uniform norm. The quantity $\|E(U^s)\|_\tau$ will be estimated by $\varepsilon_s>0$, a sequence tending to zero rapidly. Because of the form of the estimates (3.14) for L_s, we measure $E=E(U)$ in the scale of the norm of $H^{\tau+r}$ $(r=0,1,\ldots)$, but $U^{s+1}-U^s$ in the norms of $H^{-\tau+r}$. The loss of differentiability is given by $q=2\tau+2$; namely, for $r\geq 0$, E takes $H^{-\tau+r+q}$ into $H^{-\tau+r+q-2}=H^{\tau+r}$ since E is a differential operator of second order, and L_s^{-1}, considered as operator from $H^{\tau+r}\to H^{-\tau+r}$, has according to (3.14) a bound independent of ν. Thus from $U^s-x_{n+1}\in$

$H^{-\tau+r+q}$ we obtain ν-independent bounds only for $\|U^{s+1}-U^s\|_{-\tau+r}$, illustrating the loss of q derivatives.

For the following we fix the sequence ε_s by

$$\varepsilon_{s+1}=\varepsilon_s^\kappa, \qquad \kappa=\tfrac{4}{3}$$

(although we could take any number $\kappa\in(1,2)$), and reserve the freedom to choose $\varepsilon_0>0$ sufficiently small. We set

$$m=4q, \qquad q=2\tau+2 \tag{4.6}$$

and

$$N_s=c_*^{-1}\varepsilon_{s+1}^{-1/m}, \qquad N=c_*\varepsilon_0^{-1/m} \tag{4.7}$$

with a large constant $c_*\geq 1$. Finally we fix an integer k with

$$k=2m+1, \qquad l=k+q. \tag{4.8}$$

In the following we will denote by c positive constants which depend on γ, τ, r, M and $|G|_{C^b}$ but not on ν or ε_0.

(c) LEMMA 4.1. *Let U^* be a function satisfying the conditions* (2.13) *with* $a=-\tau+m+q=9\tau+10$. *Then there exists a constant $c_*>0$ such that the function U^0, defined by* (4.1), (4.7), *has, for sufficiently small ε_0, the following properties*:

(i) $U^0-x_{n+1}\in C^\infty(T^d)$, $\qquad \partial_{x_{n+1}}U^0>(2M)^{-1}$.

(ii) $|U^0-U^*|_{C^1}<c\|U^0-U^*\|_m=O(\varepsilon_0^\rho)<\eta/2$.

(iii) $\|E(U^0)-E(U^*)\|_\tau<\frac{1}{2}\varepsilon_0$.

(iv) $\|E(U^0)\|_{\tau+k}<c\varepsilon_0^{-\lambda}$, $\qquad \lambda=k/m-1>1$.

Here $\rho>0$ and c is an appropriate constant.

LEMMA 4.2. *Let U^s be the sequence defined by the recursion* (4.2) *and by* (4.1). *Assume further that $\|E(U^0)\|_\tau<\varepsilon_0$. Then U^s remains in the domain of definition of G and satisfies the estimates*

(a) $\|U^s-U^{s-1}\|_{-\tau}<c'\varepsilon_{s-1}$,

(b) $\|U^s-U^{s-1}\|_{-\tau+l}<\varepsilon_s^{-\lambda}$, $\qquad \lambda=k/m-1$,

(c) $\|E(U^s)\|_\tau<\varepsilon_s$

for all $s\geq 1$ with some constant c', provided ε_0 is chosen small enough.

From lemma 4.1 and lemma 4.2 we will conclude readily that U^s converges in H^m to the desired solution U of $E(U)=0$. Actually, this sequence converges in H^r for any r, as will be shown at the end of this section, but first we give the proofs of these two lemmas.

(d) *Proof of lemma* 4.12. Clearly, by definition of S_N, $U^0-x_{n+1}\in C^\infty(T^d)$ and by (2.24)

$$\|U^0-U^*\|_m\leq c_1N^{\tau-q}\|U^*-x_{n+1}\|_a\leq c_2N^{-(\tau+2)}=O(\varepsilon_0^{(\tau+2)/m})\to 0$$

and (i), (ii) are proven with $\rho=(\tau+2)/m$ and for ε_0 sufficiently small.

To verify (iv), we estimate

$$\|U^0-x_{n+1}\|_{-\tau+l}\leq c_3N^{l-\tau-a}\|U^*-x_{n+1}\|_a\leq c_4N^{k-m}\leq c_4c_*^{k-m}\varepsilon_0^{-\lambda} \tag{4.9}$$

since $l-\tau-a=l-q-m=k-m$; here we used (2.24). By (2.22) we obtain

$$\|E(U^0)\|_{\tau+k}\le c_5|G|_{C^{\tau+k+2}}(1+\|U^0-x_{n+1}\|_{\tau+k+2}).$$

Since $\tau+k+2=-\tau+q+k=-\tau+l$, we conclude

$$\|E(U^0)\|_{\tau+k}\le c_6\varepsilon_0^{-\lambda}.$$

Note that $b=\tau+k+2=17\tau+19$, where b is the constant of theorem 3.

To verify (iii), we show first that

$$\|E(U^0)-E(U^*)\|_\tau\le c_7\|U^0-U^*\|_{\tau+2} \tag{4.10}$$

and then use (2.24) to get with (4.7)

$$\|U^0-U^*\|_{\tau+2}=\|U^0-U^*\|_{-\tau+q}\le c_8N^{-m}\|U^*-x_{n+1}\|_a\le c_8c_*^{-m}\varepsilon_0M.$$

Hence if we choose c_* so that $c_*^m\ge 2c_7c_8M$, then (iii) follows.

To verify (4.10), we use (2.23), where f involves a first or second derivative of G in the Euler-Lagrange expression E, ϕ represents $(U^0-x_{n+1}, \bar{D}U^0, \bar{D}^2U^0)$, and ψ represents $(U^*-x_{n+1}, \bar{D}U^*, \bar{D}^2U^*)$. From (ii) we conclude that $\|U^0-x_{n+1}\|_m\le M+\frac{1}{2}\eta\le c$; hence, since $m>\tau+2$,

$$\|\phi\|_\tau\le\|\phi\|_{m-2}\le\|U^0-x_{n+1}\|_m\le c,$$

so that (2.23) yields

$$\|E(U^0)-E(U^*)\|_\tau\le c|G|_{C^{\tau+3}}\|U^0-U^*\|_{\tau+2}.$$

This proves (4.10) with a constant depending on $|G|_{C^{\tau+3}}$, $\tau+3\le b$. This completes the proof of lemma 4.1.

(e) The proof of lemma 4.2 will be carried out by induction. We begin with $s=0$, when (a), (b) are meaningless and (c) is satisfied by assumption. We assume now that (a), (b), (c) hold for all $0, 1, \dots, s$ and prove it for s replaced by $s+1$.

Step 1. To show that $U=U^s$ remains in the domain of definition, i.e. satisfies (4.4), we conclude from (a), (b) and (2.16) that for $0\le r\le l$

$$\|U^s-U^{s-1}\|_{-\tau+r}\le(c'\varepsilon_{s-1})^{1-r/l}\varepsilon_s^{-\lambda r/l}\le\varepsilon_{s-1}\varepsilon_s^{-(\lambda+1)r/l}, \tag{4.11}$$

where we have absorbed the constant c' by choosing ε_0, hence ε_s, small enough. If we take $r=3q$, it follows from $k=l-q<l$, $\lambda+1=k/m$ that

$$(\lambda+1)\frac{r}{l}=\frac{k}{m}\frac{r}{l}<\frac{r}{m}=\frac{3q}{m}=\frac{3}{4}=\kappa^{-1},$$

i.e.

$$\|U^s-U^{s-1}\|_{-\tau+3q}\le\varepsilon_s^\rho, \qquad \rho>0, \tag{4.12'}$$

and hence by (2.10)

$$|U^s-U^{s-1}|_{2q}\le c\varepsilon_s^\rho$$

since $-\tau+3q-\tau=2q+2>2q$. Therefore

$$|U^s-U^*|_{2q}\le|U^0-U^*|_{2q}+\sum_{\sigma=1}^{s}|U^\sigma-U^{\sigma-1}|_{2q}\le\frac{\eta}{2}+c\sum_{\sigma=1}^{s}\varepsilon_\sigma^\rho\le\frac{\eta}{2}+\tilde{c}\varepsilon_0^\rho<\eta \tag{4.12''}$$

if ε_0 is small enough. Since $2q>2$, it is clear that (4.4) is satisfied for $U=U^s$.

From (b) and (4.9) we find

$$\|U^s - x_{n+1}\|_{-\tau+l} \le \|U^0 - x_{n+1}\|_{-\tau+l} + \sum_{\sigma=1}^{s} \|U^\sigma - U^{\sigma-1}\|_{-\tau+l} \le c\varepsilon_0^{-\lambda} + \sum_{\sigma=1}^{s} \varepsilon_\sigma^{-\lambda} \le c\varepsilon_s^{-\lambda},$$

which with (2.22) implies

$$\|E(U^s)\|_{\tau+k} \le c|G|_{\tau+k+2}(1+\|U^s - x_{n+1}\|_{-\tau+l}) \le \hat{c}\varepsilon_s^{-\lambda}.$$

We combine this inequality with (c). Introducing

$$h_s = \varepsilon_s^{1/m}$$

and using the interpolation estimates (2.16), we get

$$\|E(U^s)\|_{\tau+r} \le c\varepsilon_s/h_s^r \qquad \text{for } 0 \le r \le k. \tag{4.13}$$

For later purposes we show that

$$\|U^s - x_{n+1}\|_{k+1} \le ch_s^{-(k-\tau)}. \tag{4.14}$$

This follows from (4.11) for $r = k+1+\tau < l$, where we use that

$$(\lambda+1)\frac{r}{l} < \lambda + 1 = \frac{k}{m},$$

hence

$$\|U^s - U^0\|_{k+1} \le \sum_{\sigma=1}^{s} \varepsilon_{\sigma-1}\varepsilon_\sigma^{-k/m} \le \varepsilon_s^{-k/m+3/4} < h_s^{-(k-\tau)},$$

since $m > \frac{4}{3}\tau$. To obtain the desired estimate (4.14), we need an appraisal of $\|U^0 - x_{n+1}\|_{k+1}$ which is obtained by interpolating between (4.9) and the assumed estimate (2.13):

$$\|U^0 - x_{n+1}\|_a \le \|U^* - x_{n+1}\|_a \le M, \qquad a = -\tau + m + q = 9\tau + 10.$$

With $\lambda = (k-m)/m$ this leads to

$$\|U^0 - x_{n+1}\|_{k+1} \le c\varepsilon_0^{-\lambda(k+1-a)/(k-m)} = ch_0^{-(k+1-a)} \le ch_0^{-(k-\tau)}$$

since $a \ge \tau + 1$. Combining this with the previous estimate yields indeed (4.14).

Step 2. Next we verify (a), (b) for s replaced by $s+1$ using (4.2). We claim that W satisfies

$$\|W\|_{-\tau+r} \le c\varepsilon_s/h_s^r \qquad \text{for } 0 \le r \le k. \tag{4.15}$$

It suffices to check this for $r=0$ and $r=k$. From the inequality (3.14) we find

$$\|W\|_{-\tau} \le c\|(\partial_{x_{n+1}}U^s)E(U^s)\|_\tau \le c\|\partial_{x_{n+1}}U^s\|_\tau \|E(U^s)\|_\tau \le \tilde{c}\varepsilon_s$$

since by (4.12″) the first factor is bounded. By the second inequality of (3.14) for $t = \tau$, $r = k$ we find

$$\|W\|_{-\tau+k} \le c\{\|(\partial_{x_{n+1}}U^s)E(U^s)\|_{\tau+k} + A^{k/(k-\tau)}\|E(U^s)\|_\tau\},$$

where A is a bound for the coefficients (see (3.11)). Using (2.22) and (4.14), we can estimate A by

$$A \le c|G|_{k+2}(1+\|U^s - x_{n+1}\|_{k+1}) \le \hat{c}h_s^{-(k-\tau)}.$$

Therefore we obtain with (4.13)

$$\|W\|_{-\tau+k} \le c\{\|E(U^s)\|_{\tau+k} + \|\partial_{x_{n+1}}U^s\|_{\tau+k}\varepsilon_s + h_s^{-k}\varepsilon_s\} \le \tilde{c}h_s^{-k}\varepsilon_s.$$

Here we have used that for $U=U^s$

$$\|\partial_{x_{n+1}}U\|_{\tau+k}=\|1+\partial_{x_{n+1}}(U-x_{n+1})\|_{\tau+k}\le 1+\|U-x_{n+1}\|_{\tau+k+1}\le c/h_s^k.$$

This proves (4.15). Similarly we find

$$\|(\partial_{x_{n+1}}U^s)W\|_{-\tau+k}\le c\varepsilon_s/h_s^k, \tag{4.16}$$

and therefore from (4.2) and (2.24)

$$\|U^{s+1}-U^s\|_{-\tau+l}\le N_s^q c\frac{\varepsilon_s}{h_s^k}=ch_{s+1}^{-q}\frac{\varepsilon_s}{h_s^k}$$

since $N_s\le h_{s+1}^{-1}$ by (4.7). The right-hand side can be estimated by

$$\varepsilon_{s+1}^{-\lambda}=\varepsilon_{s+1}h_{s+1}^{-k}$$

since

$$\frac{\varepsilon_s}{h_s^k}=h_s^{m-k}<h_{s+1}^{m-k+q}=\frac{\varepsilon_{s+1}}{h_{s+1}^{k-q}},\qquad k>2m=m+4q.$$

This establishes (b). To prove (a), we use (2.21) for $t=\tau$ to get

$$\|U^{s+1}-U^s\|_{-\tau}\le\|(\partial_{x_{n+1}}U^s)W\|_{-\tau}\le\|\partial_{x_{n+1}}U^s\|_\tau\|W\|_{-\tau}\le c\varepsilon_s,$$

where we have used (4.15) with $r=0$.

Step 3. It remains to prove (c) for $s+1$ in place of s. We write $V=U^{s+1}-U^s$ and use that

$$\|E(U^{s+1})-E(U^s)-E'(U^s)V\|_\tau\le c\|V\|_{\tau+2}^2.$$

Now

$$\|V\|_{\tau+2}=\|V\|_{-\tau+q}\le N_s^q\|(\partial_{x_{n+1}}U)W\|_{-\tau}\le cc_*^{-q}h_{s+1}^{-q}\varepsilon_s.$$

Since

$$m=4q=\frac{2\kappa}{2-\kappa}q,$$

we have

$$(h_{s+1}^{-q}\varepsilon_s)^2=\varepsilon_{s+1},$$

and, by taking c_* large enough, we obtain

$$\|E(U^{s+1})-E(U^s)-E'(U^s)V\|_\tau\le c\|V\|_{\tau+2}^2<\tfrac{1}{2}\varepsilon_{s+1}.$$

To estimate the remaining terms, we use the identity (3.17) to get

$$\|E(U^{s+1})\|_\tau\le\|W\partial_{x_{n+1}}E(U^s)\|_\tau+\|E'(U^s)(I-S_{N_s})(U^s_{x_{n+1}}W)\|_\tau+\varepsilon_{s+1}/2.$$

The first term is estimated using (4.13) for $r=1$ and (4.15) for $r=2\tau$:

$$\|W\partial_{x_{n+1}}E(U^s)\|_\tau\le\|W\|_\tau\|E(U^s)\|_{\tau+1}\le c\varepsilon_s^2/h_s^{2\tau+1}.$$

This term is dominated by $\varepsilon_s^2/h_s^{2q}<\varepsilon_{s+1}$ since $2q>2\tau+1$, and therefore can be made less than $\frac{1}{4}\varepsilon_{s+1}$. For the last term we find

$$\begin{aligned}\|E'(U^s)(I-S_{N_s})(U^s_{x_{n+1}}W)\|_\tau&\le c\|(I-S_{N_s})(U^s_{x_{n+1}}W)\|_{\tau+2}\\&\le cN_s^{-k+q}\|U^s_{x_{n+1}}W\|_{-\tau+k}\le ch_{s+1}^{k-q}\frac{\varepsilon_s}{h_s^k},\end{aligned}$$

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where we have used (4.16). Since by (4.6) and (4.8)

$$k > m + \frac{\kappa}{\kappa - 1} q = m + 4q = 2m$$

or $(k-m)(\kappa-1) > \kappa q$, we find that

$$\frac{h_{s+1}^{k-q}}{h_s^k} \frac{\varepsilon_s}{\varepsilon_{s+1}} \to 0 \qquad \text{as } \varepsilon_0 \to 0$$

and we can make the last term $< \varepsilon_{s+1}/4$ so that

$$\|E(U^{s+1})\|_\tau \le c\left(\frac{\varepsilon_s^2}{h_s^{2\tau+1}} + h_{s+1}^{k-q} \frac{\varepsilon_s}{h_s^k}\right) + \tfrac{1}{2}\varepsilon_{s+1} < \varepsilon_{s+1},$$

which completes the induction.

(f) To prove theorem 3, we assume ε_0 is chosen so small that the lemmas 4.1 and 4.2 hold. If we set $\delta = \varepsilon_0/2$ in theorem 3, it follows from (2.14) and lemma 4.1(iii) that

$$\|E(U^0)\|_\tau \le \|E(U^*)\|_\tau + \tfrac{1}{2}\varepsilon_0 < \delta + \varepsilon_0/2 < \varepsilon_0$$

so that the assumption of lemma 4.2 holds. Thus we obtain a sequence U^s which by (4.12′) converges in $H^{-\tau+3q} \subset C^2$ to an element U. Since $E(U^s) \to E(U)$, we find from lemma 4.2(c) that $E(U) = 0$, i.e. U is the desired solution. Finally, from (4.12′) and lemma 4.1(ii) we find

$$|U - U^*|_{C^1} \le c\|U - U^*\|_{-\tau+3q} < \eta \le \varepsilon,$$

where ε is the preassigned number of theorem 3. This proves (2.15). The smallness condition on $\varepsilon_0 = 2\delta$ depends on $|G|_{C^b}$, where $b = \tau + k + 2$, i.e. $b = \tau + 2m + 3 = 17\tau + 19$.

(g) Thus the proof of theorem 3 is established except for the smoothness proof and the estimates of the higher derivatives. Here the point is that we do not want to impose further smallness restrictions on $\|E(U^*)\|_\tau$ but find estimates for $\|U - x_{n+1}\|_{-\tau+r}$ which are independent of $\nu \in (0, 1)$. We make use of the above sequence U^s and show that it converges in $H^{-\tau+r}$ for any r. Here we follow the approach given in [**24**].

For the following we fix ε_0 and use the sequence $\varepsilon_s = \varepsilon_0^{(\kappa^s)}$, $\kappa = \frac{4}{3}$, as before and define N_s, N by (4.7). With C we denote constants depending on the previous constants, on F and r but not on ν or s. However, k and $l = k + q$ will have a different meaning from before, and k will be chosen so large that

$$\sigma = \frac{k}{k - \tau} \le \frac{16}{15} < \kappa = \frac{4}{3}. \tag{4.17}$$

LEMMA 4.3. *If U^s is the sequence constructed above and $l = k + q$, where k satisfies* (4.17), *then there exists a constant C such that*

$$\|U^s - x_{n+1}\|_{-\tau+l} \le C\varepsilon_s^{-4/3}.$$

Here C depends on l but not on $\nu \in (0, 1)$ or s.

Before proving this lemma, we will show that it implies that $U = \lim_{s\to\infty} U^s$ belongs to C^∞ by proving that U^s converges in $H^{-\tau+r}$ for any r. We set $l = 3r$ and pick r so large that $k = 3r - q$ satisfies (4.17). Then

$$\|U^{s+1} - U^s\|_{-\tau+l} \leq \|U^{s+1} - x_{n+1}\|_{-\tau+l} + \|U^s - x_{n+1}\|_{-\tau+l} \leq 2C\varepsilon_{s+1}^{-4/3}.$$

On the other hand, by lemma 4.2(a) we have

$$\|U^{s+1} - U^s\|_{-\tau} \leq c'\varepsilon_s$$

and therefore by interpolation

$$\|U^{s+1} - U^s\|_{-\tau+r} \leq (c'\varepsilon_s)^{2/3}(2C\varepsilon_{s+1}^{-4/3})^{1/3} = C'\varepsilon_s^\rho, \qquad \rho = \tfrac{2}{27},$$

which proves that U^s converges in $H^{-\tau+r}$ to U and that

$$\|U - U^0\|_{-\tau+r} \leq C' \sum_{s=0}^{\infty} \varepsilon_s^\rho = C''.$$

Since also

$$\|U^0 - x_{n+1}\|_{-\tau+r} \leq N^{r-m-q}\|U^* - x_{n+1}\|_a \leq C''', \tag{4.18}$$

we conclude that

$$\|U - x_{n+1}\|_{-\tau+r} \leq C_r$$

with some constant $C = C_r$ and $U - x_{n+1} \in C^\infty$. This proves (2.15) and theorem 3 completely.

It remains to prove lemma 4.3. We set

$$M_s = \|U^s - x_{n+1}\|_{-\tau+l}$$

and derive a recursive estimate from (4.2):

$$M_{s+1} \leq M_s + \|U^{s+1} - U^s\|_{-\tau+l} \leq M_s + N_s^q\|(\partial_{x_{n+1}} U^s)W\|_{-\tau+k}.$$

Applying the basic inequality (3.14) for $\phi = W$ and $LW = (\partial_{x_{n+1}} U^s)E(U^s)$, we obtain

$$M_{s+1} \leq M_s + cN_s^q\{\|(\partial_{x_{n+1}} U^s)E(U^s)\|_{\tau+k} + A^{k/(k-\tau)}\|E(U^s)\|_\tau\}.$$

Now we use that

$$\|\partial_{x_{n+1}} U^s\|_{\tau+k} \leq 1 + \|U^s - x_{n+1}\|_{\tau+k+1} \leq 1 + M_s,$$
$$A \leq C(1 + \|U^s - x_{n+1}\|_{k+2}) \leq C(1 + M_s),$$

so that with σ as defined in (4.17)

$$M_{s+1} \leq M_s + CN_s^q\{(1 + M_s) + (1 + M_s)^\sigma\varepsilon_s\}.$$

We may assume that $M_s \geq 1$ and can simplify this inequality to

$$M_{s+1} \leq CN_s^q(M_s + M_s^\sigma\varepsilon_s). \tag{4.19}$$

The statement of lemma 4.3 is obtained by analysing this inequality. Using $\varepsilon_s \leq 1$, we obtain

$$M_{s+1} \leq 2CN_s^q M_s^\sigma.$$

If λ is any number $> q/(\kappa - \sigma)$, then there exists a $C = C_\lambda$ such that

$$M_s \leq CN_s^\lambda \leq C\varepsilon_{s+1}^{-\lambda/m}.$$

Since $q/(\kappa-\sigma)\le\frac{15}{4}q=\frac{15}{16}m$, the exponent $\lambda/m>\frac{15}{16}$, and if we choose $\lambda\in(\frac{15}{16}m, m)$ we have

$$M_s\le C\varepsilon_{s+1}^{-1}=C\varepsilon_s^{-4/3}. \tag{4.20}$$

5. *A uniqueness theorem*

(a) In this section we establish a local uniqueness property for the solutions of $E(U)=0$ also for $\nu=0$. For $\nu>0$ this followed from the elliptic character of the equations and their invariance under the translation $x_{n+1}\to x_{n+1}+\lambda$ (see theorem 2 in § 2).

In the following we assume that the solution U satisfies $U-x_{n+1}\in H^{-\tau+2q+1}$ and

$$\|U-x_{n+1}\|_{-\tau+2q+1}<M, \qquad \partial_{x_{n+1}}U>M^{-1} \tag{5.1}$$

for some positive constant M, where again $\tau>\frac{1}{2}(n+1)$ and $q=2\tau+2$. Then we have the following local uniqueness result:

THEOREM 4. *Let α satisfy* (1.9). *Then there exists a positive constant ε^* depending on M and the bounds for $|\alpha|$, $|F|_{C^2}$, γ, τ with the property: if U satisfies* (5.1), *and if the functions U and $\tilde U\in H^{-\tau+2q}$ are solutions of*

$$E(U)=0, \qquad E(\tilde U)=0 \qquad \text{for } \nu=0$$

and satisfy

$$\|\tilde U-U\|_{-\tau+2q}<\varepsilon^*,$$

then

$$\tilde U(x, x_{n+1}+\lambda)=U(\bar x)$$

for some $\lambda\in\mathbb{R}$.

This theorem has several implications for the solution $U=U(\bar x,\nu)$ of the equation $E^\nu(U)=0$. Since also $U(\bar x+\lambda e_{n+1},\nu)$ is a solution, it is convenient to normalize the solution by requiring that

$$[U-x_{n+1}]=0. \tag{5.2}$$

If $U=U(\bar x,\nu)$ is the solution constructed in theorem 3 for $\nu\in[0,\nu^*]$ and normalized by (5.2), then it is unique if the number ε in (2.15) is chosen small enough. For $\nu>0$ this follows from theorem 2, and for $\nu=0$ from theorem 4. From the uniqueness and boundedness of

$$U=U(x,\nu)\in H^{-\tau+q}$$

we conclude that U is a *continuous function of* ν, i.e.

$$U(\cdot,\nu)\in C([0,\nu^*], H^{-\tau+q}).$$

Finally, we obtain a *regularity result.* If the approximate solution U^* of theorem 1 is actually a solution of

$$E(U^*)=0 \qquad \text{for } \nu=0,$$

then we conclude from the uniqueness theorem that the solution U with $U-x_{n+1}\in C^\infty$ constructed in theorem 3 agrees, for $\omega=0$, with U^*. Hence we see that a solution satisfying (2.13) is necessarily in C^∞.

It would be desirable and more natural to derive this regularity result, as well as the ν-independent estimates of theorem 3, directly from the basic estimates (3.10) and (3.13) applied to the solutions $U = U(x, \nu)$ without the detour constructing the approximations U^s. However, we did not succeed in carrying this out.

(b) The proof of theorem 4 is based on the following simple lemma:

LEMMA 5.1. *Let $r < s < t$ be real numbers and let $\phi \in H^t$ and*

$$\|\phi\|_r \le c\|\phi\|_s^2, \qquad \|\phi\|_t < a$$

for some constant c. If also $t \ge t^* = 2s - r$, $a \le a^* = c^{-1}$, then $\phi = 0$. The constants t^*, a^* are optimal.

Proof. Since by assumption $s \le \frac{1}{2}(t+r)$, it follows for $\phi \ne 0$ from (2.16) that

$$\|\phi\|_s^2 \le \|\phi\|_t \|\phi\|_r < a\|\phi\|_r,$$

hence

$$\|\phi\|_r < ac\|\phi\|_r,$$

which gives a contradiction for $ac \le 1$.

One sees that the parameters t^*, a^* are optimal by taking $d = 1$, and with a large positive integer j,

$$\phi = c^{-1}(1+j^2)^{-t^*/2}\, e^{2\pi i j x}.$$

If either $t < t^*$ or $t = t^*$, $a > c^{-1}$, this function satisfies both hypotheses, but $\phi \ne 0$.

Proof of theorem 4. We set

$$V(\bar{x}, \lambda) = \tilde{U}(\bar{x} + \lambda e_{n+1}) - U(\bar{x}), \qquad W(\bar{x}, \lambda) = (\partial_{x_{n+1}} U)^{-1} V(\bar{x}, \lambda)$$

and determine $\lambda = \lambda^*$ so that the mean value

$$f(\lambda) = [W] = \int_T W(\bar{x}, \lambda)\, d\bar{x}$$

vanishes for $\lambda = \lambda^*$. To show the existence of λ^* and to find an estimate for it, we note that

$$\frac{df}{d\lambda} = [(\partial_{x_{n+1}} U)^{-1}(\partial_{x_{n+1}} \tilde{U}(\bar{x} + \lambda e_{n+1}))] \ge c_1^{-1} > 0$$

with some constant c_1 depending on M. Indeed, since

$$0 < \partial_{x_{n+1}} U \le 1 + \|U - x_{n+1}\|_{\tau+1} \le c_2$$

is bounded by such a constant and if ε^* is small enough, we conclude from the assumption of theorem 4 that

$$|\partial_{x_{n+1}}(\tilde{U} - U)| \le \|\tilde{U} - U\|_{\tau+1} \le c'\varepsilon^* < (2M)^{-1},$$

hence $\partial_{x_{n+1}} \tilde{U} > (2M)^{-1}$ and

$$\frac{df}{d\lambda} \ge (2Mc_2)^{-1} = c_1^{-1}.$$

This ensures the existence of a unique zero λ^* of f satisfying

$$|\lambda^*| \le c_1 |f(0)| \le c_1 M \|\tilde{U} - U\|_0 < c_1 M \varepsilon^*.$$

From this we derive for $V=V(x,\lambda^*)$ the estimate

$$\|V\|_{-\tau+2q}<c_3\varepsilon^*. \tag{5.3}$$

This follows from

$$\|V\|_{-\tau+2q}\le \|(U')^{-1}(\tilde U(\bar x+\lambda^* e_{n+1})-U(\bar x+\lambda^* e_{n+1}))\|_{-\tau+2q} + \|(U')^{-1}(U(\bar x+\lambda^* e_{n+1})-U(\bar x))\|_{-\tau+2q}.$$

The first term is $<c'\varepsilon^*$ by the assumption and by the observation that

$$\|(U')^{-1}\|_{-\tau+2q}\le cM^{-\tau+2q+2}\|U'\|_{-\tau+2q}\le c',$$

where we have used (2.22) with $f(\phi)=\phi^{-1}$. The second term can with (5.1) be estimated by

$$c\left\|\int_0^{\lambda^*}\partial_{x_{n+1}}U(\bar x+\lambda e_{n+1})\,d\lambda\right\|_{-\tau+2q}\le c'|\lambda^*|\,\|U-x_{n+1}\|_{-\tau+2q+1}\le c'M\lambda^*,$$

which gives (5.3).

Thus the function $W=W(x,\lambda^*)$ has mean value zero, and we have to show that $V=V(x,\lambda^*)$, or equivalently W, vanishes if ε^* is chosen small enough. For this purpose we use that

$$\|E'(U)V\|_\tau=\|E(U+V)-E(U)-E'(U)V\|_\tau\le c_4\|V\|^2_{\tau+2}$$

since $E(U)=0$ and $E(U+V)=0$. On the other hand, the identity (3.2) gives

$$U'E'(U)V=-LW,\qquad \text{where } U'=\partial_{x_{n+1}}U,$$

so that

$$c_5^{-1}\|LW\|_\tau\le\|(U')^{-1}LW\|_\tau=\|E'(U)V\|_\tau\le c_4\|V\|^2_{\tau+2}.$$

Finally, the basic estimate (3.14), which holds also for $\nu=0$, yields

$$\|V\|_{-\tau}\le c_6\|W\|_{-\tau}\le c_7\|LW\|_\tau\le c_4c_5c_7\|V\|^2_{\tau+2},$$

where we have used that the mean value of W is zero. Now we can apply lemma 5.1 with $r=-\tau$, $s=\tau+2=-\tau+q$, $t=-\tau+2q$ to conclude that $V=0$ if only $\|V\|_{-\tau+2q}$ is small enough, which proves theorem 4.

6. *The quasi-periodic case*

(a) We conclude with a simple generalization of the above results to integrands $F=F(x,u,p)$ which depend only quasi-periodically on x but still periodically on u. We illustrate the statement with the example

$$\Delta u-\lambda f(x,u),\qquad \Delta=\sum_{\mu=1}^n\partial^2_{x_\mu}, \tag{6.1}$$

where $f=f(x,u)$ is given in terms of a periodic function $\Phi=\Phi(\xi_1,\ldots,\xi_K,u)$ by

$$f(x,u)=\Phi((\omega_1,x),(\omega_2,x),\ldots,(\omega_K,x),u),$$

$$\Phi\in C^\infty(T^{K+1}),\qquad (\omega_k,x)=\sum_{\mu=1}^n\omega_{k\mu}x_\mu,\qquad k=1,2,\ldots,K.$$

This means that f is quasi-periodic in x_μ with a frequency basis

$$\omega_{k\mu}(k=1,2,\ldots,K)$$

and of period 1 in u. We seek solutions $u = u(x)$ of (6.1) for which $e^{2\pi i u}$ is quasi-periodic in x and for which $u - (\alpha, x)$ is bounded for some given $\alpha \in \mathbb{R}^n$. We will simply call such solutions quasi-periodic even though they grow linearly.

More specifically, we require that these solutions u admit a representation in terms of a function $U = U(\bar{\xi})$, $\bar{\xi} = (\xi_1, \xi_2, \ldots, \xi_{K+1}) \in \mathbb{R}^{K+1}$, so that

$$U(\bar{\xi}) - \xi_{K+1} \in C^\infty(T^{K+1}), \qquad \partial_{\xi_{K+1}} U > 0, \tag{6.2}$$

in the form

$$u(x) = U \circ \Omega(x), \tag{6.3}$$

where $\Omega : \mathbb{R}^n \to \mathbb{R}^{K+1}$ is a linear mapping given by

$$\begin{aligned} &x \to \Omega(x) = \bar{\xi}, \\ &\xi_k = \sum_{\mu=1}^{n} \omega_{k\mu} x_\mu, \qquad k = 1, 2, \ldots, K+1, \\ &\omega_{K+1,\mu} = \alpha_\mu \end{aligned} \tag{6.4}$$

so that $\xi_{K+1} = (\alpha, x)$. Thus

$$u(x) - (\alpha, x) = (U - \xi_{K+1}) \circ \Omega(x)$$

is quasi-periodic in the sense required.

We claim that such quasi-periodic solutions of (6.1) also exist if $|\lambda|$ is sufficiently small, provided Ω satisfies the Diophantine condition

$$\sum_{\mu=1}^{n} \left(\sum_{k=1}^{K+1} \omega_{k\mu} j_k \right)^2 \geq \gamma \left(\sum_{k=1}^{K+1} \xi_k^2 \right)^{-\tau} \tag{6.5}$$

for all integers $j_1, \ldots, j_{K+1}$, which are not all zero. This is a direct generalization of example 1 in § 1, which corresponds to

$$K = n, \qquad \omega_{k\mu} = \delta_{k\mu} \qquad \text{for } k = 1, 2, \ldots, K = n.$$

(b) We formulate this statement for a general integrand $F = F(x, u, p)$. We assume that $\omega : \mathbb{R}^{n+1} \to \mathbb{R}^{K+1}$ is a linear map given by

$$\xi_k = \sum_{\mu=1}^{n} \omega_{k\mu} x_\mu \qquad (k = 1, 2, \ldots, K), \qquad \xi_{K+1} = x_{n+1},$$

and that

$$\Phi \in C^\infty(T^{K+1} \times \mathbb{R}^n), \qquad \sum_{\nu,\mu=1}^{n} \Phi_{p_\nu p_\mu}(\bar{\xi}, p) \lambda_\nu \lambda_\mu \geq \sum_{\nu=1}^{n} \lambda_\nu^2 \tag{6.6}$$

for all real $\lambda_1, \lambda_2, \ldots, \lambda_n$. The integrand $F = F(\bar{x}, p)$ is given by

$$F(\bar{x}, p) = \Phi(\omega(\bar{x}), p).$$

For a given vector $\alpha \in \mathbb{R}^n$ we seek quasi-periodic solutions u of the Euler equation

$$\sum_{\mu=1}^{n} \partial_{x_\mu} F_{p_\mu}(x, u, u_x) - F_u(x, u, u_x) = 0,$$

where we require that $u = u(x)$ can be represented in the form (6.3), Ω being defined by (6.4).

This gives rise to the differential equation

$$E(U)=\sum_{\nu=1}^{n} D_\mu \Phi_{p_\mu}(\xi, U, DU)-\Phi_u(\xi, U, DU)=0, \qquad D_\mu=\sum_{k=1}^{K+1} \omega_{k\mu}\partial_{\xi_k} \tag{6.7}$$

for U with the periodicity and monotonicity condition (6.2).

For this equation we have a straightforward generalization of theorem 1. *If* Ω *satisfies the Diophantine condition* (6.5), *and if* U^* *satisfies* (6.2) *and is an approximate solution of* (6.7), *i.e. if* $\|E(U^*)\|_\tau<\delta$, *then for a sufficiently small* δ *there exists an exact solution* U *of* $E(U)=0$ *satisfying* (6.2).

The quantitative formulation is precisely the same as given in theorem 1 and we do not repeat it. Also the proof is the same, and one may think that equation (6.7) can be viewed as a special case of the Euler equation of § 1 with n replaced by K. For $K>n$ this is actually not the case since the Legendre condition (6.6) is expressed in terms of the n variables $p\in\mathbb{R}^n$ and not in terms of the K variables π_k, where

$$p_\mu=\sum_{k=1}^{K+1} \omega_{k\mu}\pi_k.$$

In other words, the corresponding variational problem on the torus T^{K+1} is degenerate. For this reason a global theory (as developed in [**16**]) is not avaiable for the quasi-periodic case. Even for $n=1$ an analogue of Mather's theory for generalized quasi-periodic solutions of the ordinary differential equations

$$d^2x/dt^2=\lambda f(t,x), \qquad f(t,x)=\phi(\omega_1 t, \omega_2 t, \ldots, \omega_K t, x), \qquad \phi\in C^\infty(T^{K+1}), \tag{6.8}$$

is not available.

However, the local theory of this paper is not so sensitive. It provides quasi-periodic solutions of (6.8) with a frequency basis $\omega_1, \omega_2, \ldots, \omega_K$ and α if $|\lambda|$ is sufficiently small and if

$$|\omega_1 j_1+\omega_2 j_2+\cdots+\omega_K j_K+\alpha j_{K+1}|\geq \gamma' |\bar{j}|^{-\tau}$$

for all $\bar{j}\in\mathbb{Z}^{K+1}\backslash(0)$. This is a special case of a result in [**14**] which is here generalized to partial differential equations.

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